FIBONACCI FEVER

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This is an account of a strange case of infibonacciation suffered recently by the author, the only remedy for which was found to be a dose of HP-35 followed by SR-50 taken at intervals of 1.618 hours. It all began in Egypt, of course, as so many things do, and specifically with the construction of the Great Pyramid of Khufu or Cheops (no mean task—geometrically, as it turns out). Much has been written on the contributions supposedly made to its design by knowledge Egyptian mathematicians may have had of Pi or Phi, generally considered to have been pretty fibal. Imagine my surprise when, under the influence of the contagion afflicting me both phi-sickally and mentally, I looked up the values of the trigonometric functions in the neighborhood of the well-known Great Pyramid angle of approximately 51°50' (and its complement) and found what I have not seen in print anywhere, namely

$$\begin{array}{ll} \sin A = \sqrt{b} & \sin B = b \\ \cos A = b & \cos B = \sqrt{b} \\ \tan A = \sqrt{a} & \tan B = \sqrt{b} \\ \cot A = \sqrt{b} & \cot B = \sqrt{a} \\ \sec A = a & \sec B = \sqrt{a} \\ \csc A = \sqrt{a} & \csc B = a \end{array}$$

where $a = 1.618033989\ldots$ and $b = a - 1$. Interpolation in the tables or use of one of the new pocket calculators quickly yields exact values for the angles:

$$A = 51°49'38''.253 \qquad B = 38°10'21''.747$$

This observation, that the values of the trigonometric functions at which their plotted curves intersect are all, except for the familiar values 0, ±1, $\pm\sqrt{2}$, and $\pm\sqrt{2}/2$, of magnitude a, b, or their square roots, should be sufficient to launch the new science of Fibonometry or Phigonometry, according to taste. Our basic right phiangle is then the one with unit hypotenuse and base b, which has the property that its altitude is the mean proportional between its base and its hypotenuse. This altitude, $\sqrt{b}$, is the approximation to $\pi/4$ that has led to the association of the Great Pyramid with an attempt to represent π.

$$\sqrt{b} = 0.78615\ 13778$$

$$\pi/4 = 0.78539\ 81634$$

This approximation is good to 0.1%. Some other phigonometric approximations that have been noted by pyramidographers qualify as genuine Fibonacci curiosities. They are:

$$\begin{array}{lll} A \approx 1/7 & \text{circle} & \text{(error 0.8\%)} \\ A \approx 9/10 & \text{radian} & \text{(error 0.5\%)} \\ B \approx 2/3 & \text{radian} & \text{(error 0.06\%)} \end{array}$$

Further numerical approximations that have been noted are

$$6a^2/5 = 3.1416408 \approx \pi \quad \text{(error 0.0015\%)}$$

and

$$\text{arc tan } \sqrt{2}/2 \approx b \text{ radian}$$

$$0.61547971 \approx 0.61803399 \quad \text{(error 0.4\%)}$$

this latter deliriously close, but to what, is uncertain.

It is, and very likely will remain, an open question as to which of these approximations the Egyptians may have had in mind, if any, but it is nevertheless extremely curious that most of them fall within the probable limits imposed on the precision of construction of the Pyramid by the technology and surveying techniques available at the time.

The numerological ramifications of this question are quite prodigious, and demand the introduction at this point of some measurements of the actual Pyramid. Values published by Petrie and by Bruchet have been chosen here as representative of determinations made in both English and metric units, respectively, and the rounded values in cubits, given in the last column are based on generally accepted conversion factors.

	Feet	Meters	Cubits
Half-base	377.86	115.24	220
Height	481.33	146.60	280
Apothem of face	611.93	186.47	356

One cubit = 7 palms = 28 fingers, and the inclination of a pyramid's side is expressed (e.g., in the Rhind papyrus) as so many palms horizontal recession of the face for one cubit vertical rise. This quantity, the *seked*, is thus 7 cot A, where A is the inclination angle measured at the foot of the apothem. The angle A has actually been measured from one or two intact casing stones from the buried portion of the Pyramid and compares well with the estimates made from overall measurements. From the values given in the last column of the above table, we determine the *seked* of Cheops to be precisely 5.5 (or 5 palms, 2 fingers). This is not only a nice simple number but in relation to the cubit of 7 palms suggests, as do the ratios of the sides of the Pyramid triangle, the value of 22/28 as an approximation to either $\pi/4$ or $\sqrt{b}$, or both, as you wish. The error in either case is less than 0.06%.

$$A = \text{arc cot } 22/28 = .90482\ 70894 \text{ radian} = 51°50'34''$$

$$\sin A = 0.78631\ 83388$$

Measured angle of casing-stones = 51°51'

It has also been suggested that the Pyramid was designed to have a rise of 9 units in 10 taken along the edge of a face rather than at its center. This is easily checked for the triangle that forms a vertical section through a diagonal of the (almost perfectly) square base. Calling this corner angle of inclination C, we find

$$\tan C = 280/311.127 = 0.89995\ 40851$$

which verifies this hypothesis as well, to within 0.01%! The angle C turns out to be 41°59'09" or only 0.03% from the neat angle of 42° (which may recommend itself to hexagesimalists because it is 7/60 of a circle).

In case these excursions into the real world prove too enervating, let us indulge in a lettle ideal-pyramid designing, starting from our basic right phiangle whose sides are in the ratio $1:\sqrt{b} = 1.27202$ very nearly. Rounding this to 1.272, we might let our base be 1000 units and our height 1272, giving us a face apothem for the pyramid, or hypotenuse of our triangle, of 1618, a familiar number indeed. These numbers are all divisible by 2, so we get 500, 636, 809 (the last is prime). If we choose to be a little sloppy (will the Greeks detect it?), we can settle for 50, 64, 81, which has the beauty that the full base is then 100 units, and our numbers are simply 8^2, 9^2, 10^2. However, we would then have to settle for a pyramid angle of almost exactly 52° (only good for 13-fetishists or card players) with its rather poor 3.15 for π and 0.621 for b. There are those who will claim this design is justified for its 81/64 approximation to $\sqrt{a}$, which squares to 1.6018 for a itself. But then, some prefer bent pyramids to straight.

Now there is one place where all may find good values for the extrema in our triangular section, the base and hypotenuse, because their ratio if 1.618, that of the Fibonacci sequence. The sequence itself yields the pairs we need, and they get progressively better as we go to higher members, only requiring that we select for near-integer values of the mean proportional. A little play with early members is rewarding: we immediately find the ancient 3, 5 pair with its perfectly Pythagorean companion 4. Pyramid angle 53°08' and very primitive 3.2 for π. We might dream of 8, 5 with its convenient 1.6 ratio, but we left 3.2 behind in the last triangle, so we can't work a deal for $\pi = 2a$. It is at this point that it just dawns on us for no apparent reason that we can get a fair estimate of the middle value (the height of our pyramid) from the expression

$$\frac{\frac{1}{2}F_{i+2} + 2F_{i-1}}{2}$$

which shows us that F_{i+2} must be divisible by 4 to give an integer middle term. The Pythagoreans insist we write this as

$$\frac{5F_{i+1} - 3F_i}{4}$$

for obvious reasons, and as it is the same thing, we don't object. Now we cannot only construct right phiangles, but Phiophantine ones as well. (Except for the 3, 4, 5 case, we must not call them Diophantine, as the closer approximation to Phi precludes Di—still they will serve the useful purpose of providing a suitable tomb should a Pharaoh Die.) Since every F_n for $n = 6, 12, 18, 24, \ldots$ is divisible by 4 we have an unlimited supply of models.

And what do we find almost as soon as we begin the painstaking task of examining this infinity of models? The first one after the 3, 4, 5 model is the actual Great Pyramid of Khufu! 55, 89 with the interpolated 70, upon multiplication by 4, yield the values 220, 280, 356, the very dimensions in Egyptian royal cubits that most people have found acceptable. But already the astute observer will have found a hint of another pair that looks interesting (do we all have our Fibonacci sequences out?), namely 377, 610, simply because these numbers are so close to the Pyramid measurements in English feet! True, this pair doesn't yield an integer value for the middle term (here 479.75) but the Fibonacci expression does give a value about halfway between the Pythagorean 479.55 and the round 480. Good British foot-rules must have been scarce in ancient Egypt, and the architect had the bad luck to choose one that was too long by 0.27%, though whether this was an effect of the higher mean temperature at Giza or due to more esoteric considerations, such as the ratio of the sidereal to the solar day (1.00274), the inhabitants of sunny Egypt preferring the longer solar foot and thus assigning fewer units to a given length than Britons, whose work beneath the moon and in the cooler northern dawns at Stonehenge and Avebury might naturally have led them to employ the sidereal foot, remains to be determined by future investigators. In any case, the reduction required is so slight that it can scarcely conceal the fact that Khufu was built on the English system.

But wait! Had you already noticed that doubling our measurements in meters, or expressing them in semimeters (perhaps in deference to Semiramis, always so phinegy with details): 230.48, 293.20, 372.94 begins to look alarmingly as if the French too had landed on the banks of the Nile and had the situation well in hand—compare 233, 377 and the Fibonterpolated value 296.5? The expansion of the French rule appears to have been greater, amounting to 1.1%, though it might be argued it was no less just.

It may be that further study will show vaguer correspondences with rough-hewn Norse wooden rulers or sly yardsticks of China, but at least we have pointed the way. On the vexed question of what the Egyptians hoped to achieve by their design, my own opinion is that their architects made a wise decision to split the difference between a very accurate representation of π and a very exact approximation to the Golden Ratio by choosing the very neat 55, 70, 89 triad with its traditional 22/7 compromise, showing that after all they knew perfectly well you can't square the circle but you can come as close as a scarab-beetle's left front leg to doing it, and in the process keep thousands of generations of people, amateurs and savants alike, guessing and struggling with the data to resolve the issue. No edifice of lesser mass and durability than Cheops could have been relied upon to do the job of preserving the sharp edge of the blade of discrimination between subtle geometric hypotheses for thousands of years.

In a lighter vein, we noticed one day as the fever was wearing off and we were relaxing to the sound of the oud, that much of the world's music can be represented, with regard to pitches of degrees of the scale, by simple powers of ratios between 1 and 2 (the unison and octave), with the perfect fifth (3/2) doing yeoman's work ever since the days of Pythagoras, who probably learned about it in Egypt, according to legend. Musics of China, India, Persia, Arabia, Byzantium, and Greece can be represented by using sufficiently high powers of 1.5 alone (try it some time, merely taking care to reduce values that exceed 2 by the appropriate division by a power of 2 so that the set of tones remains within the octave—negative powers should be included in a symmetrical manner). Those who appreciate the value of common cents in musicology will want to see results expressed in this medium of exchange currently being favored at 1200 to the octave according to the formula

$$\text{Cents} = (1200/\log_{10}2.0)\log_{10}R$$

where R is the frequency ratio of two pitches of interest, say any note and the fundamental or tonic. If R is some power of a constant ratio between 1 and 2, say

$$R = r^j/2^k \qquad j = 0, \pm1, \pm2, \ldots, \pm n$$

and k is chosen such that $1 < R < 2$,

$$\text{Cents} = 3986.314(j \log_{10}r - k \log_{10}2).$$

The point for Fibonaccians is, of course, what happens if we choose $r = 1.618\ldots$? The result is curious. After reordering successive powers into a monotonic sequence, we have, in cents:

30.2	69.1	<u>99.3</u>	129.4	168.4	<u>198.5</u>	
237.5	276.6	<u>297.8</u>	336.7	366.9	<u>397.1</u>	
436.0	466.2	<u>496.4</u>	535.3	565.5	<u>595.6</u>	(604.4)

and so on for the upper half of the octave. These values are within a few cents of forming a

36-tone tempered scale, so that every third member of the sequence is very nearly one of the twelve tones of our present musical scale. For perfect correspondence, such that every third tone is 100, 200, 300, etc. cents, the value of r should be 1.618261.

The usual method of constructing tempered scales is to use a ratio r which is the nth root of 2 to obtain a scale of n equidistant tones. $\sqrt[36]{2} = 1.019440644$. The ratio 1.618261 is a power of this, in fact the 25th power. It is interesting to note that 1.618... itself is not a frequency ratio that corresponds to a tone of our 12-tone scale, for it gives 833 cents, far enough from 800 to sound sharp and give discords. Other attempts to relate the Golden Ratio to musical pitch have overlooked this hard musical fact. The present discussion may serve to reinstate the Divine Proportion into the Divine Harmony.

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EXPONENTIAL GENERATION OF BASIC LINEAR IDENTITIES*

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Generalizing results of Fibonacci and Lucas numbers has been an occupation of a large number of mathematicians down through the years. Frequently, one approach taken is to first prove a result involving the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and the Lucas sequence $\{L_n\}_{n=0}^{\infty}$ and then extend it to a result or results of special cases of the sequences $\{F_{nk+r}\}_{n=0}^{\infty}$ and $\{L_{nk+r}\}_{n=0}^{\infty}$, where k and r are fixed integers. In this paper attention is focused on deriving identities related to these latter sequences. Such results, called linear because of the subscripts, are surveyed in [1]. The exponential generating functions for these latter sequences are now shown to be most productive in deriving basic linear identities that the author believes to be new. In addition, alternate derivations of several known results will be given to show the great usefulness of these generating functions in attacking a variety of Fibonacci and Lucas problems.

Recalling the Maclaurin series expansion for e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and hence

$$e^{Ax} = 1 + \frac{Ax}{1!} + \frac{(Ax)^2}{2!} + \cdots = \sum_{n=0}^{\infty} A^n \frac{x^n}{n!}, \tag{1}$$

for any constant A, we note that the exponential generating functions for the first mentioned sequences are

$$\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta}$$

and

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The exponential generating functions of the sequences of interest in this paper are found by use of (1) to be

$$\sum_{n=0}^{\infty} F_{nk+r} \frac{x^n}{n!} = \frac{\alpha^r e^{\alpha^k x} - \beta^r e^{\beta^k x}}{\alpha - \beta} \tag{2}$$

$$\sum_{n=0}^{\infty} L_{nk+r} \frac{x^n}{n!} = \alpha^r e^{\alpha^k x} + \beta^r e^{\beta^k x} \tag{3}$$

$$\sum_{n=0}^{\infty} (-1)^n F_{nk+r} \frac{x^n}{n!} = \frac{\alpha^r e^{-\alpha^k x} - \beta^r e^{-\beta^k x}}{\alpha - \beta} \tag{4}$$

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*This paper was presented at the Fifth Annual Spring Conference of The Fibonacci Association, April 23, 1972.