GENERALIZED FIBONACCI-LUCAS DIFFERENCE EQUATIONS

GREGORY WULCZYN
Bucknell University, Lewisburg, PA 17837

Let p be an integer of the form $(2m + 1)^2 + 4$, $m = 0, 1, 2, \ldots$. The finite difference equation

(1) $$P_{n+2} = (2m + 1)P_{n+1} + P_n; \; P_1 = 1, \; P_2 = 2m + 1$$

has solutions given by

$$P_n = \frac{1}{\sqrt{p}}(\alpha^n - \beta^n); \; \alpha = \frac{2m + 1 + \sqrt{p}}{2}, \; \beta = \frac{2m + 1 - \sqrt{p}}{2}.$$

The finite difference equation

(2) $$Q_{n+2} = (2m + 1)Q_{n+1} + Q_n; \; Q_1 = 2m + 1, \; Q_2 = p - 2$$

has solutions given by

$$Q_n = \alpha^n + \beta^n; \; \alpha = \frac{2m + 1 + \sqrt{p}}{2}, \; \beta = \frac{2m + 1 - \sqrt{p}}{2}.$$

The following relations can be found.

$$\alpha\beta = -1$$

$$\alpha = \frac{2m + 1 + \sqrt{p}}{2} = \frac{Q_1 + P_1\sqrt{p}}{2}$$

$$\alpha^2 = \frac{p - 2 + (2m + 1)\sqrt{p}}{2} = \frac{Q_2 + P_2\sqrt{p}}{2}$$

$$\alpha^3 = \frac{(2m + 1)(p - 1) + (p - 3)\sqrt{p}}{2} = \frac{Q_3 + P_3\sqrt{p}}{2}$$

$$\alpha^4 = \frac{Q_4 + P_4\sqrt{p}}{2}$$

$$\alpha^5 = \frac{Q_5 + P_5\sqrt{p}}{2}$$

$$\alpha^6 = \frac{Q_6 + P_6\sqrt{p}}{2}$$

P_n and Q_n are both even if $n \equiv 0 \pmod 3$; otherwise they are both odd. The basic Fibonacci-Lucas identities can also be generalized.

1. $\displaystyle\sum_{i=1}^{n} F_i = F_{n+2} - 1$

1′ $\displaystyle 2(2m + 1)\sum_{i=1}^{n} P_i = Q_{n+1} - (2m - 1)P_{n+1} - 2$

2. $\displaystyle\sum_{i=1}^{n} L_i = L_{n+2} - 3$

2′ $\displaystyle 2(2m + 1)\sum_{i=1}^{n} Q_i = pP_{n+1} - (2m - 1)Q_{n+1} - 4m - 6$

3. $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ 3′ $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$

4. $L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}$ 4′ $Q_{n+1}Q_{n-1} - Q_n^2 = p(-1)^{n+1}$

5. $L_n = F_{n+1} + F_{n-1}$ 5′ $Q_n = P_{n+1} + P_{n-1}$

6. $F_{2n+1} = F_{2n+1}^2 + F_n^2$

7. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$

8. $F_{2n} = F_n L_n$

9. $F_{n+p+1} = F_{n+1}F_{p+1} + F_n F_p$

10. $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

11. $L_n^2 - 5F_n^2 = 4(-1)^n$

12. $F_{-n} = (-1)^{n+1}F_n$

6′ $P_{2n+1} = P_{n+1}^2 + P_n^2$

7′ $(2m + 1)P_{2n} = P_{n+1}^2 - P_{n-1}^2$

8′ $P_{2n} = P_n Q_n$

9′ $P_{n+t+1} = P_{n+1}P_{t+1} + P_n P_t$

10′ $(2m + 1)\sum_{i=1}^{n} P_i^2 = P_n P_{n+1}$

11′ $Q_n^2 - pP_n^2 = 4(-1)^n$

12′ $P_{-n} = (-1)^{n+1}P_n$

$$\left.\begin{matrix} x = Q_1, & y = P_1 \\ x = Q_3, & y = P_3 \\ x = Q_5, & y = P_5 \end{matrix}\right\} \text{ are particular solutions of } x^2 - py^2 = -4$$

$$\left.\begin{matrix} x = Q_2, & y = P_2 \\ x = Q_4, & y = P_4 \\ x = Q_6, & y = P_6 \end{matrix}\right\} \text{ are particular solutions of } x^2 - py^2 = 4$$

Since $x = Q_1$, $y = P_1$ is not a solution of $x^2 - py^2 = -1$ but $x = \frac{1}{2}Q_3$, $y = \frac{1}{2}P_3$ is, $x = \frac{1}{2}Q_3$, $y = \frac{1}{2}P_3$ is the primitive solution of $x^2 - py^2 = -1$ and $x = \frac{1}{2}Q_6$, $y = \frac{1}{2}P_6$ is the primitive solution of $x^2 - py^2 = 1$. Also,

$$\alpha^r \cdot \alpha^s = \alpha^{r+s} = \frac{Q_{r+s} + P_{r+s}\sqrt{p}}{2}.$$

Theorem 1: An integer y is an integer of the sequence $P_1, P_2, \ldots,$ if and only if $py^2 - 4$ or $py^2 + 4$ is an integer square.

Proof (1):

(3) $$x^2 - py^2 = -4$$

The three solution chains of $x^2 - py^2 = -4$ are given by

$$(Q_1 + P_1\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+1} + P_{6t+1}\sqrt{p}$$
$$(Q_3 + P_3\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+3} + P_{6t+3}\sqrt{p}$$
$$(Q_5 + P_5\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+5} + P_{6t+5}\sqrt{p},\ t = 0, 1, 2, \ldots .$$

Letting $t = 0, 1, 2, \ldots,$ the y integer values are $P_1, P_3, P_5, \ldots,$ the successive odd P numbers.

(4) $$x^2 - py^2 = 4$$

The three solution chains of $x^2 - py^2 = 4$ are given by

$$(Q_2 + P_2\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+2} + P_{6t+2}\sqrt{p}$$
$$(Q_4 + P_4\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+4} + P_{6t+4}\sqrt{p}$$
$$(Q_6 + P_6\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = Q_{6t+6} + P_{6t+6}\sqrt{p},\ t = 0, 1, 2, \ldots .$$

Letting $t = 0, 1, 2, \ldots,$ the y integer values are $P_2, P_4, P_6, \ldots,$ the successive even P numbers.

Proof (2):

(5) $$y = P_{2m+1}$$

$$py^2 - 4 = pP_{2m+1}^2 - 4 = (\alpha^{2m+1} - \beta^{2m+1})^2 - 4$$

$$= \alpha^{4m+2} + 2(\alpha\beta)^{2m+1} + \beta^{4m+2} = (\alpha^{2m+1} + \beta^{2m+1})^2$$

$$= Q_{2m+1}^2, \text{ an integer square.}$$

(6) $$y = P_{2m}$$

$$py^2 + 4 = pP_{2m}^2 + 4 = (\alpha^{2m} - \beta^{2m})^2 + 4$$

$$= \alpha^{4m} + 2(\alpha\beta)^{2m} + \beta^{4m} = (\alpha^{2m} + \beta^{2m})^2$$

$$= Q_{2m}^2, \text{ an integer square.}$$

Theorem 2: An integer y is an integer of the sequence $Q_1, Q_2, Q_3, \ldots,$ if and only if $pn^2 - 4p$ or $pn^2 + 4p$ is an integer square.

Proof (3):

(7) $$x^2 - py^2 = -4p$$

The three solution chains of $x^2 - py^2 = -4p$ are given by

$$(pP_1 + Q_1\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+1} + Q_{6t+1}\sqrt{p}$$

$$(pP_3 + Q_3\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+3} + Q_{6t+3}\sqrt{p}$$

$$(pP_5 + Q_5\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+5} + Q_{6t+5}\sqrt{p}, \quad t = 0, 1, 2, 3, \ldots .$$

Letting $t = 0, 1, 2, 3, \ldots,$ the y integer values are $Q_1, Q_3, Q_5, \ldots,$ the successive odd Q numbers.

(8) $$x^2 - py^2 = 4p$$

The three solution chains of $x^2 - py^2 = 4p$ are given by

$$(pP_2 + Q_2\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+2} + Q_{6t+2}\sqrt{p}$$

$$(pP_4 + Q_4\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+4} + Q_{6t+4}\sqrt{p}$$

$$(pP_6 + Q_6\sqrt{p})(\tfrac{1}{2}Q_6 + \tfrac{1}{2}P_6\sqrt{p})^t = pP_{6t+6} + Q_{6t+6}\sqrt{p}, \quad t = 0, 1, 2, \ldots .$$

Letting $t = 0, 1, 2, 3, \ldots,$ the y integer values will be $Q_2, Q_4, Q_6, \ldots,$ the successive even Q numbers.

Proof (4):

(9) $$y = Q_{2m+1}$$

$$py^2 + 4p = p(Q_{2m+1}^2 + 4) = p[\alpha^{4m+2} + 2(\alpha\beta)^{2m+1} + \beta^{4m+2} + 4]$$

$$= p[\alpha^{4m+2} - 2(\alpha\beta)^{2m+1} + \beta^{4m+2}]$$

$$= p(\alpha^{2m+1} - \beta^{2m+1})^2$$

$$= p^2P_{2m+1}^2, \text{ an integer square.}$$

(10) $$y = Q_{2m}$$

$$pn^2 - 4p = p[(\alpha^{2m} + \beta^{2m})^2 - 4]$$

$$= p(\alpha^{2m} - \beta^{2m})^2$$

$$= p^2P_{2m}^2, \text{ an integer square.}$$

*****