$M(n, 4k + 1, 4k - 1)$ is an abelian group on n generators $A_1, A_2, \ldots, A_n$ subject to the n defining relations $kA_i - (2k + 1)A_{i+1} + kA_{i+2} = 0$ ($i = 1, 2, \ldots, n$), subscripts reduced mod n when necessary. Similarly, the homology group of $M(n, 4k + 3, 4k + 1)$ has defining relations $(k + 1)A_i - (2k + 1)A_{i+1} + (k + 1)A_{i+2} = 0$ ($i = 1, 2, \ldots, n$). Thus, $C_n(k, -(2k + 1), k)$ and $C_n(k + 1, -(2k + 1), k + 1)$ are the determinants of the "relation matrices" of these groups. When these circulants are nonzero, they are (in absolute value) equal to the orders of these groups (compare Fox [3, p. 149]). Note that $C_n(k + 1, -(2k + 1), k + 1)$ and $-C_n(k, -(2k + 1), k)$ are perfect squares for odd values of n, in agreement with the theorem of Plans [9]. In the case $k = 1$ [equations (37′) and (38′) above], the two-bridge knot of type {5, 3} is just the figure-eight knot. The homology groups of the branched cyclic coverings of this knot have been determined by Fox and agree with (37′) and (38′) (see [4, p. 193]).

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AN EXPANSION OF GOLUBEV'S 11 × 11 MAGIC SQUARE OF PRIMES TO ITS MAXIMUM, 21 × 21

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Edgar Karst, in the December 1972 issue of *The Fibonacci Quarterly* presented Golubev's magic square of order 11 consisting of prime numbers of the form $30x + 17$ and asked whether someone is able to attach a frame of order 13. The characteristics in Golubev's square are additionally "magic" in several ways which are repeated from the article cited. The stated requirements imposed were that:

1. All n rows, n columns, and 2 major diagonals have the same sum equal to n x the central number (n x 63317 in Golubev's square).
2. All included numbers be prime numbers equal to 17 plus an integral multiple of 30, with the multiple not divisible integrally by 17.
3. The sums of each pair of opposite (top and bottom or left and right) borders, excepting corner numbers, equal 2 x (the order less 2) x the central number [here 2 x 7 x 63317 or 2 x (n - 2) x 63317].
4. The sums of opposite outer elements in any row or column equal 2 x the central number, for any order.
5. The opposite corner primes in the squares of each order have the sum 2 x the central prime (2 x 63317).

The addition of frames of the order 13 through 21 was as far as I could go with positive primes of form $30x + 17$ centered about 63317, following the rules imposed above. There were about 46 unused primes left over in the series. This is of course not enough for another (23rd-order) frame, but the availability of more primes in the progression suggests the possibility of rearrangements of complementary pairs and that an additional degree of magicality might be accomplished in the 21 x 21 square.

The 21 x 21 square is shown in Figures 1a, 1b, and 1c, which are to be considered as the left, middle, and right thirds of the square, respectively.

87587	38867	91757	34457	95087	30137	98897
87887	64037	62417	18257	48947	84377	41687
37397	79337	79817	45587	85577	40787	91097
92297	37217	83177	78797	47777	85247	41117
34127	91367	43427	79427	63647	62627	65147
99527	33767	88667	46877	64667	73547	52757
23567	97607	37547	84737	49367	80177	59447
103217	27767	94397	41777	80747	80897	73127
22637	103307	31907	89867	17957	81077	53117
106907	23057	100787	36527	92987	81647	52727
19577	121157	24677	94907	33587	44927	74507
109517	18047	106487	31607	104327	44417	51257
16787	112577	19997	99707	22037	43787	101537
112247	13877	111977	25367	112877	84437	46187
2957	116867	2897	106277	13217	27917	57947
114827	9467	117497	20327	120977	53657	73877
11087	120647	8837	111347	46727	64007	61487
119027	5717	122867	2207	78857	41387	85517
6047	124847	1427	81047	41057	85847	35537
121487	2357	64217	108377	77687	42257	84947
947	87767	34877	92177	31547	96497	27737

FIGURE 1a. Left-Hand Third of Square

27617	102677	23627	49937	19697	110567	14537
90527	35747	96137	30467	102317	24137	107687
35507	96587	29837	43037	24197	107837	18287
32327	34667	98327	27827	104987	21467	111497
57077	67427	56807	70157	49157	75227	49877
52457	74567	51287	75767	49787	49727	24527
54767	71987	54167	72647	53597	50147	84407
67217	60527	60257	58427	59387	70937	66467
75437	64877	60497	54347	71147	65717	51197
55967	60017	64577	61637	63737	66617	70667
69737	72707	62477	63317	64157	53927	56897
57737	58067	62897	64997	62057	68567	68897
56957	60917	66137	72287	55487	61757	69677
60167	66107	66377	68207	67247	55697	59417
71867	54647	72467	53987	73037	76487	42227
74177	52067	75347	50867	76847	76907	102107
69557	59207	69827	56477	77477	51407	76757
94307	91967	28307	98807	21647	105167	15137
91127	30047	96797	83597	102437	18797	108347
36107	90887	30497	96167	24317	102497	18947
99017	23957	103007	76697	106937	16067	112097

FIGURE 1b. Middle Third of Square

116027	10457	119057	7187	122117	3677	125687
18917	113147	13457	118757	7727	124277	38747
114197	12227	119657	6947	125207	47297	89237
14867	118037	8387	124427	43457	89417	34337
77867	48197	79907	47207	83207	35267	92507
119087	72977	61967	79757	37967	92867	27107
68687	46457	77267	41897	89087	29027	103067
53507	45737	45887	84857	32237	98867	23417
73517	45557	108677	36767	94727	23327	103997
73907	44987	33647	90107	25847	103577	19727
52127	81707	93047	31727	101957	5477	107057
75377	82217	22307	95027	20147	108587	17117
25097	82847	104597	26927	106637	14057	109847
80447	42197	13757	101267	14657	112757	14387
67187	98717	113417	20357	123737	9767	123677
7547	53087	5657	106307	9137	117167	11807
48767	78437	62987	15287	117797	5987	115547
111767	8597	118247	47837	3767	120917	7607
12437	114407	6977	119687	46817	1787	120587
107717	13487	113177	7877	118907	62597	5147
10607	116177	7577	119447	4517	122957	39047

FIGURE 1c. Right-Hand Third of Square

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SOME EXTENSIONS OF PROPERTIES OF THE SEQUENCE OF FIBONACCI POLYNOMIALS

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Sequences of functions, $\langle g_n \rangle$, that satisfy the recursion formula

$$g_{n+2}(x) = axg_{n+1}(x) + bg_n(x) \tag{1}$$

where a and b are constants, inherit many of the properties of the sequence of Fibonacci polynomials [1]. This paper is intended to present some of these extensions.

1. BASIC DEFINITIONS AND PROPERTIES

Suppose that a and b are numbers. Let R denote the set of real numbers and C denote the set of complex numbers.

Definition 1: If $V \subseteq R$, $\mathcal{S}_{(a,b)}(V) = \{\langle g_n \rangle |$. For each natural number $p, g_p : V \to C$ and $g_{p+2}(x) = axg_{p+1}(x) + bg_p(x)$ for each $x \in V\}$.

If $V_1 \subseteq V_2$, it is easy to verify that if $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V_2)$, the corresponding sequence of restrictions is an element of $\mathcal{S}_{(a,b)}(V_1)$.

Theorem 1: If $\langle g_n \rangle$ and $\langle h_n \rangle$ are members of $\mathcal{S}_{(a,b)}(V)$ and $s:V \to C$ and $t:V \to C$, then $\langle sg_n + th_n \rangle \in \mathcal{S}_{(a,b)}(V)$. The proof for Theorem 1 is a straightforward computation.

Theorem 2: If $\{\langle g_n \rangle, \langle h_n \rangle\} \subseteq \mathcal{S}_{(a,b)}(V)$, then $\langle g_n \rangle = \langle h_n \rangle$ if and only if $g_1 = h_1$ and $g_2 = h_2$.

The proof of one of the implications of Theorem 2 is an application of the definition of equality of sequences. The other implication is an easy induction proof.

The elements of $\mathcal{S}_{(a,b)}(V)$ share a common summation formula.

Theorem 3: Suppose that for each natural number p, $g_p : V \to C$. $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$ if and only if for each natural number p,

$$(ax + b - 1)\sum_{j=1}^{p} g_j(x) = g_{p+1}(x) + bg_p(x) + (ax - 1)g_1(x) - g_2(x).$$

Proof: If $\langle g_n \rangle \in \mathcal{S}_{(a,b)}(V)$, the summation formula can be proved by a simple inductive argument. If $\langle g_n \rangle$ is a sequence of complex-valued functions on V with the given summation formula, then for each natural number p, the identity

$$(ax + b - 1)g_{p+1}(x) = (ax + b - 1)\left[\sum_{j=1}^{p+1} g_j(x) - \sum_{j=1}^{p} g_j(x)\right]$$