Example 3: $\sum_{r=1}^{n} r^3 L_r = \sum_{r=1}^{n} r(r+1)(r+2)L_r - 3\sum_{r=1}^{n} r(r+1)L_r + \sum_{r=1}^{n} rL_r$

$$= n^3 L_{n+2} - (3n^2 - 3n + 1)L_{n+3} + (6n - 6)L_{n+4} - 6L_{n+5} + 112.$$

REFERENCES

1. Leonard Dickson. *History of the Theory of Numbers*. Chapter 17.
2. V. C. Harris. "Fibonacci-Lucas Identities." *The Fibonacci Quarterly*.

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A GENERALIZATION OF SOME L. CARLITZ IDENTITIES

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Leonard Carlitz [1], by factoring $(x + y)^p - x^p - y^p$, developed the following identities:

1. $F_{n+1}^3 - F_n^3 - F_{n-1}^3 = 3F_{n-1}F_nF_{n+1}$
   $L_{n+1}^3 - L_n^3 - L_{n-1}^3 = 3L_{n-1}L_nL_{n+1}$
2. $F_{n+1}^4 + F_n^4 + F_{n-1}^4 = 2[F_{n+1}^2 - F_nF_{n-1}]^2$
   $L_{n+1}^4 + L_n^4 + L_{n-1}^4 = 2[L_{n+1}^2 - L_nL_{n-1}]^2$
3. $F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n-1}F_nF_{n+1}(F_{n+1}^2 - F_nF_{n-1})$
   $L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n-1}L_nL_{n+1}(L_{n+1}^2 - L_nL_{n-1})$
4. $F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n-1}F_nF_{n+1}(F_{n+1}^2 - F_nF_{n-1})^2$
   $L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n-1}L_nL_{n+1}(L_{n+1}^2 - L_nL_{n-1})^2$

The common subscript difference is 1. A generalization consists in forming identities with (a) a common subscript difference $2r + 1$; (b) a common subscript difference $2r$.

1. $F_{n+2r+1}^3 - L_{2r+1}^3F_n^3 - F_{n-2r-1}^3 = 3L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}$
   $L_{n+2r+1}^3 - L_{2r+1}^3L_n^3 - L_{n-2r-1}^3 = 3L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}$

   $L_{2r}^3F_n^3 - F_{n+2r}^3 - F_{n-2r}^3 = 3L_{2r}F_{n-2r}F_nF_{n+2r}$
   $L_{2r}^3L_n^3 - L_{n+2r}^3 - L_{n-2r}^3 = 3L_{2r}L_{n-2r}L_nL_{n+2r}$

2. $F_{n+2r+1}^4 + L_{2r+1}^4F_n^4 + F_{n-2r-1}^4 = 2[F_{n+2r+1}^2 - L_{2r+1}F_nF_{n-2r-1}]^2$
   $L_{n+2r+1}^4 + L_{2r+1}^4L_n^4 + L_{n-2r-1}^4 = 2[L_{n+2r+1}^2 - L_{2r+1}L_nL_{n-2r-1}]^2$

   $F_{n+2r}^4 + L_{2r}^4F_n^4 + F_{n-2r}^4 = 2(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$
   $L_{n+2r}^4 + L_{2r}^4L_n^4 + L_{n-2r}^4 = 2(L_{n+2r}^2 + L_{2r}L_nL_{n-2r})^2$

3. $F_{n+2r+1}^5 - L_{2r+1}^5F_n^5 - F_{n-2r-1}^5 = 5L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F_{n+2r+1}^2 - L_{2r+1}F_nF_{n-2r-1})$
   $L_{n+2r+1}^5 - L_{2r+1}^5L_n^5 - L_{n-2r-1}^5 = 5L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}(L_{n+2r+1}^2 - L_{2r+1}L_nL_{n-2r-1})$

   $L_{2r}^5F_n^5 - F_{n+2r}^5 - F_{n-2r}^5 = 5L_{2r}F_{n-2}\ F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})$
   $L_{2r}^5L_n^5 - L_{n+2r}^5 - L_{n-2r}^5 = 5L_{2r}L_{n-2}\ L_nL_{n+2r}(L_{n+2r}^2 + L_{2r}L_nL_{n-2r})$

4. (a) $F_{n+2r+1}^7 - L_{2r+1}^7F_n^7 - F_{n-2r-1}^7 = 7L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F_{n+2r+1}^2 - L_{2r+1}F_nF_{n-2r-1})^2$
   (b) $L_{n+2r+1}^7 - L_{2r+1}^7L_n^7 - L_{n-2r-1}^7 = 7L_{2r+1}L_{n-2r-1}L_nL_{n+2r+1}(L_{n+2r+1}^2 - L_{2r+1}L_nL_{n-2r-1})^2$

(c) $L_{2r}^7 F_n^7 - F_{n+2r}^7 - F_{n-2r}^7 = 7L_{2r}F_{n-2r}F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$

(d) $L_{2r}^7 L_n^7 - L_{n+2r}^7 - L_{n-2r}^7 = 7L_{2r}L_{n-2r}L_nL_{n+2r}(L_{n+2r}^2 + L_{2r}L_nL_{n-2r})^2$

The proofs of 4(a) and 4(c) could serve as proof models for the remaining identities.

4(a): $F_{n+2r+1}^7 - L_{2r+1}F_n^7 - F_{n-2r-1}^7 = -(L_{2r+1}F_n)^7 + F_{n+2r+1}^7 - F_{n-2r-1}^7$

$$= -(F_{n+2r+1} - F_{n-2r-1})^7 + F_{n+2r+1}^7 - F_{n-2r-1}^7$$

$$= 7F_{n+2r+1}^6F_{n-2r-1} - 21F_{n+2r+1}^5F_{n-2r+1}^2 + 35F_{n+2r-1}^4F_{n-2r+1}^3 - 35F_{n+2r+1}^3F_{n-2r-1}^4 + 21F_{n+2r+1}^2F_{n-2r-1}^5 - 7F_{n+2r+1}F_{n-2r-1}^6$$

$$= 7F_{n+2r+1}F_{n-2r-1}(F_{n+2r+1}^5 - 3F_{n+2r+1}^4F_{n-2r-1} + 5F_{n+2r+1}^3F_{n-2r-1}^2 - 5F_{n+2r+1}^2F_{n-2r-1}^3 + 3F_{n+2r+1}F_{n-2r-1}^4 - F_{n-2r-1}^5)$$

$$= 7F_{n+2r+1}F_{n-2r-1}(F_{n+2r+1} - F_{n-2r-1})(F_{n+2r+1}^4 - 2F_{n+2r+1}^3F_{n-2r-1} + 3F_{n+2r+1}^2F_{n-2r-1}^2 - 2F_{n+2r+1}F_{n-2r-1}^3 + F_{n-2r-1}^4)$$

$$= 7F_{n+2r+1}F_{n-2r-1}L_{2r+1}F_n(F_{n+2r+1}^2 - F_{n+2r+1}F_{n-2r-1} + F_{n-2r-1}^2)^2$$

$$= 7L_{2r+1}F_{n-2r-1}F_nF_{n+2r+1}(F_{n+2r+1}^2 - L_{2r+1}F_nF_{n-2r-1})^2$$

4(c): $F_n^7L_{2r}^7 - F_{n+2r}^7 - F_{n-2r}^7 = (F_nL_{2r})^7 - F_{n+2r}^7 - F_{n-2r}^7 = (F_{n+2r} + F_{n-2r})^7 - F_{n+2r}^7 - F_{n-2r}^7$

$$= 7F_{n-2r}F_{n+2r}(F_{n+2r}^5 + 3F_{n+2r}^4F_{n-2r} + 5F_{n+2r}^3F_{n-2r}^2 + 5F_{n+2r}^2F_{n-2r}^3 + 3F_{n+2r}F_{n-2r}^4 + F_{n-2r}^5)$$

$$= 7F_{n-2r}F_{n+2r}(F_{n+2r} + F_{n-2r})(F_{n+2r}^4 + 2F_{n+2r}^3F_{n-2r} + 3F_{n+2r}^2F_{n-2r}^2 + 2F_{n+2r}F_{n-2r}^3 + F_{n-2r}^4)$$

$$= 7F_{n-2r}F_{n+2r}L_{2r}F_n(F_{n+2r}^2 + F_{n+2r}F_{n-2r} + F_{n-2r}^2)^2$$

$$= 7L_{2r}F_{n-2r}F_nF_{n+2r}(F_{n+2r}^2 + L_{2r}F_nF_{n-2r})^2$$

NOTE: On the assumption that Type I primitive units are given by

$$\left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},$$

these sixteen generalized F-L identities are valid Type I identities.

REFERENCE

1. Problem H-112 (and its solution), proposed by Leonard Carlitz. *The Fibonacci Quarterly* 7 (1969).

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A CHARACTERIZATION OF THE PYTHAGOREAN TRIPLES

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The Pythagorean triples are all the systems of positive integers x, y, z which satisfy the "Pythagorean equation"

(1) $$x^2 + y^2 = z^2.$$

It is well known (see Uspensky and Heaslet [2]) that the Pythagorean triples can be characterized by the formulas

(2) $$x = M(r^2 - s^2),\ y = M2rs,\ z = M(r^2 + s^2),$$

where r and s are any two relatively prime numbers of different parity with $r > s$ and M is an arbitrary positive integer.