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The Average Gap Distribution For Generalized Zeckendorf Decompositions,

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Abstract

An interesting characterization of the Fibonacci numbers is that if we write the m as $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5, \dots$, then every positive integer can be written uniquel y as a sum of non-adjacent Fibonacci numbers. This is now known as Zeckendorf's Theorem [21], and simi lar decompositions exist for many other sequences $\{G_{n+1} = c_1G_n + \cdots + c_LG_{n+1-L}\}$ arising from recurrence relations. Much more is known. Using continued fraction approaches, Lekkerkerker [15] proved the average number of summands needed for integers in $[G_n, G_{n+1})$ is on the order of $C_{\rm Lek} n$ for a non-zero constant; this was improved by others to show the number of summands has Gaussian fluctuations about this mean.

Koloğlu, Kopp, Miller and Wang [13, 17] recently recast the problem combinatorially, reproving and generalizing these results. We use this new perspective to investigate the distribution of gaps between summands. We explore the average behavior over all $m \in [G_n, G_{n+1})$ for special choices of the c_i's. Specifically, we study the case where each $c_i \in \{0, 1\}$ and there is a g such that there are always exactly $g - 1$ zeros between two non-zero c_i's; note this includes the Fibonacci, Tribonacci and many other important special cases. We prove there are no gaps of length less than g, and the probability of a gap of length $j > g$ decays geometrically, with the decay ratio equal to the largest root of the recurrence relation. These methods are combinatorial and apply to related problems; we end with a discussion of similar results for far-difference (i.e., signed) decompositions.