and

$$(31)\qquad \sum_{j=0}^{n}\binom{n}{j}F_m^jL_m^{n-j}\left[F_{jk+r}L_{(n-j)k+r} + (-1)^{jk+r}F_{(n-2j)k}\right] = 2^nF_{m+1}^nF_{nk+2r}.$$

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IDENTITIES OF A GENERALIZED FIBONACCI SEQUENCE

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The purpose of this note is to give identities of third power and above of the generalized Fibonacci sequence with nth term H_n satisfying the recurrence relation $H_n = pF_n + qF_{n-1}$ and $H_0 = q$ where F_n denotes the nth classical Fibonacci number.

We refer to the following identities of A. F. Horadam [1]:

$$(1)\qquad H_nH_{n+2} - H_{n+1}^2 = (-1)^ne$$

$$(2)\qquad H_{m+h}H_{m+k} - H_mH_{m+h+k} = (-1)^meF_hF_k$$

$$(3)\qquad H_m = F_{k+1}H_{m-k} + F_kH_{m-k-1}$$

and also use

$$(4)\qquad H_{k+1}H_{k+2}H_{k+4}H_{k+3} = H_{k+5}^4 - e^2$$

where $e = p^2 - pq - q^2$.

Identity 1: $H_n^4 - 2H_{n+1}^3H_n - H_{n+1}^2H_n^2 + 2H_n^3H_{n+1} + H_{n+1}^4 = e^2$.

Identity 2: $H_{n+4}^4 - 4H_{n+3}^4 - 19H_{n+2}^4 - 4H_{n+1}^4 + H_n^4 = -6e^2$.

Identity 3: $H_{n+5}^4 = 5H_{n+4}^4 + 15H_{n+3}^4 - 15H_{n+2}^4 - 5H_{n+1}^4 + H_n^4$.

Identity 4: $25\sum_{k=0}^{n}H_k^4 = H_{n+3}^4 - 3H_{n+2}^4 - 22H_{n+1}^4 - H_n^4 + 6e^2(n-1) + A$

where $A = 15p^4 - 32p^3q - 12p^2q^2 + 16pq^3 + 34q^4$.

Identity 5: A. $18\sum_{k=1}^{n}(-1)^kH_k^4 = (-1)^n(H_{n+4}^4 - 6H_{n+3}^4 - 9H_{n+2}^4 + 24H_{n+1}^4 - H_n^4)$;

B. $9\sum_{k=1}^{n}(-1)^kH_k^4 = (-1)^n(-H_{n+3}^4 + 5H_{n+2}^4 + 14H_{n+1}^4 - H_n^4 - 3e^2)$.

Identity 6: $25\sum H_{k+1}H_{k+2}H_{k+4}H_{k+5} = 26H_{n+3}^4 + 22H_{n+2}^4 + 3H_{n+1}^4 - H_n^4 - C$,

where $C = 19e^2n + (66p^4 + 70p^3q + 131p^2q^2 + 146pq^3 + 47q^4)$.

Identity 7: $9\sum_{0}^{2n-1}(-1)^k H_{k+1}H_{k+2}H_{k+4}H_{k+5} = H_{2n+5}^4 - 5H_{2n+4}^4 - 14H_{2n+3}^4 + H_{2n+2}^4 + 3e^2 + D,$

where $D = q(4p^3 + 6p^2q + 4pq^2 + q^3)$.

The proof of Identities 1-7 follow along the same lines as in [1], hence the details are omitted here.

Some more identities that are easily verifiable by induction follow:

(a) $2\sum_{0}^{n}(-1)^r H_{m+3r} = (-1)^n H_{m+3n+1} + H_{m-2}$ $\quad m = 2, 3, \ldots;$

(b) $3\sum_{0}^{n}(-1)^r H_{m+4r} = (-1)^n H_{m+4n+2} + H_{m-2}$ $\quad m = 2, 3, \ldots;$

(c) $11\sum_{0}^{n}(-1)^r H_{m+5r} = (-1)^n (5H_{m+5n+1} + 2H_{m+5n}) + 4H_m - 5H_{m-1}$ $\quad m = 1, 2, \ldots;$

(d) $4\sum_{0}^{n}H_k H_{2k+1} + 2H_0^2 = H_{2n+3}H_n + H_{2n}H_{n+3};$

(e) $3\sum_{0}^{n}(-1)^r H_{m+2r}^2 = (-1)^n H_{m+2n}H_{m+2n+2} + H_m H_{m-2}$ $\quad m = 2, 3, \ldots;$

(f) $7\sum_{0}^{n}(-1)^r H_{m+4r}^2 = (-1)^n H_{m+4n}H_{m+4n+4} + H_m H_{m-4}$ $\quad m = 4, 5, \ldots;$

(g) $2\sum_{1}^{n}H_{k+2}H_{k+1}^2 = H_{n+3}H_{n+2}H_{n+1} - H_0 H_1 H_2;$

(h) $2\sum_{1}^{n}(-1)^r H_r H_{r+1}^2 = (-1)^n H_n H_{n+1}H_{n+2} - H_0 H_1 H_2;$

(i) $2\sum_{1}^{n}(-1)^r H_{r+1}^3 = (-1)\ (H_{n+1}^2 H_{n+4} - H_n H_{n+2}H_{n+3}) - E,$

where $E = p^3 - 3pq^2 - q^3$.

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DIVISIBILITY PROPERTIES OF A GENERALIZED FIBONACCI SEQUENCE

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This note gives some divisitility properties of the generalized Fibonacci numbers viz $H_0 = q$, $H_1 = p$, $H_{n+1} = bH_n + cH_{n-1}$ $(n \geq 1)$, denoted henceforth by (b, c, p, q) GF sequence. The results have similarity to those of Dov Jarden [1].

For the Horadam generalized Fibonacci sequence: $H_0 = q$, $H_1 = p$, $H_{n+1} = H_n + H_{n-1}$ $(n \geq 1)$, we have

Theorem 1: $H_{n+k} + (-1)\ H_{n-k}$ is divisible by H for all $n \geq k$.

Proof: The proof easily follows from the identity

(1) $$H_{n+k} + (-1)^k H_{n-k} = L_k H_n.$$

Corollary a: $H_{n+k}^2 + (-1)^{2k+1}H_{n-k}^2$ is divisible by H_n; and

Corollary b: $H_{n+k}^3 + (-1)^{3k+2}H_{n-k}^3$ is divisible by H_n.

Divisibility properties of (b, c, p, q) GF sequence.

Theorem 2: If $(m,n) = 1$ and $q = 0$, $H_m H_n / H_{mn}$.

Proof: $H_n = (gr^n - hs^n)/(r - s)$ and $H_{mn} = (gr^{mn} - hs^{mn})/(r - s)$, where r and s are the roots of $x^2 - bx - c = 0$ and $g = p - sq$ and $h = p - rq$.