REMARKS ON THE DIOPHANTANIAN EQUATIONS $a^2 \pm ab + b^2 = c^2$

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The present note is concerned with properties of ordered triples (a,b,c) of nonnegative integers which satisfy one of the two equations given in the title. Such solutions of

(1) $$a^2 + ab + b^2 = c^2$$

will be referred to as obtuse Pythagorian triples; the corresponding solutions of

(2) $$a^2 - ab + b^2 = c^2$$

will be called acute Pythagorian triples. If a, b, and c are relatively prime, the triples will be termed "primitive."

These two Diophantanian equations arise in a variety of ways; as it will be shown, even the Fibonacci numbers can generate and be generated by solutions thereof. The following problems will further exemplify this diversity. The reader is encouraged to pursue them at least to the point of recognizing their relevance:

1. Find three Pythagorian triangles of the same area. This problem was resolved by Euler in about 1781 [3].
2. Find solutions for the Diophantanian equation $x^2 + y^2 + z^2 = 2A^2$. In doing so, A. Gerardin [4] resolved several other equations as well.
3. Find three squares as consecutive terms of an arithmetic progression with common difference k. This problem along with its ramifications was discussed by R. L. Goodstein [5].
4. Remove a square of side x from each corner of a rectangular cardboard so that the remaining portion can be folded into an open box of maximum volume. What dimensions for the rectangle yield integral x? The first part is an old calculus problem probably dating back to Lamb [8] or earlier.
5. Find fourth-degree polynomials with integral coefficients whose extrema and inflection points have integral coordinates and are easily found (i.e., the constant term of the first derivative is zero).
6. Find integral triangles (triangles, all of whose sides are of integral length) with a 60° or 120° angle. According to Dickson [3], this problem was first considered by A. Girard, whose solutions were rediscovered dozens of times over the past three hundred years.

In fact, except for rediscoveries of various formulas generating their solutions, the Diophantanian equations under consideration are almost totally neglected in the mathematical literature. We hope to fill this gap at least partially. As a basis for the results to follow, we restate here without proof a procedure that was originally given by Zuge [13] in a slightly different format:

Representation Theorem: Let m and n be relatively prime positive integers of different parity and assume that $3 \nmid m$. Let $(a,b,c) = (4mn, 2mn + |m^2 - 3n^2|, m^2 + 3n^2)$. Then all primitive acute Pythagorian triples are either of the form (a,b,c) or of the form $(|a - b|, \max\{a,b\}, c)$ and all primitive obtuse Pythagorian triples are of the form $(|a - b|, \min\{a,b\}, c)$.

During the course of this work, using this representation theorem, a computer program was prepared by Russell Still, an undergraduate student, generating all primitive acute and obtuse Pythagorian triples for which $m,n \leq 50$. The author's gratitude is hereby expressed to Mr. Still for his valuable assistance. Copies of the printout are available from the author upon request.

The three types of triples given by the representation theorem may also be related by observing that if (a,b,c) is a solution of equation (1), then both $(a, a + b, c)$ and $(a + b, b, c)$ will satisfy equation (2). Consequently, in light of the geometrical interpretation afforded by Problem 6, they may be obtained from one another by the addition and/or subtraction of equilateral triangles. We shall further utilize this geometrical interpretation in regarding the triples as triangles and, in particular, in referring to c as the hypotenuse and to a and b as the legs of (a,b,c).

We first observe that since m and n are relatively prime, of different parity and $3 \nmid m$, the pair (m,n) must be congruent modulo 6 to one of the following pairs of numbers: (1,0), (1,2), (1,4), (2,1), (2,3), (2,5), (4,1), (4,3), (4,5), (5,0), (5,2), (5,4). Simple

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calculations show that in each case $m^2 + 3n^2 \equiv 1 \pmod 6$, that is, the hypotenuse of primitive obtuse and acute Pythagorian triples is always of the form $6k + 1$. This proves a conjecture by McArdle [9].

In fact, not only c, but every divisor of it must be of the same form. To prove this, let p be a prime divisor of $c = m^2 + 3n^2$. Observe first that $p \neq 2$ since m and n are of different parity, $p \neq 3$ since $3 \nmid m$, and $p \nmid m$ and $p \nmid n$ due to the relative primeness of m and n. Consequently, by raising both members of the congruence $m^2 \equiv -3n^2 \pmod p$ to the $(p-1)/2$th power and upon applying Fermat's theorem, one finds that $(-3)^{(p-1)/2} \equiv 1 \pmod p$.

Assume that $p = 6k + 5$. If k is even, say $k = 2s$, then $3^{6s+2} \equiv 1 \pmod{12s + 5}$ follows. If k is odd, say $k = 2s - 1$, one similarly obtains $3^{6s-1} \equiv -1 \pmod{12s - 1}$. Since both of these conclusions are contrary to known facts (see, for example, Theorem 20 on page 32 of [10]), the assumption that $p = 6k + 5$ is indeed untenable.

Conversely, if $c = 6k + 1$ is a prime, then it has a unique representation of the form $m^2 + 3n^2$ (see, for example, Theorem 5 on page 323 of [12]). Such m and n must clearly satisfy the restrictions stated in the Representation Theorem, hence each prime must appear as the hypotenuse of exactly one (two) primitive obtuse (acute) Pythagorian triple(s).

This last fact may be connected to a slight extension of Girard's results mentioned earlier, to conclude that each prime number of the form $6k + 1$ is uniquely expressible in both of the forms $x^2 \pm xy + y^2$, where x and y are positive integers. For example, one finds that the representations

$$7 = 1^2 + 1 \cdot 2 + 2^2 = 1^2 - 1 \cdot 3 + 3^2,$$
$$13 = 1^2 + 1 \cdot 3 + 3^2 = 1^2 - 1 \cdot 4 + 4^2, \text{ and}$$
$$19 = 2^2 + 2 \cdot 3 + 3^2 = 2^2 - 2 \cdot 5 + 5^2,$$

are unique.

If c has r distinct prime divisors, each of the form $6k + 1$, then repeated application of the well-known formula

$$(3) \qquad (m_1^2 + 3n_1^2)(m_2^2 + 3n_2^2) = (m_1m_2 \pm 3n_1n_2)^2 + 3(m_1n_2 \pm m_2n_1)^2$$

will yield exactly 2^{r-1} (2^r) primitive obtuse (acute) triples with hypotenuse c. Correspondingly, c will also have 2^{r-1} representations of each of the forms $x^2 \pm xy + y^2$. Equation (3) may also be regarded as a method of obtaining new triples out of old ones. Another such method is afforded by the matrix

$$M = \begin{pmatrix} -3 & 7 & 1 \\ 15 & 5 & 17 \\ 18 & 2 & 20 \end{pmatrix};$$

if (a,b,c) is an obtuse Pythagorian triple, then so is $(a,b,c)M$—viewed as a product of matrices.

Obtuse and acute Pythagorian triples may also be generated from Pythagorian triples by matrices. If we define

$$N = \begin{pmatrix} 2 & -2 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad L = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

then $(a,b,c)N$ is an obtuse and $(a,b,c)K$ and $(a,b,c)L$ are acute Pythagorian triples whenever $a^2 + b^2 = c^2$. Since N, K, and L are nonsingular, their inverses can also be utilized in transforming our results into the pythagorian setting.

The well-known [11] mechanical generation of sequences of Pythagorian triples from (21,220,221) and (41,840,841) by a systematic insertion of zeros may also be paralleled; each of the six sequences of triples given below are obtuse Pythagorian:

(120, 23, 133),
(10200, 203, 10303),
(1002000, 2003, 1003003), ...;

(129, 391, 469),
(1209, 39991, 40609),
(12009, 3999991, 4006009), ...;

(81, 1599, 1641),
(801, 159999, 160401),
(8001, 155999999, 16004001), ...;

(41, 399, 421),
(401, 39999, 40201),
(4001, 3999999, 4002001), ...;

(21 , 99 , 111), (201 , 9999 , 10101), (2001, 999999, 1001001), ...; (80 , 19 , 91), (9800 , 199 , 9901), (998000, 1999, 999001), ...

Other interesting sequences of obtuse and acute Pythagorian triples were discussed in two earlier notes by the author [1, 2] in a more geometric setting. Still other modes of generating infinite sequences (a_k, b_k, c_k) of primitive obtuse Pythagorian triples with special properties are depicted in the tables below.

On the basis of Table 1, one may prove, for example, that there are infinitely many obtuse Pythagorian triples whose legs differ by unity. The proof of this fact has been posed by the author as a problem in *The Fibonacci Quarterly* in a slightly different setting. The corresponding problem concerning the existence of acute Pythagorean triples (a,b,c) with $a - b = 1$ has a totally different solution: there are no such triples. The proof of this fact is also left to the reader.

k	m	n	a	b	c
1	1	2	8	7	13
2	13	2	104	105	181
3	13	28	1456	1455	2521
4	181	28	20272	20273	35113
5	181	390	282360	282359	489061
6	2521	390	3932760	3932761	6811741
7	2521	5432	54776288	54776287	94875313
8	35113	5432	762935264	762935265	1321442641

TABLE 1

In Table 2, $a_k - b_k = 2$ for each k. Again, an infinite number of such triples can be recursively generated from the ones displayed. It may also be noticed that each m_{2k-1} (n_{2k}) of Table 2 is twice as large as the corresponding m_{2k-1} (n_{2k}) of Table 1, thus the two tables could be obtained from one another. The proof of the fact that in each case $m_{2k} = c_k$ reveals some analogy to the well-known Fibonacci identity $F_{2n+1} = F_n^2 + F_{n+1}^2$.

k	m	n	a	b	c
1	2	1	5	3	7
2	7	4	57	55	97
3	26	15	781	779	1351
4	97	56	10865	10863	18817
5	362	209	368517	368515	908287

TABLE 2

Continuing with the obtuse case, one may further observe that for each $k = 2, 3, 4, \ldots,$ there exists a primitive obtuse Pythagorian triple (a,b,c) for which $c - b = k$; in fact, one such triple is given by

$$(2k - 1,\ 3k^2 - 4k + 1,\ 3k^2 - 3k + 1).$$

If, in addition, k is not a multiple of 3, then

$$(2k - 3,\ k^2 - 4k + 3,\ k^2 - 3k + 3)$$

is another such triple.

These triples may also serve as the basis for yet another observation: each odd number appears at least once as the shorter leg of a primitive obtuse Pythagorian triple. The two formulas above exhaust all such triples for powers of odd primes; with an increase in the number of divisors, one can observe a corresponding increase in the number of such triples.

One can also identify those primitive obtuse Pythagorian triples both of whose legs are odd. They are of the form

$$(2mn + m^2 - 3n^2,\ 2mn - m^2 + 3n^2,\ m^2 + 3n^2),$$

where $\frac{m}{3} < n < m$ and, as usual, n and m are relatively prime, of different parity, and $3 \nmid m$.

Conversely, if $n < \frac{m}{3}$ or $m < n$, then the primitive obtuse Pythagorian triples obtained via the Representation Theorem have an even leg. In fact, such a leg must be a multiple of 8 as it is readily shown via equation (1). For, suppose that a is even, say $a = 2x$. Then b and c must both be odd, say $b = 2y + 1$ and $c = 2z + 1$, and hence, from equation (1) we obtain that $2[x^2 + y(y + 1) - z(z + 1)] = x(2y + 1)$. This implies that x must be even. But then the left member of this equality is a multiple of 4, since $y(y + 1)$ and $z(z + 1)$ are clearly even. Therefore x is a multiple of 4 and, hence, a is a multiple of 8.

Incidentally, these observations provided a solution to a problem posed in the *American Mathematical Monthly* [7].

Furthermore, each multiple of 8 appears as the leg of a primitive obtuse Pythagorian triple. One such triple is given by the formula

$$(8k,\ 12k^2 - 4k - 1,\ 12k + 1)$$

where $k = 1, 2, 3, \ldots$. Again, not all such triples are given by this formula; for example, with the help of the printout one may verify that there are six different triples with a leg of 280.

If the triples are not required to be primitive, one may further observe that each of the following formulas yields obtuse Pythagorian triples for each $k = 1, 2, 3, \ldots$:

$$(8k + 2,\ 24k^2 + 8k,\ 24k^2 + 12k + 2),$$
$$(8k + 4,\ 12k^2 + 8k,\ 12k^2 + 12k + 4),$$
$$(8k + 6,\ 24k^2 + 32k + 10,\ 24k^2 + 36k + 14).$$

Since (6,10,14) is also such a triple, we may conclude that each positive integer except 1, 2, 4, and 8 can appear as the shorter leg of an obtuse Pythagorian triple (see [7]).

Concerning divisibility properties, we have the following two facts, which may be established by a case-by-case examination of all possible congruences:

(i) If (a,b,c) is an obtuse Pythagorial triple, then of the four numbers, a, b, $a + b$, and c, one is divisible by 3, one by 5, one by 7, and one by 8. Since (3,5,7) is one such triple, this result is the best possible.

(ii) If (a,b,c) is a primitive acute Pythagorian triple, and if $a + b$ is even, one has $a + b \equiv \pm 2 \pmod{12}$, while if $a + b$ is odd, the congruences $a + b \equiv \pm 1 \pmod{12}$ result.

In conclusion, paralleling results of Horadam [6], we associate the generalized Fibonacci sequences with the triples under consideration as follows. Let k be an arbitrary positive integer and assume that m and n satisfy the requirements set forth in the Representation Theorem. Define H_0 and H_1 by

$$H_0 = (-1)^{k+1}(F_k m - F_{k+1} n),\quad H_1 = (-1)^k (F_{k-1} m - F_k n),$$

and for $i \geq 2$ let $H_i = H_{i-1} + H_{i-2}$. Then it is easily shown that

$$H_k = n \quad \text{and} \quad H_{k+1} = m,$$

and thus H_k and H_{k+1} generate primitive obtuse and acute Pythagorian triples in the sense of the Representation Theorem. For example, the Fibonacci numbers may be associated with the triple (8,5,7) in the following manner:

$$(8,5,7) = (4F_2F_3,\ 2F_2F_3 + F_3^2 - 3F_2^2,\ F_3^2 + 3F_2^2).$$

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