3. C. L. Liu. *Introduction to Combinatorial Mathematics*. New York: McGraw-Hill Book Company, Inc., 1968.
4. R. A. Rosenbaum. "An Application of Matrices to Linear Recursion Relations." *American Math. Monthly* 66 (1959):792-793.
5. C. Ryavec. "An Application of Spectral Theory to Fibonacci Numbers." *The Fibonacci Quarterly* 13, No. 4 (1975):307-308.
6. Marcellus E. Waddill. "Matrices and Generalized Fibonacci Sequences." *The Fibonacci Quarterly* 12, No. 4 (1974):381-386.

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CONVERGENCE PROPERTIES OF LINEAR RECURSION SEQUENCES

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1. INTRODUCTION

The object of this paper is to examine convergence properties of linear recursion sequences of complex numbers. Included are several theorems providing necessary and sufficient conditions, in terms of solutions of an associated auxiliary equation, for various cases and types of convergence.

The question of convergence of linear recursion sequences was raised by Singmaster in Advanced Problem H-179 [6]. The articles of Raphael [4], Shannon [5], and Jarden [3] give representations for linear recursion sequences of integers which are valid also for complex number sequences (the restriction being for aesthetic reasons) and have been useful in preparing this paper. These representations will be included without proof as the substance of the next section.

Let $a_1, a_2, \ldots, a_n$ be complex numbers, with $a_n \neq 0$. We define a linear recursion sequence $\{Q_i^{a(x),U}\}$ by

$$Q_i^{a(x),U} = \sum_{j=1}^{n} a_j Q_{i-j}^{a(x),U} \quad \text{for } n \geq 1 \tag{1}$$

where $U = [u_1, u_2, \ldots, u_n]$, $Q_{i-n}^{a(x),U} = u_i$ for $1 \leq i \leq n$, and $a(x) = x^n - a_1x^{n-1} - a_2x^{n-2} \ldots - a_n$. We will refer to $a(x) = 0$ as the auxiliary equation. The absence of the row vector U from the notation will imply that $U = [0, 0, \ldots, 0, 1]$, representing the normalized sequence we will be most concerned with in this paper. The order of the sequence $\{Q^{a(x),U}\}$ is n, and hence the restriction that $a_n \neq 0$ incures a unique definition of order.

2. REPRESENTATIONS

Some representations for linear recursion sequences will be helpful, and are presented here.

Noticing that the recursion relation (1) has a form similar to that of scalar multiplication of n-tuples leads to a matrix approach, presented for instance in Raphael [4]. Explicitly, we may write

$$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} Q_m^{a(x)} \\ Q_{m-1}^{a(x)} \\ \vdots \\ Q_{m-n+1}^{a(x)} \end{bmatrix} \quad \text{for } m \geq 0. \tag{2}$$

Another approach by Raphael [4] relates linear recursion sequences to power series in the following way:

$$\sum_{i=0}^{\infty} Q_i^{a(x)} x^i = 1/(1 - a_1x - a_2x^2 - \cdots - a_nx^n). \tag{3}$$

Let $r_1, \ldots, r_n$ be the n complex roots of $a(x)$ (repeated according to their multiplicity). Then, as Jarden [3, pp. 106-107] noted,

(4) $$D^{a(x)}Q_m^{a(x)} = D_1^{a(x)}r_{1,d_1}^{(m)} + D_2^{a(x)}r_{2,d_2}^{(m)} + \cdots + D_n^{a(x)}r_{n,d_n}^{(m)}, \qquad \text{for } m \geq 0.$$

where $D^{a(x)}$ is the constant determinant

(5) $$D^{a(x)} = \begin{vmatrix} r_{1,d_1}^{(0)} & r_{2,d_2}^{(0)} & \cdots & r_{n,d_n}^{(0)} \\ r_{1,d_1}^{(1)} & r_{2,d_2}^{(1)} & \cdots & r_{n,d_n}^{(1)} \\ \vdots & \vdots & & \vdots \\ r_{1,d_1}^{(n-1)} & \cdots & & r_{n,d_n}^{(n-1)} \end{vmatrix},$$

the constant determinants $D_i^{a(x)}$ are as in (5) with ith column deleted, and replaced by $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, d_i is the multiplicity of r_i among $r_1, \ldots, r_{i-1}$, and

(6) $$r_{i,d_i}^{(m)} = \binom{m}{d_i} r_i^{m-d_i}.$$

Also involving the roots $r_1, \ldots, r_n$ of $a(x)$, it has been shown (see [5], for example) that

(7) $$Q_m^{a(x)} = \sum_{\sum_{i=1}^{n} m_i = m} r_1^{m_1} \cdot r_2^{m_2} \cdot \cdots \cdot r_n^{m_n} \text{ for } m \geq 0,\ m_i \geq 0.$$

3. CONVERGENCE THEOREMS

In this section we will look at the convergence of $\{Q_i^{a(x)}\}_{i=0}^{\infty}$. Convergence of $\{Q_i^{a(x)}\}_{i=0}^{\infty}$ to zero will be considered first.

Theorem 3.1: Let $\{Q_i^{a(x)}\}$ be a linear recursion sequence. Then $\{Q_i^{a(x)}\}_{i=0}^{\infty}$ converges to zero if and only if all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$.

Proof: Suppose all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$. Notice that $a(x)$ is the characteristic polynomial of the matrix

(8) $$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdot & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$

Since by hypothesis, all the roots of $a(x)$, the characteristic polynomial of A, lie in $\{z \mid |z| \leq 1\}$, from Bodewig [1, p. 57], $\lim_{m\to\infty} A^m = 0$. It now follows from equation (2) that $\lim_{m\to\infty} Q_m^{a(x)} = 0$.

Let $\lim_{m\to\infty} Q_m^{a(x)} = 0$, and define $E = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Then E, $A \cdot E$, $A^2 \cdot E$, $\ldots$, $A^{n-1} \cdot E$ form a basis for the field of all n-dimensional column vectors, since $a(x)$ is also the minimum polynomial of A. Thus an arbitrary column vector X can be written as $C_0 \cdot E + C_1(A \cdot E) + C_2(A^2 \cdot E) + \cdots + C_{n-1}(A^{n-1} \cdot E)$. We compute using (2) that

(9) $$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lim_{m\to\infty} \begin{bmatrix} Q_m^{a(x)} \\ Q_{m-1}^{a(x)} \\ \vdots \\ Q_{m-n+1}^{a(x)} \end{bmatrix} = \lim_{m\to\infty} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lim_{m\to\infty}(A^m \cdot E) = (\lim_{m\to\infty} A^m) \cdot E.$$

Similarly, $\lim_{m \to \infty}(A^{m-i} \cdot A^i \cdot E) = 0$ for $1 \le i \le n - 1$. Thus,

$$(10) \qquad 0 = C_0 \cdot \lim_{m \to \infty}(A^m \cdot E) + C_1 \cdot \lim_{m \to \infty}(A^m \cdot A \cdot E) + \cdots + C_{n-1} \cdot \lim_{m \to \infty}(A^m \cdot A^{n-1} \cdot E) = \lim_{m \to \infty}(A^m \cdot X).$$

Therefore, $\lim_{m \to \infty} A^m = 0$, since X was arbitrary, and from Bodewig [1, p. 57] all the roots of $a(x)$, the characteristic polynomial of A, must lie in $\{z \mid |z| < 1\}$.

We may use this theorem and equation (3) to prove the following corollary.

Corollary 3.2: The infinite sum $\sum_{i=0}^{\infty} Q_i^{a(x)}$ exists and equals $1/(1 - a_1 - \cdots - a_n) = \frac{1}{a(1)}$ if and only if all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$.

Proof: If $\sum_{i=0}^{\infty} Q_i^{a(x)}$ exists, then $\lim_{m \to \infty} Q_i^{a(x)} = 0$, and from Theorem 3.1, all the roots of $a(x)$ must lie in $\{z \mid |z| < 1\}$.

If all the roots of $a(x)$ lie in $\{z \mid |z| < 1\}$, then all of the roots of $1 - a_1x - a_2x^2 - \cdots - a x$ must lie outside $\{z \mid |z| \le 1\}$, since the roots of $a(x)$ are the reciprocals of the roots of $1 - a_1x - a_2x^2 - \cdots - a_nx^n$. Note that $a(x)$ has no zero roots, since $a_n \ne 0$. Hence, the power series for $1/(1 - a_1x - \cdots - a_nx^n)$ is valid at $x = 1$. Thus, from equation (3), $\sum_{i=0}^{\infty} Q_i^{a(x)}$ exists, and $\sum_{i=0}^{\infty} Q_i^{a(x)} = 1/(1 - a_1x - \cdots - a_nx^n)$.

We are now ready to prove a theorem about the convergence of linear recursion sequences to nonzero complex numbers.

Theorem 3.3: Let $\{Q_i^{a(x)}\}$ be a linear recursion sequence. Then $\lim_{m \to \infty} Q_m^{a(x)} = b \ne 0$ if and only if 1 is a root of $a(x)$, and all the roots of $a(x)/(x - 1)$ lie in $\{z \mid |z| < 1\}$. Furthermore, if $a^*(x) = a(x)/(x - 1)$, then $\lim_{m \to \infty} Q_m^{a(x)} = 1/(a^*(1))$.

Proof: Let $\lim_{m \to \infty} Q_m^{a(x)} = b \ne 0$. Using equation (1), we find

$$(11) \qquad 0 \ne b = \lim_{m \to \infty} Q_m^{a(x)} = \lim_{m \to \infty}\left(\sum_{i=1}^{n} a_i Q_{m-i}^{a(x)}\right) = \sum_{i=1}^{n} a_i \lim_{m \to \infty} Q_{m-i}^{a(x)} = \left(\sum_{i=1}^{n} a_i\right) \cdot b.$$

Thus $\sum_{i=1}^{n} a_i = 1$ and hence $a(1) = 0$. Therefore, 1 is a root of $a(x)$. Let $r_1, \ldots, r_n$ be the roots of $a(x)$ (repeated to their multiplicity) with $r_n = 1$. Then if $a^*(x) = a(x)/(x - 1)$, $r_1, \ldots, r_{n-1}$ are the $(n - 1)$ roots of $a^*(x)$. From equation (7),

$$(12) \qquad Q^{a(x)} = \sum_{\sum_{i=1}^{n} m_i = m} r_1^{m_1} \cdot \cdots \cdot r_{n-1}^{m_{n-1}} \cdot r_n^{m_n} = \sum_{m_n=0}^{m}\left(1^{m_n} \sum_{\sum_{i=1}^{n-1} m_i = m - m_n} r_1^{m_1} \cdot \cdots \cdot r_{n-1}^{m_{n-1}}\right) = \sum_{m_n=0}^{m} Q_{m_n}^{a^*(x)}.$$

Thus, since $\lim_{m \to \infty} Q_m^{a(x)}$ exists, $\sum_{m=0}^{\infty} Q_m^{a^*(x)}$ exists, both equal $1/a^*(1)$, and all the roots of $a^*(x)$ lie in $\{z \mid |z| < 1\}$ by Corollary 3.2.

If $a(x)$ has a root of 1, and all the roots of $a^*(x) = a(x)/(x - 1)$ lie in $\{z \mid |z| < 1\}$, by Corollary 3.2,

$$\sum_{m=0}^{\infty} Q_m^{a^*(x)} = 1/(a^*(1))$$

and from equation (12), $\lim_{m \to \infty} Q_m^{a(x)} = 1/(a^*(1))$.

4. RELATED THEOREMS

Theorems 3.1 and 3.3 together give necessary and sufficient conditions for convergence, in the usual sense, of a linear recursion sequence to a complex number. We will now consider some other aspects of linear recursion sequences related to convergence. The next theorem concerns the ratio of consecutive terms of a linear recursion sequence.

Theorem 4.1: Let $\{Q_i^{a(x)}\}$ be a linear recursion sequence. If among the roots of largest norm for $a(x)$ there is a unique root, r_n, of greatest multiplicity, then there exists $N > 0$ such that if $m > N$, $Q_{m+1}^{a(x)}/Q_m^{a(x)}$ exists, and $\lim Q_{m+1}^{a(x)}/Q^{a(x)} = r_n$.

Proof: Let $r_1, \ldots, r_n$ be the n roots of $a(x)$ (repeated according to their multiplicity). Let r_n be as described in the theorem, and the r_i's be arranged so that $|r_i| < |r_n|$ for $i = 1, \ldots, j$ (j could be 0) and $|r_i| = |r_n|$ for $i = j + 1, \ldots, n - 1$. Using (4), we may then write

$$Q_m^{a(x)} = C_1^{a(x)} r_{1,d_1}^{(m)} + C_2^{a(x)} r_{2,d_2}^{(m)} + \cdots + C_n^{a(x)} r_{n,d_n}^{(m)} \quad \text{for } m > 0 \tag{13}$$

where $C_i^{a(x)} = D_i^{a(x)}/D^{a(x)}$ are constants depending on $a(x)$. Jarden [3, p. 107] observes that $D^{a(x)} \neq 0$, so this quotient is defined, for such $i = 1, \ldots, n$. Also notice using the definition of $D_n^{a(x)}$ that $D_n^{a(x)} = D_n^{a(x)/x - r_n} \neq 0$, again by Jarden's observation [3, p. 107], thus, $C_n \neq 0$, a fact we will need shortly. From the definition of $r_{i,d_i}^{(m)}$ in (6), we may write

$$Q_m^{a(x)} = \sum_{i=1}^{n} C_i^{a(x)} r_i^{m-d_i} \text{ for } m > n. \tag{14}$$

We next form the new equation

$$Q_m^{a(x)}/r_n^{m-d_n} = \sum_{i=1}^{n} C_i^{a(x)} \binom{m}{d_i}\left(r_i^{m-d_i}/r_n^{m-d_n}\right) = \sum_{i=1}^{n} E_i^{a(x)} \binom{m}{d_i}(r_i/r_n)^{m-d_n} \tag{15}$$

where $E_i^{a(x)} = C_i^{a(x)} r_i^{d_n-d_i}$ are constants depending on $a(x)$. Thus,

$$\lim_{m\to\infty} E_i^{a(x)} \binom{m}{d_i}(r_i/r_n)^{m-d_n} = 0 \quad \text{for } i = 1, \ldots, j. \tag{16}$$

Since $|r_i| < |r_n|$ for $i = 1, \ldots, j$ and there exists an $N > 0$ such that

$$\left|E_i^{a(x)} \binom{m}{d_i}(r_i/r_n)^{m-d_n}\right| < |G/n| \text{ for each } i = 1, \ldots, j \text{ when } m > N_0; \quad G = \max\{E_n, 1\}. \tag{17}$$

We consider two cases. If r_n is a simple root, then $j = n - 1$ and

$$\begin{aligned}
1 &= \lim_{m\to\infty}\left[E_n\binom{m+1}{d_n}(r_n/r_n)^{m+1-d_n}\right]\Big/\left[E_n\binom{m}{d_n}(r_n/r_n)^{r+1-d_n}\right] \\
&= \lim_{m\to\infty}\left[E_n\binom{m+1}{d_n}(r_n/r_n)^{m+1-d_n} + \sum_{i=1}^{n-1} E_i\binom{m+1}{d_i}(r_i/r_n)^{m-d_n}\right]\Big/\left[E_n\binom{m}{d_n}(r_n/r_n)^{m-d_n} + \sum_{i=1}^{n-1} E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right] \\
&= \lim_{m\to\infty}\left[Q_{m+1}^{a(x)}/r_n^{m+1-d_n}\right]\Big/\left[Q_m^{a(x)}/r_n^{m-d_n}\right] = \lim_{m\to\infty}(1/r_n)\cdot Q_{m+1}^{a(x)}/Q_m^{a(x)}.
\end{aligned} \tag{18}$$

Hence $\lim_{m\to\infty} Q_{m+1}^{a(x)}/Q_m^{a(x)} = r_n$. Notice that $Q_{m+1}^{a(x)}/Q_m^{a(x)}$ exists for $m > N_0$, since

$$\begin{aligned}
|Q_m^{a(x)}/r_n^{m-d_n}| &= \left|\sum_{i=1}^{n-1} E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n} + E_n\binom{m}{d_n}(r_n/r_n)^{m-d_n}\right| \\
&\geq -\sum_{i=1}^{n-1}\left|E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right| + \left|E_n\binom{m}{0}\right| \\
&\geq -\left|\frac{(n-1)\cdot E_n}{n}\right| + |E_n| > 0 \text{ for } m > N_0,
\end{aligned} \tag{19}$$

and we are finished.

If r_n is not a simple root, $d_n \neq 0$. Let $M > 0$ and define

(20) $$N_M = \left(\max\left\{N_0,\ M\left[(n - j) \cdot \max_{i=j+1}^{n-1} \{|E_i|\} + d_n + 1\right]\right\}\right)|E_n|.$$

Then we find

(21) $$\begin{aligned}\left|E_n\binom{m}{d_n}\right| &= \left||E_n|[(m - d_n)/d_n]\binom{m}{d_n - 1}\right| \\ &\geq \left|M \cdot \left((n - j)\left[\max_{i=j+1}^{n-1} \{|E_i|\}\right] + 1\right)\binom{m}{d_n - 1}\right| \\ &\geq M \cdot \left|\sum_{i=j+1}^{n-1} \left|E_i\binom{m}{d_n - 1}\right| + 1\right| \\ &\geq M \cdot \left|\sum_{i=j+1}^{n-1} \left|E_i\binom{m}{d_n - 1}(r_i/r_n)^{m-d_n}\right| + \sum_{i=1}^{j} \left|E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right|\right| \\ &\geq M \cdot \left|\sum_{i=1}^{n-1} E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right| \quad \text{for } m > N_M.\end{aligned}$$

Therefore, $Q_{m+1}^{a(x)}/Q_m^{a(x)}$ exists for $m > N_1$. Furthermore, if

(22) $$h_m = \left(\sum_{i=1}^{n-1} E_i\binom{m}{d_i}(r_i/r_n)^{m-d_n}\right)\Big/ E_n\binom{m}{d_n}$$

by (21), $\lim_{m\to\infty} h = 0$ and thus

(23) $$\begin{aligned}1 &= \lim_{m\to\infty}\binom{m+1}{d_n}\Big/\binom{m}{d_n} = \lim_{m\to\infty}\left[E_n\binom{m+1}{d_n}(1 + h_{m+1})\right]\Big/\left[E_n\binom{m}{d_n}(1 + h_m)\right] \\ &= \lim_{m\to\infty}\left(Q_{m+1}^{a(x)}/r_n^{m+1-d_n}\right)\Big/\left(Q_m^{a(x)}/r_n^{m-d_n}\right) = \lim_{m\to\infty} 1/r_n \cdot Q_{m+1}^{a(x)}/Q_m^{a(x)} \\ &= 1/r_n \lim_{m\to\infty} Q_{m+1}^{a(x)}/Q_m^{a(x)},\end{aligned}$$

and hence $\lim_{m\to\infty} Q_{m+1}^{a(x)}/Q_m^{a(x)} = r_n$.

Theorem 4.2: Let $\{Q_m^{a(x)}\}$ be a linear recursion sequence, with all of the roots of $a(x)$ in $\{z \mid |z| \leq 1\}$, and let $r_{j+1}, \ldots, r_n$ be the roots of $a(x)$ in $\{z \mid |z| = 1\}$. If for each $i = j + 1, \ldots, n$, r_i is a simple root, $r_i^m = 1$ for some integer $m > 0$, and m_i is the least positive integer with $r_i^{m_i} = 1$, then there exists $\{P_m\}_{m=1}^{\infty}$, a periodic sequence of period L.C.M. $\{m_{j+1}, \ldots, m_n\}$ such that

$$\lim_{m\to\infty}(Q_m^{a(x)} - P_m) = 0.$$

Proof: Let $r_1, \ldots, r_n$ be the roots of $a(x)$, repeated according to their multiplicity, with $r_{j+1}, \ldots, r_n$ as described in the theorem. Using (4), we may write

(24) $$Q_m^{a(x)} = \sum_{i=1}^{n} C_i^{a(x)} r_{i,d_i}^{(m)} \quad \text{for } m > 0$$

where $C_i^{a(x)} = D_i^{a(x)}/D^{a(x)}$ from equation (5). Evaluating $r_{i,d_i}^{(m)}$, we find

(25) $$Q_m^{a(x)} = \sum_{i=1}^{n} C_i^{a(x)}\binom{m}{d_i} r_i^{m-d_i} \quad \text{for } m > n.$$

Notice that

(26) $$\lim_{m\to\infty}\left(C_i^{a(x)}\binom{m}{d_i} r_i^{m-d_i}\right) = 0 \quad \text{for } i = 1, \ldots, j.$$

Hence, for each k, there exists an $N_k > 0$ such that

(27) $$\left|C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right| < 1/(j \cdot k) \text{ for each } i = 1, \ldots, j.$$

Therefore,

(28) $$0 = \lim_{m\to\infty}\left[\left(\sum_{i=1}^{n} C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right) - \left(\sum_{i=j+1}^{n} C_i^{a(x)}\binom{m}{d_i}r_i^{m-d_i}\right)\right]$$
$$= \lim_{m\to\infty}\left(Q_m^{a(x)} - \sum_{i=j+1}^{n} C_i^{a(x)}\binom{m}{0}r_i^m\right) = \lim_{m\to\infty}(Q_m^{a(x)} - P_m).$$

The powers of the r_i, $i = j + 1, \ldots, n$ are periodic sequences of period m_i, and hence

$P = \sum_{i=j+1}^{n} C_i^{a(x)} r_i^m$ is a periodic sequence of period L.C.M. $\{m_{j+1}, \ldots, m_n\}$.

5. NONNORMALIZED SEQUENCES

A sequence $\{Q_m^{a(x),U}\}$ may be written as a linear combination of the terms in the sequence $\{Q_m^{a(x)}\}$ by modeling it after (2)

(29) $$\begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix} = \begin{bmatrix} u_n \\ u_{n-1} \\ \vdots \\ u_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}^{-1}$$

where the matrix has an inverse, since its characteristic polynomial $a(x)$ has no zero roots. We may then write equation (2) for the nonnormalized sequence

(30) $$\begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix} = \begin{bmatrix} Q_m^{a(x),U} \\ Q_{m-1}^{a(x),U} \\ \vdots \\ Q_{m-n+1}^{a(x),U} \end{bmatrix}$$

or also

(31) $$Q_m^{a(x),U} = \sum_{i=1}^{n} X_{n+1-i} Q_{m+1-i}^{a(x)}, \quad m > 0.$$

Hence, we may use the normalized sequence to determine convergence for the nonnormalized sequence.

6. CONCLUSION

The theorems proved in this paper rely heavily on the relationship of the roots of the auxiliary polynomial to the region $\{z \mid |z| < 1\}$. A problem in Wall's book [7, p. 190] gives exact computations to determine this relationship. So given a linear recursion sequence and its auxiliary polynomial, it can be decided whether it converges to 0, to a nonzero complex number, or is nonconvergent in the usual sense.

Necessary and sufficient conditions for "convergence" to infinity are not given, and are not known to this author. Theorem 4.1 gives a sufficient condition that if there exists a root r_n of $a(x)$ which has norm larger than or equal to all other roots and has greatest multiplicity among the roots of its norm. The sufficiency of the conditions excluded by Theorems 3.1, 3.3, and 4.2 may be refuted by considering $a(x) = x^2 - 2$. The roots of $a(x)$ are $-\sqrt{2}$, $+\sqrt{2}$, both of which lie outside $\{z \mid |z| \le 1\}$, but the sequence begins 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, ... and thus does not "converge" to infinity. This example also shows that Theorem 4.1 cannot include certain examples where there are two or more roots of largest norm and equally great multiplicity, since the quotient $Q_{m+1}^{a(x)}/Q_m^{a(x)} = 0$ for even m, and does not exist for odd m.

The original problem of Singmaster [5] asked if the conditions that the a_i all be real $a_i \geq 0$ for $i = 1, \ldots, n$, $a_1 > 0$, and $\sum_{i=1}^{n} a_i = 1$ were sufficient for $\lim_{m \to \infty} Q_m^{a(x)} = b \neq 0$. The answer to this question is affirmative. Looking at equation (2), the matrix may be viewed as a stationary Markov transition matrix, and by Doob [2, p. 256] the powers of the matrix converge. Thus, $\lim_{m \to \infty} Q_m^{a(x)}$ exists. Since $\sum_{i=1}^{n} a_i = 1$, 1 is a root of $a(x)$, and so, by Theorem 3.3, $\lim_{m \to \infty} Q_m^{a(x)} = a^*(1)$ where $a^*(x) = \dfrac{a(x)}{(x - 1)}$.

REFERENCES

1. E. Bodewig. *Matrix Calculus.* Amsterdam: North-Holland Publishing Company, 1959, p. 57.
2. J. L. Doob. *Stochastic Processes.* New York: John Wiley & Sons, Inc., 1953, p. 256.
3. D. Jarden. *Recurring Sequences.* Jerusalem: Riveon Lematematika, 1966, pp. 105-107.
4. L. Raphael. "Linearly Recursive Sequences of Integers." *The Fibonacci Quarterly* 12, No. 1 (1974):11-38.
5. A. G. Shannon. "Some Properties of a Fundamental Recursive Sequence of Arbitrary Order." *The Fibonacci Quarterly* 12, No. 4 (1974):327-334.
6. D. Singmaster. "Problem H-179." *The Fibonacci Quarterly* 9, No. 1 (1971):62.
7. H. P. Wall. *Continued Fractions.* Princeton, N.J.: D. Van Nostrand Company, Inc., 1948, p. 190.

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