It is easily seen that H_m or H_n divides H_{mn} if $g = h$. Since $r = s$ leads to the degenerate case, we must have $q = 0$. Also, it is necessary that $(m,n) = 1$.

Theorem 3: If $p^2 - bpq - cq^2 = 0$, then $H_m H_n / H_{mn}$.

Proof: By the identity

$$H_n^2 - H_{n+1}H_{n-1} = (-c)^{n-1}e, \tag{2}$$

where $e = p^2 - bpq - cq^2$, the desired result follows.

Theorem 4: For $p = cq(1 - b)/(b^2 + c + 1 - b)$, if $c^2 = (-1 - b)(1 + 2c)$, then $H_m H_n / H_{mn}$.

It is known from [2] that $H_n = pU_n + cqU_{n-1}$, where the nth member of the U sequence is defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = bU_{n+1} + cU_n$ $(n > 0)$.

On suitably combining this relation with

$$2(pU_n + cqU_{n-1}) = (pU_{n+1} + cqU_n) + (pU_{n-1} + cqU_{n-2}), \tag{3}$$

it is easy to see that (b, c, p, q) GF sequence results in an A.P. Therefore, if $H_m H_n$ were to divide H_{mn}, we would get

$$c^2 = (1 - b)(1 + 2c).$$

Further equating the initial term of the A.P. with the common difference, we get either $c = 0$ or $p(b^2 + c + 1 - b) = cq(1 - n)$.

The case $c = 0$ is already discussed in Theorem 3; hence, the other condition gives the desired result of divisibility.

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PYTHAGOREAN PENTIDS

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1. INTRODUCTION

Let $T_n = n(n + 1)/2$ denote the nth triangular number. Then we have

$$(T_{2r})^2 + (T_{2r} + 1)^2 + (T_{2r} + 2)^2 + \cdots + (T_{2r} + r) = (T_{2r} + r + 1)^2 + (T_{2r} + r + 2)^2 + \cdots + (T_{2r} + 2r)^2 \tag{1.1}$$

and

$$(T_{2r} + 9k)^2 + (T_{2r} + 1 + 12k)^2 + \cdots + (T_{2r} + r + 12k)^2 = (T_{2r} + r + 1 + 12k)^2 + (T_{2r} + r + 2 + 12k)^2 + \cdots + (T_{2r} + 2r + 15k)^2, \tag{1.2}$$

$$r = 1, 2, 3, \ldots; \ k = 1, 2, 3, \ldots .$$

This gives a generalized identity of squares of numbers with $r + 1$ terms on the left-hand side and r terms on the right-hand side. But the triangular numbers are a particular case of the generalized Tribonacci sequence having a recurrence relation

(1.3) $X_{n+3} = 3X_{n+2} - 3X_{n+1} + X_n$, $n \geq 0$, with $X_0 = 0$, $X_1 = 1$, and $X_2 = 3$.

Therefore, the properties of the generalized Tribonacci sequence are also properties of the triangular numbers.

The case $r = 1$ in equation (1.1) gives the well-known Pythagorean triad (3, 4, 5). For $r = 2$, we have the Pythagorean pentid (10, 11, 12, 13, 14). Pythagorean triads have been studied by various authors, particularly by Teigen and Hadwin [6] and by Shannon and Horadam [5]. The object of this note is to extend the results of the above-mentioned authors to the Pythagorean pentids. Similar extensions are also possible for the general Pythagorean n-tids of (1.1).

2. GENERALIZED FIBONACCI PENTIDS

The Horadam [2] generalized Fibonacci sequence satisfies the recurrence relation

$$H_{n+2} = H_{n+1} + H_n \quad (n \geq 1).$$

For this sequence, we have the identity

$$(2.1) \qquad H_{n+3}^2 + (H_{n+2} + H_n)^2 = (H_{n+2} - H_n)^2 + (2H_{n+2})^2 + H_n^2.$$

This can be easily checked by the substitutions

$$H_n = p, \; H_{n+1} = p + q.$$

Then corresponding to a result of Shannon [4], we have the identity

$$(2.2) \qquad U_{n+4}^2 + (U_{n+3} + U_n)^2 = (U_{n+3} - U_n)^2 + (2U_{n+3})^2 + U_n^2,$$

where U_n is the nth term of the Tribonacci [1] sequence whose recurrence relation is $U_{n+3} = U_{n+2} + U_{n+1} + U_n$, with U_1, U_2, and U_3 as the initial terms.

Proof: On using the recurrence relation, we obtain

$$(A) \qquad U_{n+4} - U_n = 2(U_{n+2} + U_{n+1}) \ldots$$

and

$$(B) \qquad U_{n+4} + U_n = 2U_{n+3} \ldots .$$

On multiplying (A) and (B), we have

$$U_{n+4}^3 - U_n^2 = 4U_{n+3}[U_{n+2} + U_{n+1}]$$

or

$$U_{n+4}^2 = U_n^2 + 2U_{n+3}[U_{n+4} - U_n]$$
$$= U_n^2 + [2U_{n+3}]^2 + [U_{n+3} - U_n]^2 - [U_{n+3} + U_n]^2,$$

from which the desired result follows. Comparison of (2.1) with (2.2) suggests a similar identity for the general recurring sequence V_n of order r with

$$V_{n+r} = \sum_{i=0}^{r-1} V_{n+i}, \; n \geq 1,$$

with the initial values $V_1, V_2, \ldots, V_r$.

Identity

$$(2.3) \qquad V_{n+r+1}^2 + [V_{n+r} + V_n]^2 = [V_{n+r} - V_n]^2 + [2V_{n+r}]^2 + V_n^2,$$

in which $r = 2$, gives (2.1), and $r = 3$ gives (2.2).

For the generalized Fibonacci sequence $W_n(a, b, p, q)$ of Horadam [3], we have

$$(2.4) \qquad \{QW_{n+3}\}^2 + \{2PW_{n+2} + W_n\}^2 = \{2PW_{n+2} - W_n\}^2 + \{4PW_{n+2}\}^2 + W_n^2$$

where $Q = p/q^2$ and $P = (p^2 - q)/2q^2$. This follows easily from a lemma of Shannon [5]:

$$(2.5) \qquad (p^2 - q)W_{n+2} - pW_{n+3} = q^2 W_n.$$

But (2.4) is in a form which can be generalized for higher-order recurrence relations. Therefore, we have the following:

Theorem 1: All Pythagorean pentids are recurrence pentids.

3. PYTHAGOREAN n-TIDS

In this section, the method of Teigen and Hadwin [6] is extended to Pythagorean n-tids. Teigen and Hadwin proved that the Pythagorean triad (a, b, c) can be represented by

(3.1) $\quad a = x + z$, $b = y + z$, $c = x + y + z$, where x, y, z are positive and $2xy = z^2$, z even.

For the Pythagorean pentid (a, b, c, d, e), we have

(3.2) $\quad a = x + y + z$, $b = y + z + t$, $c = z + t + u$, $d = x + y + z + t$ and

$e = y + z + t + u$, where x, y, z, t, u are positive, and

(3.3) $\quad z^2 = 2(xy + yt + yu)$, z even.

Similarly, for the Pythagorean septid (a, b, c, d, e, f, g), we have

(3.4) $\quad a = x + y + z + t$, $b = y + z + t + u$, $c = z + t + u + v$, $d = t + u + y + w$,

$e = x + y + z + t + u$, $f = y + z + t + u + v$, and $g = z + t + u + v + w$

where all the right-hand side parameters are positive, and

(3.5) $t^2 = 2(xu + yv + zw + zu + zv)$, t even.

Similar extensions follow for the n-tids.

An alternate method of generating infinite numbers of Pythagorean n-tids from a given n-tid is discussed in [7].

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A TRIANGLE FOR THE BELL NUMBERS

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The Bell, or exponential, numbers B_n are defined by

(1) $$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} = \frac{1}{e}\left(\frac{0^n}{0!} + \frac{1^n}{1!} + \frac{2^n}{2!} + \cdots\right)$$

The first twelve Bell numbers are given in the following table:

TABLE 1. Bell Numbers

n	B_n
0	1
1	1
2	2
3	5
4	15
5	52
6	203
7	877
8	4140
9	21147
10	115975
11	678570

The Bell numbers also appear in the Maclaurin expansion of e^{e^x}:

(2) $$e^{e^x} = e\sum_{k=0}^{\infty}\frac{B_k x^k}{k!} = e\left(1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \cdots\right)$$

The Bell numbers can be generated recursively by an interesting method described in [2]. If we take the array described in this article and "flip" it about and then reorient it, the following triangle appears. This triangle is similar in form to Pascal's triangle. We shall call it the "Bell Triangle," and denote each element by $B'(n,r)$. This notation is similar to $C(n,r)$ for Pascal's triangle. There are three rules of formation for this triangle.