RECONSIDERING A PROBLEM OF M. WARD

JAN VAN LEEUWEN
University of Utrecht, Utrecht, Netherlands

ABSTRACT

In a recent issue of *The Fibonacci Quarterly*, Laxton proved a conjecture of Ward to the effect that integral linear recurrences which are not degenerate in a certain sense necessarily contain infinitely many distinct prime divisors. We point out that the result is an immediate corollary to an early theorem of Pólya published in 1921, and derive Ward's conjecture for a more general class of integral linear recurrences.

1. INTRODUCTION

Ward [6, 7] showed that nondegenerate integral linear recurrences of order 2 and 3 always contain infinitely many distinct prime divisors. Recently, Laxton [3] proved Ward's conjecture that a similar result must hold for recurrences of arbitrary higher order (again excluding some degenerated cases).

Let

$$w_{n+m} = a_{m-1}w_{n+m-1} + \cdots + a_1 w_{n+1} + a_0 w_n$$

(with $a_0 \neq 0$, $m > 0$, $n \geq 0$) be an mth order integral linear recurrence and let

$$P_w(x) = x^m - a_{m-1}x^{m-1} \cdots - a_0$$

be the associated characteristic (or spectral) polynomial.

Here is what was proved.

Theorem: Let $\{w_n\}$ be an integral linear recurrence of order $m > 1$. If all roots of $P_w(x)$ are distinct and if no ratio of distinct roots is a root of unity, then $\{w_n\}$ has infinitely many distinct prime divisors.

It turns out that the answer already did exist before the question. The very same result (and thereby the solution to Ward's conjecture) is an almost immediate corollary to a theorem of Pólya [5, Satz II'] dating back to 1921, which seems to have escaped attention.

We shall indicate how the theorem can be applied and use it to derive a stronger solution of Ward's problem.

2. POLYA'S THEOREM

We shall have to assume that the reader is familiar with some algebraic number theory (see Landau [2] or Pollard [4] for an excellent introduction).

First we observe

Lemma: Let K be an algebraic number field, D a nonzero algebraic integer in K, and $\{w_n\}$ a sequence of rational integers. $\{w_n\}$ has infinitely many prime divisors if and only if $\{Dw_n\}$ has infinitely many prime-ideal divisors.

We now combine Pólya's Satz III' [5, p. 15] and Satz II' [5, p. 17] to obtain

Theorem: Let $\alpha_1, \ldots, \alpha_r$ and all coefficients of the nontrivial polynomials $P_1(x), \ldots, P_r(x)$ be algebraic integers. Let $D \neq 0$ be an algebraic integer such that

$$F(x) = \frac{1}{D}(P_1(x)\alpha_1^x + \cdots + P_r(x)\alpha_r^x)$$

has rational integer values for $x = 0, 1, 2, \ldots$.

Assume that $r + \min \deg P(x) \leq 2$. If no ratio of distinct α's is a root of unity, then $F(x)$ has infinitely many prime-divisors.

Pólya showed the theorem for $D = 1$ (or any rational integer for that matter) but only slight modifications in the proof make it true for arbitrary algebraic integers.

For consider $G(x) = D \cdot F(x)$ and carry out the same proof. By the lemma, it follows that assuming that $F(x)$ only has finitely many prime-divisors (by way of contradiction, as Pólya does) is equivalent to assuming that $G(x)$ only has finitely many prime-ideal divisors. Where Pólya considers absolute values, one should use norms; where Pólya proceeds with analytic arguments related to the series $\Sigma F(n)z^n$, one can do exactly the same for G after factoring out D.

The theorem enables us to prove Ward's conjecture with the condition that all roots need to be distinct omitted!

Here is what we get.

Theorem: Let $\{w_n\}$ be an integral linear recurrence of order $m \geq 2$. If no ratio of distinct roots of $P_w(x)$ is a root of unity, then $\{w_n\}$ has infinitely many distinct prime divisors.

Here is how to prove it. Consider the recurrence equation for w_n. Following Gel'fond [1] (or other books on difference equations), the general solution can be expressed as

$$w_n = \frac{1}{D}(P_1(x)\alpha_1^x + \cdots + P_r(x)\alpha_r^x)$$

where $\alpha_1, \ldots, \alpha_r$ are the roots of $P_w(x)$, $P_i(x)$ a polynomial of degree equal to the multiplicity of w_i minus 1 and with algebraic integer coefficients, and D a nonzero determinant of algebraic integers (hence an algebraic integer as well). It easily follows that the conditions for Pólya's theorem are satisfied and $\{w_n\}$ must have infinitely many distinct prime divisors.

REFERENCES

1. A. O. Gel'fond. *Calculus of Finite Differences*. Delhi: Hindustan Publishing Corp., 1971.
2. E. Landau. *Vorlesungen über Zahlentheorie*. Band 3. Leipzig, 1927.
3. R. R. Laxton. "On a Problem of M. Ward." *The Fibonacci Quarterly* 12 (1974):41-44.
4. H. Pollard. *The Theorie of Algebraic Numbers*. Carus Math. Monographs 6. New York: AMS & Wiley, 1950.
5. G. Pólya. "Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen." *Journal für die reine und angewandte Mathematik* Band 151 (1921):1-31.
6. M. Ward. "Prime Divisors of Second-Order Recurring Sequences." *Duke Math. J.* 21 (1954):607-614.
7. M. Ward. "The Laws of Apparition and Repetition of Primes in a Cubic Recurrence." *Trans. AMS* 79 (1955):72-90.

*****

WHAT A DIFFERENCE A DIFFERENCE MAKES!

JERRY T. SULLIVAN

Two men are leaving the office when one remarks that both his wife and boy are celebrating their birthdays that night. The other wonders if it is his youngest son. "Yes," says the first, "but he's not so little anymore. His age, multiplied by my wife's age, is equal to the square of the difference of their ages plus one year." This problem, similar to an earlier one in *The Fibonacci Quarterly* [1], provides some surprising and amusing mathematical twists.

On the premise that many mothers are between 25 and 35 years of age, and also that a typical boy is about 10 years old, pairs of ages such as 10 and 30, 11 and 35, etc., can be tested. After a few trials, an answer is seen to be 13 and 34. Further thought shows that the problem can be handled algebraically. If the age of the wife is W and that of the boy is B, then

(1) $$WB = (W - B)^2 + 1.$$

The wife's age can be solved as a function of the boy's age:

(2) $$W = [3B \pm (5B^2 - 4)^{1/2}]/2.$$

Substituting $B = 13$ into equation (2) and using the positive square root gives the known answer $W = 34$. However, using the negative square root gives the answer $W = 5$. It is an unusual wife who is younger than her son, but the numbers 13 and 5 also satisfy equation (1). Using the number 5 in equation (2) and choosing the negative root gives the numbers 5 and 2 as another solution. Proceeding in this fashion results in the sequence

(3) $$1, 2, 5, 13, 34, 89, \ldots,$$

where each successive pair of numbers satisfies equation (1). The number 1 has the unusual property of giving the solutions 1 and 2 when substituted into equation (2). It does not give a solution lower than itself.

The above sequence is every other number of the usual Fibonacci sequence. Calling the initial age in the sequence A_0, the next A_1, etc., equation (1) may be rewritten as a difference equation,

(4) $$A_{N+1}A_N = (A_{N+1} - A_N)^2 + 1.$$