POLYNOMIAL FIBONACCI-LUCAS IDENTITIES OF THE FORM $\sum_{r=1}^{n} P(r)F_r$

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INTRODUCTION

$\sum_{r=1}^{n} P(r)$ can be evaluated by substitution in the mnemonic chain of formulas:

(1) $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(2) $$\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$$

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(K) $$\sum_{r=1}^{n} r(r+1)\ \ldots\ (r+k-1) = \frac{n(n+1)\ \cdots\ (n+k)}{k+1}.$$

The proof of (K) by mathematical induction also establishes the validity of (1), (2), ..., (K + 1).

Example 1: From (a) on the right,

$$r^3 = r(r+1)(r+2) - 3r(r+1) + r$$

(a)

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & \lfloor 0 \\ -1 & 1 & -1 & \lfloor 1 & \\ -2 & 1 & \lfloor -3 & & \end{array}$$

$$\sum_{r=1} r^3 = \sum_{r=1}^{n} r(r+1)(r+2) - 3\sum_{r=1}^{n} r(r+1) + \sum_{r=1}^{n} r$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} - n(n+1)(n+2) + \frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4}.$$

THE FIBONACCI-LUCAS CHAIN OF POLYNOMIAL IDENTITIES

I. Using the Fibonacci lists of identities, or otherwise, it is possible to write the following relations:

(1) $$\sum_{r=1}^{n} F_r = F_{n+2} - 1$$

(2) $$\sum_{r=1}^{n} rF_r = nF_{n+2} - F_{n+3} + 2$$

(3) $$\sum_{r=1}^{n} r(r+1)F_r = n(n+1)F_{n+2} - 2nF_{n+3} + 2F_{n+4} - 6$$

(4) $$\sum_{r=1}^{n} r(r+1)(r+2)F_r = n(n+1)(n+2)F_{n+2} - 3n(n+1)F_{n+3} + 6nF_{n+4} - 6F_{n+5} + 30$$

(5) $$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)F_r = n(n+1)(n+2)(n+3)F_{n+2} - 4n(n+1)(n+2)F_{n+3} + 12n(n+1)F_{n+4} - 24nF_{n+5} + 24F_{n+6} - 192$$

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(K) $$\sum_{r=1}^{n} r(r+1)\ \cdots\ (r+k-1)F_r = n(n+1)\ \cdots\ (n+k-1)F_{n+2} - kn(n+1)\ \cdots\ (n+k-2)F_{n+3} + \cdots + (-1)^{k+1}k!F_{n+k+2} + (-1)^{k+1}k!F_{k+2}.$$

Using iterated integration by parts for finite differences:

$$\sum_{r=1}^{n} r(r+1)\cdots(r+k-1)F_r = n(n+1)\cdots(n+k+1)(F_{n+1}+F_n)$$

$$-\,kn(n+1)\cdots(n+k-2\,(F_{n+1}+2F_n+F_{n-1})$$

$$+\,k(k-1)n(n+1)\cdots(n+k-3)(F_{n+1}+3F_n+3F_{n-1}+F_{n-2})$$

$$-\,k(k-1)(k-2)n(n+1)\cdots(n+k-4)(F_{n+1}+4F_n+6F_{n-1}+4F_{n-2}+F_{n-3})$$

$$+\cdots+(-1)^{k+1}k!\left[F_{n+1}+\binom{k+1}{1}F_{n+2}+\binom{k+1}{2}F_{n+3}+\cdots\right]+C_k.$$

Catalan has an operator formula $U^{n+p} = U^{n-p}(u+1)^p$. After the algebraic operations are performed on the right, all powers become subscripts of U. U_k can be replaced by either F_k or L_k. Using this formula, or mathematical induction,

$$\sum_{r=1}^{n} r(r+1)\cdots(r+k-1)F_r = n(n+1)\cdots(n+k-1)F_{n+2}$$

$$-\,kn(n+1)\cdots(n+k-2)F_{n+3}$$

$$+\cdots+(-1)^{k+1}k!F_{n+k+2}+C_k.$$

Let $n = 1$ in (K) and (K - 1), C_k, an integer, must be a multiple of $k!$ Dividing out the common factor $(k-1)!$ in (K - 1) and $k!$ in (K), $|C| = F_{k+3} - F_{k+1} = F_{k+2}$.

<u>*Example 2*</u>: $$\sum_{r=1}^{n} r^3F_r = \sum_{r=1}^{n} r(r+1)(r+2)F_r - 3\sum_{r=1}^{n} r(r+1)F_r + \sum_{r=1}^{n} F_r$$

$$= n^3F_{n+2} - (3n^2 - 3n + 1)F_{n+3} + (6n-6)F_{n+4} - 6F_{n+5} + 50.$$

II. Using a Lucas list of identities, or otherwise, it is possible to establish (using proofs similar to those in (I) the following identities:

(1) $$\sum_{r=1}^{n} L_r = L_{n+2} - 3$$

(2) $$\sum_{r=1}^{n} rL_r = nL_{n+2} - L_{n+3} + 4$$

(3) $$\sum_{r=1}^{n} r(r+1)L_r = n(n+1)L_{n+2} - 2nL_{n+3} + 2L_{n+4} - 14$$

(4) $$\sum_{r=1}^{n} r(r+1)(r+2)L_r = n(n+1)(n+2)L_{n+2} - 3n(n+1)L_{n+3} + 6nL_{n+4} - 6L_{n+5} + 66$$

(5) $$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)L_r = n(n+1)(n+2)(n+3)L_{n+2} - 4n(n+1)(n+2)L_{n+3} + 12n(n+1)L_{n+4} - 24nL_{n+5} + 24L_{n+6} - 432$$

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(K) $$\sum_{r=1}^{n} r(r+1)\cdots(r+k-1)L_r = n(n+1)\cdots(n+k-1)L_{n+2} - kn(n+1)\cdots(n+k-2)L_{n+3} + k(k-1)n(n+1)\cdots(n+k-3)L_{n+4} - \cdots + (-1)^{k+1}k!L_{n+k+2} + (-1)^{k+1}k!L'_{k+2}.$$

Example 3: $$\sum_{r=1}^{n} r^3 L_r = \sum_{r=1}^{n} r(r+1)(r+2)L_r - 3\sum_{r=1}^{n} r(r+1)L_r + \sum_{r=1}^{n} rL_r$$

$$= n^3 L_{n+2} - (3n^2 - 3n + 1)L_{n+3} + (6n - 6)L_{n+4} - 6L_{n+5} + 112.$$

REFERENCES

1. Leonard Dickson. *History of the Theory of Numbers*. Chapter 17.
2. V. C. Harris. "Fibonacci-Lucas Identities." *The Fibonacci Quarterly*.

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A GENERALIZATION OF SOME L. CARLITZ IDENTITIES

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Leonard Carlitz [1], by factoring $(x + y)^p - x^p - y^p$, developed the following identities:

1. $F^3_{n+1} - F^3_n - F^3_{n-1} = 3F_{n-1}F_nF_{n+1}$
   $L^3_{n+1} - L^3_n - L^3_{n-1} = 3L_{n-1}L_nL_{n+1}$

2. $F^4_{n+1} + F^4_n + F^4_{n-1} = 2[F^2_{n+1} - F_nF_{n-1}]^2$
   $L^4_{n+1} + L^4_n + L^4_{n-1} = 2[L^2_{n+1} - L_nL_{n-1}]^2$

3. $F^5_{n+1} - F^5_n - F^5_{n-1} = 5F_{n-1}F_nF_{n+1}(F^2_{n+1} - F_nF_{n-1})$
   $L^5_{n+1} - L^5_n - L^5_{n-1} = 5L_{n-1}L_nL_{n+1}(L^2_{n+1} - L_nL_{n-1})$

4. $F^7_{n+1} - F^7_n - F^7_{n-1} = 7F_{n-1}F_nF_{n+1}(F^2_{n+1} - F_nF_{n-1})^2$
   $L^7_{n+1} - L^7_n - L^7_{n-1} = 7L_{n-1}L_nL_{n+1}(L^2_{n+1} - L_nL_{n-1})^2$

The common subscript difference is 1. A generalization consists in forming identities with (a) a common subscript difference $2r + 1$; (b) a common subscript difference $2r$.

1. $F^3_{n+2r+1} - L^3_{2r+1}F^3_n - F^3_{n-2r-1} = 3L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}$
   $L^3_{n+2r+1} - L^3_{2r+1}L^3_n - L^3_{n-2r-1} = 3L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}$

   $L^3_{2r}F^3_n - F^3_{n+2r} - F^3_{n-2r} = 3L_{2r}F_{n-2r}F_nF_{n+2r}$
   $L^3_{2r}L^3_n - L^3_{n+2r} - L^3_{n-2r} = 3L_{2r}L_{n-2r}L_nL_{n+2r}$

2. $F^4_{n+2r+1} + L^4_{2r+1}F^4_n + F^4_{n-2r-1} = 2[F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1}]^2$
   $L^4_{n+2r+1} + L^4_{2r+1}L^4_n + L^4_{n-2r-1} = 2[L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1}]^2$

   $F^4_{n+2r} + L^4_{2r}F^4_n + F^4_{n-2r} = 2(F^2_{n+2r} + L_{2r}F_nF_{n-2r})^2$
   $L^4_{n+2r} + L^4_{2r}L^4_n + L^4_{n-2r} = 2(L^2_{n+2r} + L_{2r}L_nL_{n-2r})^2$

3. $F^5_{n+2r+1} - L^5_{2r+1}F^5_n - F^5_{n-2r-1} = 5L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1})$
   $L^5_{n+2r+1} - L^5_{2r+1}L^5_n - L^5_{n-2r-1} = 5L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}(L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1})$

   $L^5_{2r}F^5_n - F^5_{n+2r} - F^5_{n-2r} = 5L_{2r}F_{n-2}\ F_nF_{n+2r}(F^2_{n+2r} + L_{2r}F_nF_{n-2r})$
   $L^5_{2r}L^5_n - L^5_{n+2r} - L^5_{n-2r} = 5L_{2r}L_{n-2}\ L_nL_{n+2r}(L^2_{n+2r} + L_{2r}L_nL_{n-2r})$

4. (a) $F^7_{n+2r+1} - L^7_{2r+1}F^7_n - F^7_{n-2r-1} = 7L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1})^2$
   (b) $L^7_{n+2r+1} - L^7_{2r+1}L^7_n - L^7_{n-2r-1} = 7L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}(L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1})^2$