and

$$\sigma(n) - n - p_1 = n - (p_1 - 2^{\alpha+1}) < n.$$

Write (13) into the form

$$(14) \qquad p_t = \frac{(2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1) - 2^{\alpha+1}}{2^{\alpha+1}\prod_{i=1}^{t-1}p_i - (2^{\alpha+1} - 1)\prod_{i=1}^{t-1}(p_i + 1)}.$$

We see that p_t from (14) also satisfies (12) and this remains valid if we replace $2^{\alpha+1}$ in (13) and (14), e.g., by any constant $A \geq 2^{\alpha+1}$ provided that $p_1 > A$.

We can now present an algorithm for computing arbitrary long (great) primitive weird numbers n satisfying (10) and (14) if they exist.

For given α choose first the prime $p_1 > (A \geq) 2^{\alpha+1}$ and then p_2 from (14). If this is not a prime, choose p_2 an arbitrary prime $> p_1$ and calculate p_3 from (14). If this is not a prime, choose p_3 an arbitrary prime $> p_2$, and so on. The algorithm ends when we obtain a prime p_t from (14).

REFERENCE

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FIBONACCI CONCEPT: EXTENSION TO REAL ROOTS OF POLYNOMIAL EQUATIONS

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It was in November 1973 when Professor T. A. Davis was conducting a biocensus that he introduced me to the well-known Fibonacci numbers. He told me that certain limbs of a normal human body are in the Golden Ratio, viz. 1.618... . I observed that the reciprocal of the Golden Ratio (0.618...) is nothing but a root of the quadratic equation

$$(1) \qquad x^2 + x = 1$$

or

$$(2) \qquad x^2 + x - 1 = 0$$

which is formed by equating the three ratios of human limbs (each ratio, in fact, is equal to the Golden Ratio).

As is well known, this root 0.618 of (1) is the fixed ratio of the successive terms (ignoring some of the initial terms) of the Fibonacci sequence. I considered the sequence $\{U_r\}$ defined as follows:

$$(3) \qquad U_r = 1,\ \forall_r = 1, 2, 3;\ U_r = U_{r-1} + U_{r-2} + U_{r-3},\ \forall_r \geq 4.$$

Using a computer program, I found that after 21 terms of the sequence, the ratios $\left\{\frac{U_{r-1}}{U_r}\right\}$ become constant up to the 9th decimal place and is 0.543689013, which is found to be a root of the polynomial equation (cubic),

$$(4) \qquad x^3 + x^2 + x = 1.$$

Now, consider the sequences defined, analogously, as follows:

Sequence (Definitions):

(i) $U_r = 1,\ \forall_r = 1, 2, 3, 4;\ U_r = U_{r-1} + U_{r-2} + U_{r-3} + U_{r-4},\ \forall_r \geq 5;$

(ii) $U_r = 1,\ \forall_r = 1, 2, 3, 4, 5;\ U_r = U_{r-1} + U_{r-2} + U_{r-3} + U_{r-4} + U_{r-5},\ \forall_r \geq 6;$

⋮

(vii) $U_r = 1,\ \forall_r = 1, 2, 3, \ldots, 10;\ U_r = U_{r-1} + U_{r-2} + \cdots + U_{r-10},\ \forall_r \geq 11.$

The approximate limit points (which do exist) of sequences of ratios $\left\{\frac{U_{r-1}}{U_r}\right\}$ as obtained by computer, are 0.518790064, 0.508660392, 0.504138258, 0.502017055, 0.500994178, 0.500493118, and

0.500245462, respectively. We see that the nature of these sequences of ratios is also similar. These fixed ratios are the roots of the following polynomials (q) where q can be any one of the symbols i, ii, iii, ..., vii.

(i) $x^4 + x^3 + x^2 + x = 1$

(ii) $x^5 + x^4 + x^3 + x^2 + x = 1$

$\vdots$

(vii) $x^{10} + x^9 + x^8 + \cdots + x^2 + x = 1$

[I also observed that these ratios are tending to 0.5 ($=\frac{1}{2}$) as n becomes larger and larger, where n is the number of prefixed terms, each equal to unity, and also is the number of terms on R.H.S. of recurrence relations used in the definitions of the sequences $\{U_r\}$.] This can be explained mathematically on the ground that

$$\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \left(\frac{1}{2}\right)^n + \frac{1}{2} \to 1 \text{ as } n \to \infty.$$

I observed this link only up to $n = 10$. For $n = 1$, it is obvious; also, for $n = 2$, it is easily seen to be valid. Intuitively, it can be stated that this fact is valid for all infinite n.

Then I considered the sequence:

(5) $$U_r = 1,\ \forall_r = 1,\ 2;\ U_r = 2U_{r-1} + 3U_{r-2},\ \forall_r \geq 3.$$

I studied the ratios of the consecutive terms of this sequence and I found that the ratio tends to a root (= 0.3333...) of the equation

(6) $$3x^2 + 2x = 1.$$

Now consider the cubic equation

(7) $$2x^3 + x^2 + x = 1,$$

and form the sequence $\{U_r\}$ as follows:

(8) $$U_r = 1,\ \forall_r = 1,\ 2,\ 3;\ U_r = U_{r-1} + U_{r-2} + 2U_{r-3},\ \forall_r \geq 4.$$

Take the ratios of the consecutive terms of this sequence. The sequence of these ratios comes out to be tending to 0.5, which is a root of the cubic equation.

Let us now slightly generalize the concept. Consider the polynomial equation

(9) $$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x = 1$$

where all a_i's are positive real constants and n is any positive integer. Construct a corresponding sequence $\{U_r\}$ as follows:

(10) $$U_r = 1,\ \forall_r = 1,\ 2,\ 3,\ \ldots,\ n;\ U_r = a_1U_{r-1} + a_2U_{r-2} + \cdots + a_nU_{r-n},\ \forall_r \geq n + 1.$$

Take the consecutive ratios of the terms of this sequence. The sequence $\left\{\frac{U_{r-1}}{U_r}\right\}$ of ratios comes out to be tending to a root of the polynomial equation (9).

Now consider the following polynomial

(11) $$x^2 - 2x = 1$$

involving negative coefficients also. Construct the sequence as follows:

(12) $$U_r = 1,\ \forall_r = 1,\ 2;\ U_r = -2U_{r-1} + U_{r-2}.$$

Find the ratios of the consecutive terms of this sequence. These again tend to -0.414213..., a root of this polynomial equation.

In case the roots of a quadratic equation are not real, the said sequence of ratios does not converge. In some cases it fluctuates in a manner that is readily observed. In other cases it is quite difficult to know the fluctuation pattern. I believe that there is some mathematical relation between this fluctuation pattern of the sequence of the ratios and the discrimanant of the quadratic equation when the constant term is made -1, by suitably disposing the coefficients. As an example, one can observe the quadratic equation

(13) $$-x^2 - x = 1.$$

Sometimes it happens that the sequence of the said ratios behaves in such a manner that it is quite difficult to assess even whether it is converging to some constant or fluctuating in some pattern. In such cases, with the help of a computer, one can assess the nature of the sequence of the ratios observing fairly large numbers of terms of this sequence (say 200 and 300, etc.). For example, I could not see any pattern easily in the sequence of the said ratios for the polynomial equation

(14) $$2x^3 + x^2 - x = 1.$$

However, when the fluctuations pattern is readily observable, it is my belief that there is a relation between the oscillating ratios and an imaginary root of the considered polynomial.

I could not get any case where the sequence of the said ratios converged to some constant, say x_0, when x_0 was not a root of the considered polynomial equation. This led me to state the following:

"Given a polynomial equation of the type

(15) $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x = 1$$

(all a_n's are real and n any positive integer)."

We observe the sequence of the ratios of the successive terms of the sequence $\{U_r\}$ defined as follows:

(16) $$U_r = 1,\ \forall_r = 1, 2, \ldots, n;\ U_r = a_1 U_{r-1} + a_2 U_{r-2} + \cdots + a_n U_{r-n},\ \forall_r \geq n + 1.$$

If the sequence $\left\{\frac{U_{r-1}}{U_r}\right\}$ converges to some fixed number x_0 (and I believe if there is a real root it converges more often than not), then x_0 satisfies this polynomial equation.

This fact can be utilized to attempt to find out the roots of a polynomial equation

(α) $$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0 = 0,$$

where the A_i's are all real and n is a positive integer. The method is summarized as follows: "If $A_0 = 0$, clearly $x = 0$ satisfies (α). So zero is one root of (α). Divide, then, (α) by x to get an equation of $(n - 1)$st degree, and again treat this new polynomial equation of degree $(n - 1)$ as (α). If $A_0 \neq 0$, we can write (α) in the following form:

(17) $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x = 1, \text{ where } a_i = -\frac{A_i}{A_0}.$$

Now form a sequence $\{U_r\}$ as in (16). A fixed quantity x_0, to which the sequence $\left\{\frac{U_{r-1}}{U_r}\right\}$ tends, is one of the roots of (α). I am sure that it does tend, at least when all a_i's are positive. Divide (α) by $(x - x_0)$ to obtain a polynomial equation of degree $(n - 1)$. Again treat this new polynomial equation as (17) and (if possible) obtain another root, and so on."

I believe that the whole phenomenon is not merely a magic of numbers; instead, there is some mathematics behind this, though I could not get hold of it. Also, I could not find the system by which the sequence of the ratios chooses one of the roots to converge to.

Lastly, I quote an interesting example. Call the set of unities used in defining the sequence $\{U_r\}$ as "generators." In fact, this example will show the importance of generators.

"Consider the quadratic equation

(18) $$2x^2 - x - 1 = 0$$
or
$$2x^2 - x = 1.$$

Form $\{U_r\}$ as usual (i.e., taking 1,1 as generator) so as to get 1, 1, 1, 1, 1, 1, ..., which gives us 1 as one of the roots of (18). Now form another sequence $\{U_r\}$ taking the coefficients 2 and -1 as generators to get 2, -1, 5, -7, 17, -31, 65, ..., which converges to $-\frac{1}{2}$ and, interestingly, $-\frac{1}{2}$ is another root of (18)."

I shall conclude this article by posing a problem regarding the Fibonacci sequence. Up to the 36th term of the sequence (F_{36} = 24, 157, 817 ...) none is a perfect number. It remains to be solved whether any Fibonacci number is a perfect one. If not, then what is the mathematical logic behind it?

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