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DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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ENUMERATION OF TWO-LINE ARRAYS

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1. We consider the enumeration of two-line arrays of positive integers

$$\begin{array}{|cccc|} \hline a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \hline \end{array} \tag{1.1}$$

subject to certain conditions. We assume first that

$$\max(a_i, b_i) \le \min(a_{i+1}, b_{i+1}) \qquad (1 \le i < n) \tag{1.2}$$

and

$$\max(a_i, b_i) \le i \qquad (1 \le i \le n). \tag{1.3}$$

Let $f(n,k)$ denote the number of arrays (1.1) satisfying (1.2) and (1.3) and in addition

$$a_n = b_n = k; \tag{1.4}$$

let $g(n,k)$ denote the number of arrays (1.1) satisfying (1.2) and (1.3) and

$$\max(a_n, b_n) = k. \tag{1.5}$$

Also put

$$f(n) = f(n,n), \qquad g(n) = g(n,n). \tag{1.6}$$

Next let $h(n,k)$ denote the number of arrays (1.1) that satisfy the conditions

$$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n = k \tag{1.7}$$

and

$$a_i \le i \qquad (1 \le i \le n). \tag{1.8}$$

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Also put

$$h(n) = \sum_{k=1}^{n} h(n,k) \ . \tag{1.9}$$

We shall determine the enumerants f, g, h explicitly. In particular, we show that

$$f(n+1) = \frac{1}{n}\sum_{t=1}^{n}\binom{n}{t}\binom{2n+t}{t-1} \ , \tag{1.10}$$

$$h(n) = \frac{1}{n}\binom{3n}{n-1} \ . \tag{1.11}$$

Note that $f(n+1)$ is the total number of arrays satisfying (1.2) and (1.3), while $h(n)$ is the total number of arrays satisfying (1.8) and

$$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n \ . \tag{1.7$'$}$$

The conditions (1.2), (1.3) are suggested by one formulation of the ballot problem (for references see [2]). On the other hand, (1.2) has also occurred in a problem in multipartite partitions [1], [4].

2. To begin with, we consider the functions $f(n,k)$, $g(n,k)$. We state some preliminary results.

$$f(n+1,k) = \sum_{j=1}^{k} (2k-2j+1)f(n,j) \qquad (k \le n) \ , \tag{2.1}$$

$$f(n+1,k) = \sum_{j=1}^{k} g(n,j) \qquad (k \le n+1) \ , \tag{2.2}$$

$$g(n+1,k) = \sum_{j=1}^{k} (2k-2j+1)g(n,j) \qquad (k \le n) \ , \tag{2.3}$$

$$g(n,k) = f(n,k) + 2\sum_{j=1}^{k-1} f(n,j) \qquad (k \le n+1) \ , \tag{2.4}$$

$$g(k+1,\ k) = \sum_{j=1}^{k} g(j,j)g(k-j+1,\ k-j+1)\ . \tag{2.5}$$

To prove (2.1), consider the array

$$\boxed{\begin{array}{ccccc} a_1 & a_2 & \cdots & a_n & k \\ b_1 & b_2 & \cdots & b_n & k \end{array}}$$

where

$$\max(a_i,\ b_i) \le \min(a_{i+1},\ b_{i+1}) \qquad (1 \le i \le n)\ ,$$
$$\max(a_n,\ b_n) \le k\ ,$$
$$\max(a_i,\ b_i) \le i \qquad (1 \le i \le n)\ .$$

Let $j = \min(a_n,\ b_n)$. For fixed $j \le k$, we can pick a_n, b_n in $2k - 2j + 1$ ways. This evidently implies (2.1).

Equation (2.2) is an immediate consequence of the definitions. The proof of (2.3) is similar to the proof of (2.1). We consider the array

$$\boxed{\begin{array}{ccccc} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ b_1 & b_2 & \cdots & b_n & b_{n+1} \end{array}}$$

where now

$$\max(a_n,\ b_n) = j, \qquad \max(a_{n+1},\ b_{n+1}) = k\ .$$

For fixed j,k, we can pick a_{n+1}, b_{n+1} in $2k - 2j + 1$ ways. This yields (2.3).

As for (2.4), it is only necessary to observe that corresponding to the array

$$\boxed{\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array}}$$

where $\max(a_n,\ b_n) = k$, we have the set of arrays

$$\boxed{\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{n-1} & j \\ b_1 & b_2 & \cdots & b_{n-1} & j \end{array}}$$

where $j = \min(a_n,\ b_n)$.

To prove (2.5), consider

(2.6)
$$\begin{array}{|ccc|ccc|}\hline 1 & \cdots & j & j & \cdots & k \\ 1 & \cdots & \cdot & j & \cdots & \cdot \\ \hline \end{array}$$
$$\underbrace{\qquad\qquad}_{j} \qquad \underbrace{\qquad\qquad}_{k-j+1}$$

Since

$$\max(a_1, b_1) = 1, \qquad \max(a_{k+1}, b_{k+1}) = k,$$

there is a least j such that

$$\max(a_j, b_j) = \max(a_{j+1}, b_{j+1}).$$

Thus $a_{j+1} = b_{j+1} = j$. Subtracting $j - 1$ from each element in the right-hand sub-array of (2.6), we get (2.5).

A more general result is

$$g(n + k, k) = \sum_{j=1}^{k} g(j, j)g(n + k - j, k - j + 1) \qquad (n \geq 1). \tag{2.7}$$

To prove (2.7), we consider the array

$$\begin{array}{|ccc|ccc|}\hline 1 & \cdots & j & j & \cdots & k \\ 1 & \cdots & \cdot & j & \cdots & \cdot \\ \hline \end{array}$$
$$\underbrace{\qquad\qquad}_{j} \qquad \underbrace{\qquad\qquad}_{n+k-j}$$

and pick j as in the proof of (2.5).

Next we have

$$f(k + 1, k) = \sum_{j=1}^{k} g(j, j)f(k - j + 1, k - j + 1) \tag{2.8}$$

and

$$f(n + k, k) = \sum_{j=1}^{k} g(j, j)f(n + k - j, k - j + 1) \qquad (n \geq 1). \tag{2.9}$$

The proof of these formulas is similar to the proof of (2.5) and (2.7).

3. Put

$$f(n) = f(n,n) = f(n, n-1), \qquad g(n) = g(n,n), \tag{3.1}$$

$$F(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f(n,k)x^n y^k, \tag{3.2}$$

$$G(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} g(n,k)x^n y^k, \tag{3.3}$$

$$F(x) = \sum_{n=1}^{\infty} f(n)\, x^n, \tag{3.4}$$

$$G(x) = \sum_{n=1}^{\infty} g(n)\, x^n. \tag{3.5}$$

We rewrite (2.8) in the form

$$f(k+1) = \sum_{j=1}^{k} g(j)f(k-j+1). \tag{3.6}$$

Then by (3.4),

$$\begin{aligned} F(x) &= x + \sum_{k=1}^{\infty} f(k+1)x^{k+1} \\ &= x + \sum_{k=1}^{\infty} x^{k+1} \sum_{j=1}^{k} g(j)f(k-j+1) \\ &= x + \sum_{j=1}^{\infty} g(j)x^j \sum_{k=1}^{\infty} f(k)x^k, \end{aligned}$$

so that

$$F(x) = x + F(x)G(x). \tag{3.7}$$

Next, by (2.7),

$$\sum_{k=1}^{\infty} g(n+k,k)x^{n+k} = \sum_{k=1}^{\infty} x^{n+k} \sum_{j=1}^{\infty} g(j,j)g(n+k-j,k-j+1) = \sum_{j=1}^{\infty} g(j,j)x^j \sum_{k=1}^{\infty} g(n+k-1,k)x^{n+k-1}.$$

Hence, if we put

$$G_n(x) = \sum_{k=1}^{\infty} g(n+k-1,k)\, x^{n+k-1} \qquad (n \geq 1)\,, \tag{3.8}$$

we have

$$G_{n+1}(x) = G(x)G_n(x) \qquad (n \geq 1)\,. \tag{3.9}$$

Since

$$G_1(x) = \sum_{k=1}^{\infty} g(k,k)x^k = G(x)\,,$$

it follows that

$$G_n(x) = G^n(x)\,. \tag{3.10}$$

Next consider

$$\begin{aligned} G(x,y) &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} g(n,k)x^n y^k \\ &= \sum_{j,k=1}^{\infty} g(j+k-1,\,k)\, x^{j+k-1}\, y^k \\ &= \sum_{j=1}^{\infty} y^{-j+1} \sum_{k=1}^{\infty} g(j+k-1,\,k)(xy)^{j+k-1} \\ &= \sum_{j=1}^{\infty} y^{-j+1} G^j(xy)\,, \end{aligned}$$

by (3.10). Therefore

$$G(x,y) = \frac{G(xy)}{1 - y^{-1}G(xy)}\,. \tag{3.11}$$

On the other hand, by (2.3),

$$\begin{aligned} G(x,y) &= xy + x\sum_{n=1}^{\infty}\sum_{k=1}^{n+1} g(n+1,\,k)\, x^n y^k \\ &= xy + \sum_{n=1}^{\infty} g(n+1,\,n+1)(xy)^{n+1} + x\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{j=1}^{k} (2k-2j+1)g(n,j)x^n y^k = \end{aligned}$$

$$= G(xy) + x\sum_{n=1}^{\infty}\sum_{j=1}^{n} g(n,j)x^n y^j \sum_{k=0}^{n-j}(2k+1)y^k$$

$$= G(xy) + x\sum_{n,j=1}^{\infty} g(n+j-1,\ j)x^{n+j-1} y^j \sum_{k=0}^{n-1}(2k+1)y^k$$

$$= G(xy) + x\sum_{n=1}^{\infty} y^{-n+1}\sum_{k=0}^{n-1}(2k+1)y^k \sum_{j=1}^{\infty} g(n+j-1,\ j)(xy)^{n+j-1}$$

$$= G(xy) + x\sum_{n=1}^{\infty} y^{-n+1}\sum_{k=0}^{n-1}(2k+1)y^k \cdot G^n(xy)$$

$$= G(xy) + x\sum_{n,k=0}^{\infty}(2k+1)y^{-n}\, G^{n+k+1}(xy)$$

$$= G(xy) + \frac{xG(xy)}{1 - y^{-1}G(xy)}\sum_{k=0}^{\infty}(2k+1)G^k(xy)\ .$$

Since

$$\sum_{k=0}^{\infty}(2k+1)z^k = \frac{1+z}{(1-z)^2}\ ,$$

it follows that

$$G(x,y) = G(xy) + \frac{xG(xy)}{1 - y^{-1}G(xy)}\ \frac{1 + G(xy)}{\big(1 - G(xy)\big)^2} \tag{3.12}$$

Comparing (3.12) with (3.11), we get

$$\frac{1}{1 - y^{-1}G(xy)} = 1 + \frac{x}{1 - y^{-1}G(xy)}\ \frac{1 + G(xy)}{\big(1 - G(xy)\big)^2}\ .$$

For $y = 1$, this reduces to

$$\frac{1}{1 - G(x)} = 1 + \frac{x}{1 - G(x)}\ \frac{1 + G(x)}{\big(1 - G(x)\big)^2}$$

Therefore

$$G(x)\big(1 - G(x)\big)^2 = x\big(1 + G(x)\big)\ . \tag{3.13}$$

Now consider the equation

$$z(1 - z)^2 = x(1 + z)\ , \tag{3.14}$$

where $z = 0$ when $x = 0$. By [3, p. 125], the equation

$$w = \frac{z}{\phi(z)} \qquad (\phi(0) = 1), \tag{3.15}$$

where $\phi(z)$ is analytic in the neighborhood of $z = 0$, has the solution

$$z = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}\phi^n(x)}{dx^{n-1}}\right]_{x=0} . \tag{3.16}$$

Later we shall require the more general result:

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} f'(x)\phi^n(x)\right]_{x=0} . \tag{3.17}$$

If we take

$$\phi(x) = (1 + x)(1 - x)^{-2},$$

then

$$\phi^n(x) = \sum_{s=0}^{\infty} \binom{n}{s} x^s \sum_{t=0}^{\infty} \binom{2n + t - 1}{t} x^t$$

$$= \sum_{m=0}^{\infty} x^m \sum_{s+t=m} \binom{n}{s}\binom{2n + t - 1}{t} ,$$

so that

$$\left[\frac{d^{n-1}}{dx^{n-1}} \phi^n(x)\right]_{x=0} = (n - 1)! \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t - 1}{t} .$$

Therefore, by (3.13) and (3.16), we get

$$G(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t - 1}{t} . \tag{3.18}$$

Thus

$$g(n) = \frac{1}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1}\binom{2n + t - 1}{t} . \tag{3.19}$$

4. In the next place, by (3.7),

(4.1) $$F(x) = \frac{x}{1 - G(x)} .$$

Then, making use of (3.17),

(4.2) $$\frac{F(x)}{x} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} \frac{(1 + x)^n}{(1 - x)^{2n+2}} \right]_{x=0} .$$

It is easily verified that

$$\left[\frac{d^{n-1}}{dx^{n-1}} \frac{(1 + x)^n}{(1 - x)^{2n+2}} \right]_{x=0} = (n - 1)! \sum_{t=0}^{n-1} \binom{n}{t + 1} \binom{2n + t + 1}{t} ,$$

so that (4.2) yields

(4.3) $$F(x) = x + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1} \binom{2n + t + 1}{t} .$$

Hence

(4.4) $$f(n + 1) = \frac{1}{n} \sum_{t=0}^{n-1} \binom{n}{t + 1} \binom{2n + t + 1}{t} .$$

To determine $g(n,k)$ we use (3.10), that is

(4.5) $$\sum_{k=1}^{\infty} g(j + k - 1, k) x^{j+k-1} = G^j(x) .$$

Taking $f(z) = z^j$ in (3.17), we get

$$G^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} \left(x^{j-1} \frac{(1 + x)^n}{(1 - x)^{2n}} \right) \right]_{x=0} .$$

Since

$$\frac{d^{n-1}}{dx^{n-1}} \left(x^{j-1} \frac{(1 + x)^n}{(1 - x)^{2n}} \right) = (n - 1)! \sum_{t=0}^{n-j} \binom{n}{j + t} \binom{2n + t - 1}{t} ,$$

it follows that

(4.6) $$G^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n} \sum_{t=0}^{n-j} \binom{n}{j + t} \binom{2n + t - 1}{t} .$$

Hence, by (4.5),

$$g(n, n - j + 1) = \frac{j}{n} \sum_{t=0}^{n-j} \binom{n}{j+t} \binom{2n+t-1}{t} . \tag{4.7}$$

Next if we put

$$F_n(x) = \sum_{k=1}^{\infty} f(n + k - 1, k)x^{n+k-1} \qquad (n \geq 1) , \tag{4.8}$$

it follows from (2.9) that

$$\begin{aligned} F_{n+1}(x) &= \sum_{k=1}^{\infty} f(n + k, k)x^{n+k} \\ &= \sum_{k=1}^{\infty} x^{n+k} \sum_{j=1}^{k} g(j)f(n + k - j, k - j + 1) \\ &= \sum_{j=1}^{\infty} g(j)x^j \sum_{k=1}^{\infty} f(n + k - 1, k)x^{n+k-1} , \end{aligned}$$

so that

$$F_{n+1}(x) = F_n(x)G(x) \qquad (n \geq 1) . \tag{4.9}$$

Hence, by (4.9) and (4.1),

$$F_{n+1}(x) = F(x)G^n(x) = \frac{xG^n(x)}{1 - G(x)} \qquad (n \geq 0) . \tag{4.10}$$

We now apply (3.17) with

$$f(x) = \frac{x^j}{1 - x} .$$

Since

$$f'(x) = \frac{jx^{j-1}}{1 - x} + \frac{x^j}{(1 - x)^2} ,$$

we get

$$\frac{1}{x} F_{j+1}(x) = f(0) + \sum_{n=j}^{\infty} \frac{x^n}{n} \left\{ j \sum_{t=0}^{n-j} \binom{n}{j+t} \binom{2n+t}{t} + \sum_{t=0}^{n-j-1} \binom{n}{j+t+1} \binom{2n+t+1}{t} \right\} .$$

It follows that

$$f(n + 1, n - j + 1) = \frac{1}{n} \left\{ j \sum_{t=0}^{n-j} \binom{n}{j+t} \binom{2n+t}{t} + \sum_{t=0}^{n-j-1} \binom{n}{j+t+1} \binom{2n+t+1}{t} \right\} ,$$

or if we prefer

$$f(n+1,\ k+1) = \frac{1}{n}\left\{(n-k)\sum_{t=0}^{k}\binom{n}{k-t}\binom{2n+t}{t} + \sum_{t=1}^{k}\binom{n}{k-t}\binom{2n+t}{t-1}\right\}. \tag{4.11}$$

Note that, when $n = k$, (4.11) becomes

$$f(k+1) = f(k+1,\ k+1) = \frac{1}{k}\sum_{t=1}^{k}\binom{k}{t}\binom{2k+t}{t-1},$$

in agreement with (4.4).

5. The total number of arrays

$$\begin{array}{|cccc|}\hline a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \hline\end{array} \tag{5.1}$$

such that

$$\begin{cases} \max(a_i,\ b_i) \le \min(a_{i+1},\ b_{i+1}) & (1 \le i < n) \\ \max(a_i,\ b_i) \le i & (1 \le i \le n), \end{cases} \tag{5.2}$$

is equal to

$$\sum_{j=1}^{n} g(n,j) = f(n+1,\ n) = f(n+1) = \frac{1}{n}\sum_{t=0}^{n-1}\binom{n}{t+1}\binom{2n+t+1}{t}. \tag{5.3}$$

Similarly the total number of arrays (5.1) satisfying (5.2) and $a_n = b_n$ is equal to

$$\sum_{k=1}^{n} f(n,k) = \frac{1}{2}\big(g(n,n) + f(n,n)\big). \tag{5.4}$$

The numbers $f(n,k)$, $g(n,k)$ can be computed by means of the recurrences (2.1), (2.3). Checks are furnished by (2.2) and (2.4).

$f(n,k)$:

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	4				
4	1	7	21	21			
5	1	10	47	126	126		
6	1	13	82	324	818	818	
7	1	16	126	642	2300	5594	5594

$g(n,k)$:

n \ k	1	2	3	4	5	6
1	1					
2	1	3				
3	1	6	14			
4	1	9	37	79		
5	1	12	69	242	494	
6	1	15	110	516	1658	3294

6. We turn now to $h(n,k)$, the number of arrays

$$\begin{array}{|cccc|} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array}$$

such that

(6.1) $$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n = k$$

and

(6.2) $$a_i \le i \qquad (1 \le i \le n).$$

It is clear from the scheme

$$\begin{array}{|ccccc|} a_1 & a_2 & \cdots & a_{n-1} & k \\ b_1 & b_2 & \cdots & b_{n-1} & j \end{array}$$

that

$$h(n,k) = \sum_{j=1}^{k} \sum_{s=1}^{j} h(n-1, s).$$

This yields the recurrence

(6.3) $$h(n,k) = \sum_{s=1}^{k} (k - s + 1)h(n-1, s) \qquad (1 \le k \le n).$$

When $k = n$, it is understood that (6.3) becomes

(6.4) $$h(n,n) = \sum_{s=1}^{n-1} (n - s + 1)h(n-1, s).$$

It is clear from the definition that

(6.5) $$h(n) = \sum_{k=1}^{n} h(n,k)$$

is equal to the total number of arrays that satisfy (6.2) and

$$1 = b_1 = a_1 \le b_2 \le a_2 \le \cdots \le b_n \le a_n \tag{6.6}$$

The first few values of $h(n,k)$ can be computed by means of (6.2):

$h(n,k)$:

n \ k	1	2	3	4	5	h(n)
1	1					1
2	1	2				3
3	1	4	7			12
4	1	6	18	30		55
5	1	8	33	88	143	273

The numbers in the right-hand column are obtained by summing in the rows. Thus the entries are $h(n)$ as defined by (6.5).

We shall now show that

$$h(k+1, k) = \sum_{j=1}^{k} h(j,j)h(k-j+1, k-j+1) . \tag{6.7}$$

<u>Proof.</u> Consider the scheme

$$\underbrace{\begin{array}{ccc} 1 & \cdots & j \\ 1 & \cdots & \cdot \end{array}}_{j} \;\Big|\; \underbrace{\begin{array}{ccc} j & \cdots & k \\ j & \cdots & \cdot \end{array}}_{k-j+1}$$

We choose j as the least positive integer such that

$$a_{j+1} = b_{j+1} = a_j .$$

Such an integer exists because $a_{k+1} = k$. Subtracting $j - 1$ from each element in the right-hand sub-array, we get (6.7).

Next we have

$$h(n+k,k) = \sum_{j=1}^{k} h(j,j)h(n+k-j, k-j+1) \qquad (n \ge 1) . \tag{6.8}$$

To prove (6.8), consider the scheme

$$\underbrace{\begin{array}{ccc} 1 & \cdots & j \\ 1 & \cdots & \cdot \end{array}}_{j} \;\Big|\; \underbrace{\begin{array}{ccc} j & \cdots & k \\ j & \cdots & \cdot \end{array}}_{n+k-j}$$

We choose j as above. The rest of the proof is the same.

Now put

$$H(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^n y^k , \tag{6.9}$$

$$H(x) = \sum_{k=1}^{\infty} h(k,k)x^k = H_1(x) , \tag{6.10}$$

$$H_n(x) = \sum_{k=1}^{\infty} h(n+k-1, k)\, x^{n+k-1} \qquad (n \geq 1) . \tag{6.11}$$

Then, by (6.8),

$$\begin{aligned} H_{n+1}(x) &= \sum_{k=1}^{\infty} h(n+k, k)x^{n+k} \\ &= \sum_{k=1}^{\infty} x^{n+k} \sum_{j=1}^{k} h(j,j)h(n+k-j, k-j+1) \\ &= \sum_{j=1}^{\infty} h(j,j)x^j \sum_{k=1}^{\infty} h(n+k-1, k)x^{n+k-1} , \end{aligned}$$

so that

$$H_{n+1}(x) = H(x)H_n(x) \qquad (n \geq 1) . \tag{6.12}$$

Therefore

$$H_n(x) = H^n(x) \qquad (n \geq 1) . \tag{6.13}$$

In the next place

$$\begin{aligned} H(x,y) &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} h(n,k)x^n y^k \\ &= \sum_{j,k=1}^{\infty} h(j+k-1, k)x^{j+k-1} y^k \\ &= \sum_{j=1}^{\infty} y^{-j+1} \sum_{k=1}^{\infty} h(j+k-1, k)(xy)^{j+k-1} \\ &= \sum_{j=1}^{\infty} y^{-j+1} H_j(xy) . \end{aligned}$$

Thus, by (6.13),

$$H(x,y) = \frac{H(xy)}{1 - y^{-1}H(xy)} \quad . \tag{6.14}$$

On the other hand, by (6.3),

$$\begin{aligned}
H(x,y) &= xy + x\sum_{n=1}^{\infty}\sum_{k=1}^{n+1} h(n+1,\,k)x^n y^k \\
&= xy + \sum_{n=1}^{\infty} h(n+1,\,n+1)(xy)^{n+1} + x\sum_{n=1}^{\infty}\sum_{k=1}^{n} h(n+1,\,k)x^n y^k \\
&= H(xy) + x\sum_{n=1}^{\infty}\sum_{k=1}^{n} x^n y^k \sum_{j=1}^{k} (k-j+1)h(n,j) \\
&= H(xy) + x\sum_{n=1}^{\infty}\sum_{j=1}^{n} h(n,j)x^n y^j \sum_{k=0}^{n-j} (k+1)\, y^k \\
&= H(xy) + x\sum_{n,j=1}^{\infty} h(n+j-1,\,j)x^{n+j-1} y^j \sum_{k=0}^{n-1} (k+1)y^k \\
&= H(xy) + x\sum_{n=1}^{\infty} y^{-n+1} \sum_{k=0}^{n-1} (k+1)\, y^k \sum_{j=1}^{\infty} h(n+j-1,\,j)(xy)^{n+j-1} \\
&= H(xy) + x\sum_{n=1}^{\infty} y^{-n+1} \sum_{k=0}^{n-1} (k+1)\, y^k H^n(xy) \\
&= H(xy) + x\sum_{n,k=0}^{\infty} (k+1)y^{-n} H^{n+k+1}(xy) \\
&= H(xy) + xH(xy)\sum_{k=0}^{\infty} (k+1)H^k(xy) \sum_{n=0}^{\infty} y^{-n} H^n(xy)
\end{aligned}$$

and therefore

$$H(x,y) = H(xy) + \frac{xH(xy)}{\left(1 - y^{-1}H(xy)\right)\left(1 - H(xy)\right)^2} \quad . \tag{6.15}$$

We now compare (6.15) with (6.14) and take $y = 1$. This yields

$$\frac{1}{1 - H(x)} = 1 + \frac{x}{\left(1 - H(x)\right)^3}$$

which reduces to

$$H(x)\left(1 - H(x)\right)^2 = x \ . \tag{6.16}$$

Applying (3.16), we get

$$H(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dt^{n-1}} (1 - t)^{-2n}\right]_{t=0} .$$

Since

$$\left[\frac{d^{n-1}}{dt^{n-1}} (1 - t)^{-2n}\right]_{t=0} = (n - 1)! \binom{3n - 2}{n - 1} ,$$

we have

$$H(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \binom{3n - 2}{n - 1} . \tag{6.17}$$

Applying (3.17), we get

$$H^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n!} \left[\frac{d^{n-1}}{dt^{n-1}} \left(t^{j-1}(1 - t)^{-2n}\right)\right]_{t=0} \qquad (j \geq 1) .$$

This reduces to

$$H^j(x) = j \sum_{n=j}^{\infty} \frac{x^n}{n} \binom{3n - j - 1}{n - j} \qquad (j \geq 1) . \tag{6.18}$$

It follows from (6.11), (6.13) and (6.18) that

$$h(n, n - j + 1) = \frac{j}{n} \binom{3n - j - 1}{n - j} \qquad (1 \leq j \leq n) . \tag{6.19}$$

In particular, for $j = 1$, we have

$$h(n,n) = \frac{1}{n} \binom{3n - 2}{n - 1} . \tag{6.20}$$

We shall now compute

$$h(n) = \sum_{k=1}^{n} h(n,k) = \sum_{j=1}^{n} h(n, n - j + 1) . \tag{6.21}$$

By (6.19) and (6.21),

$$h(n) = \sum_{j=1}^{n} \frac{j}{n}\binom{3n-j-1}{n-j} = \frac{1}{n}\sum_{j=0}^{n-1}(n-j)\binom{2n+j-1}{j}$$

$$= \sum_{j=0}^{n-1}\binom{2n+j-1}{j} - \frac{1}{n}\sum_{j=1}^{n-1} j\binom{2n+j-1}{j}$$

$$= \sum_{j=0}^{n-1}\binom{2n+j-1}{j} - 2\sum_{j=1}^{n-1}\binom{2n+j-1}{j-1}$$

$$= \sum_{j=0}^{n-1}\binom{2n+j-1}{2n-1} - 2\sum_{j=0}^{n-2}\binom{2n+j}{2n}$$

$$= \binom{3n-1}{2n} - \binom{3n-1}{2n+1} .$$

This reduces to

$$h(n) = \frac{1}{n}\binom{3n}{n-1} . \tag{6.22}$$

7. By (5.3),

$$f(n+1) = \frac{1}{n}\sum_{t=1}^{n}\binom{n}{t}\binom{2n+t}{t-1} \tag{7.1}$$

enumerates the number of arrays that satisfy (5.1) and (5.2).

Consider the quantity

$$U(n,t) = \frac{1}{n}\binom{n}{t}\binom{2n+t}{t-1} \qquad (1 \le t \le n) . \tag{7.2}$$

Clearly $nU(n,t)$ is an integer Moreover

$$U(n,t) = \frac{1}{t}\binom{n-1}{t-1}\binom{2n+t}{t-1} = \frac{1}{2n+1}\binom{n-1}{t-1}\binom{2n+1}{t} ,$$

so that $(2n+1)U(n,t)$ is also an integer. Since both $nU(n,t)$ and $(2n+1)U(n,t)$ are integers, it follows that $U(n,t)$ is itself an integer. The question then arises whether $U(n,t)$ can be given a simple combinatorial interpretation. In the special case $t = n$, we have, by (6.22)

$$U(n,n) = h(n) ; \tag{7.3}$$

however the general case remains open.

A curious relation between $G(x)$ and $H(x)$ is implied by (3.13) and (6.16):

$$G(x)\big(1 - G(x)\big)^2 = x\big(1 + G(x)\big) , \tag{7.4}$$

(7.5) $$H(x)\big(1 - H(x)\big)^2 = x .$$

Since the equation

(7.6) $$z(1 - z)^2 = u, \qquad z(0) = 0$$

has the unique solution $z = H(u)$, it follows from (7.4) and (7.5) that

(7.7) $$H\Big(x\big(1 + G(x)\big)\Big) = G(x) .$$

By (3.8), (3.10), (6.11) and (6.13), we have

$$\begin{aligned} H\Big(x\big(1 + G(x)\big)\Big) &= \sum_{k=1}^{\infty} h(k,k)x^k \big(1 + G(x)\big)^k \\ &= \sum_{k=1}^{\infty} h(k,k)x^k \sum_{j=0}^{k} \binom{k}{j} G^j(x) \\ &= H(x) + \sum_{k=1}^{\infty} h(k,k)x^k \sum_{j=1}^{k} \binom{k}{j} \sum_{s=1} g(j + s - 1, s)x^{j+s-1} \\ &= H(x) + \sum_{n=2}^{\infty} x^n \sum_{j+k+s-1=n} \binom{k}{j} h(k,k)g(j + s - 1, s) . \end{aligned}$$

Thus (7.7) yields

(7.8) $$g(n) = h(n,n) + \sum_{j+k\le n} \binom{k}{j} h(k,k)g(n - k, n - j - k + 1) .$$

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UNIT DETERMINANTS IN GENERALIZED PASCAL TRIANGLES

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There are many ways in which one can select a square array from Pascal's triangle which will have a determinant of value one. What is surprising is that two classes of generalized Pascal triangles which arose in [1] also have this property: the multinomial coefficient triangles and the convolution triangles formed from sequences which are found as the sums of elements appearing on rising diagonals within the binomial and multinomial coefficient triangles. The generalized Pascal triangles also share sequences of $k \times k$ determinants whose values are successive binomial coefficients in the kth column of Pascal's triangle.

1. UNIT DETERMINANTS WITHIN PASCAL'S TRIANGLE

When Pascal's triangle is imbedded in matrices throughout this paper, we will number the rows and columns in the usual matrix notation, with the leftmost column the first column. If we refer to Pascal's triangle itself, then the leftmost column is the zeroth column, and the top row is the zeroth row.

First we write Pascal's triangle in rectangular form as the $n \times n$ matrix $P = (p_{ij})$.

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 1 & 3 & 6 & 10 & 15 & 21 & \cdots \\ 1 & 4 & 10 & 20 & 35 & 56 & \cdots \\ 1 & 5 & 15 & 35 & 70 & 126 & \cdots \\ 1 & 6 & 21 & 56 & 126 & 252 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n}, \tag{1.1}$$

where the element p_{ij} in the ith row and j^{th} column can be obtained by

$$p_{ij} = p_{i-1,j} + p_{i,j-1} = \binom{i+j-2}{i-1},$$

and the generating function for the elements appearing in the jth column is $1/(1-x)^j$, $j = 1, 2, \cdots, n$.

Pascal's triangle written in left-justified form can be imbedded in the $n \times n$ matrix $A = (a_{ij})$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}_{n \times n} \tag{1.2}$$

where

$$a_{ij} = \binom{i-1}{j-1},$$

and the column generators are

$$x^{j-1}/(1-x)^j .$$

The sums of the elements found on the rising diagonals of A, found by beginning at the left-most column and moving right one and up one throughout the array, are 1, 1, 2, 3, 5, 8, 13, $\cdots$, $F_{n+2} = F_{n+1} + F_n$, the Fibonacci numbers.

<u>Theorem 1.1.</u> Any $k \times k$ submatrix of A which contains the column of ones and has for its first row the ith row of A has a determinant of value one.

This is easily proved, for if the preceding row is subtracted from each row successively for $i = k, k-1, \cdots, 2$, and then for $i = k, k-1, \cdots, 3, \cdots$, and finally for $i = k$, the matrix obtained equivalent to the given submatrix has ones on its main diagonal and zeroes below. For example, for $k = 4$ and $i = 4$,

$$\begin{vmatrix} 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \\ 1 & 6 & 15 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 1 & 5 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

which is an interesting process in itself, since each row becomes the same as the first row except moved successively one space right.

Let the $n \times n$ matrix A^T be the transpose of A, so that Pascal's triangle appears on and above the main diagonal with its rows in vertical position. Then $AA^T = P$, the rectangular Pascal's matrix of (1.1). That is, for $n = 4$,

$$AA^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = P .$$

As proof, the generating functions for the columns of A are

$$\frac{1}{1-x} \cdot \left(\frac{x}{1-x}\right)^{j-1} ,$$

and for A^T, $(1+x)^{j-1}$, $j = 1, 2, \cdots, n$, so that the generating functions for the columns of AA^T are

$$\frac{1}{1-x}\cdot\left(1+\frac{x}{1-x}\right)^{j-1} = \left(\frac{1}{1-x}\right)^{j}, \qquad j = 1, 2, \cdots, n,$$

which we recognize as the generating functions for the columns of P. Notice that $\det A = \det A^T = 1$, so that $\det AA^T = \det P = 1$ for any n. That is,

Theorem 1.2. The $k \times k$ submatrix, formed from Pascal's triangle written in rectangular form to contain the upper left-hand corner element of P, has a determinant value of one.

In P, if the $(n-1)^{st}$ column is subtracted from the nth column, the $(n-2)^{nd}$ from the $(n-1)^{st}$, $\cdots$, the 2nd from the 3rd, and the first from the 2nd, a new array is formed:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n\times n}$$

In the new array, the determinant is still one but it equals the determinant of the $(n-1)\times(n-1)$ array formed by deleting the first row and first column, which array can be found within the original array P by moving one column right and using the original row of ones as the top row. If we again subtract the $(k-1)^{st}$ column from the kth column successively for $k = n-1, n-2, \cdots, 3, 2$, the new array has the determinant value one and can be found as the $(n-2)\times(n-2)$ array using the original row of ones as its top row and two columns right in P. By the law of formation of Pascal's triangle, we can subtract thusly, k times beginning with an $(n+k)\times(n+k)$ matrix P to show that the determinant of any $n\times n$ array within P containing a row of ones has determinant equivalent to $\det P$ and thus has value one. By the symmetry of Pascal's triangle and P, any $n\times n$ square array taken within P to include a column of ones as its edge also has a unit determinant.

We have proved

Theorem 1.3. The determinant of any $n\times n$ array taken with its first row along the row of ones, or with its first column along the column of ones in Pascal's triangle written in rectangular form, is one.

Returning to the matrices A and A^T, AA^T gave us Pascal's triangle in rectangular form. Consider A^mA^T

$$A^2A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 2 & 3 & 4 & 5 & 6 & \cdots \\ 4 & 8 & 13 & 19 & 26 & \cdots \\ 8 & 20 & 38 & 63 & 96 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n},$$

where the first column is $1, 2, 4, 8, \cdots, 2^k, \cdots,$ with column generators

$$\frac{(1 - x)^{j-1}}{(1 - 2x)^j},$$

$j = 1, 2, \cdots, n$. Notice that the sums of elements appearing on the rising diagonals are $1, 3, 8, 21, 55, \cdots,$ the Fibonacci numbers with even subscripts.

In general, A^mA^T has first column $1, m, m^2, \cdots, m^k, \cdots,$ with column generators

$$\frac{[1 - (m - 1)x]^{j-1}}{(1 - mx)^j}, \quad j = 1, 2, \cdots, n.$$

Notice that any $k \times k$ submatrix of A^mA^T containing the first row (which is a row of ones) has a unit determinant, since that submatrix is the product of submatrices with unit determinants from A and $A^{m-1}A^T$, where $A^1A^T = P$.

In general, if we consider determinants whose rows are composed of successive members of an arithmetic progression of the rth order, we can predict the determinant value as well as give a second proof of the foregoing unit determinant properties of Pascal's triangle. The following theorem is due to Howard Eves [2].

Let us define an <u>arithmetic progression of the rth order</u>, denoted by $(AP)_r$, to be a sequence of numbers whose r^{th} row of differences is a row of constants, but whose $(r - 1)^{th}$ row is not. For example, the third row of Pascal's triangle in rectangular form is an $(AP)_3$, since

$$\begin{array}{ccccccccccccc} 1 & & 4 & & 10 & & 20 & & 35 & & 56 & & \cdots \\ & 3 & & 6 & & 10 & & 15 & & 21 & & & \cdots \\ & & 3 & & 4 & & 5 & & 6 & & & & \cdots \\ & & & 1 & & 1 & & 1 & & & & & \cdots \end{array}$$

A row of repeated constants will be called an $(AP)_0$. The constant in the rth row of differences of an $(AP)_r$ will be called the constant of the progression.

<u>Theorem 1.4.</u> (Eves' Theorem). Consider a determinant of order n whose i^{th} row $(i = 1, 2, \cdots, n)$ is composed of any n successive terms of an $(AP)_{i-1}$ with constant a_i. Then the value of the determinant is the product $a_1a_2 \cdots a_n$.

The proof is an armchair one, since, by a sequence of operations of the type where we subtract from a column of the determinant the preceding column, one can reduce the matrix of the determinant to one having $a_1, a_2, \cdots, a_n$ alongthe main diagonal and zeros everywhere above the main diagonal.

Eves' Theorem applies to all the determinants formed from Pascal's triangle discussed in this section, with $a_1 = a_2 = \cdots = a_n = 1$, whence the value of each determinant is one. As a corollary to Eves' Theorem, notice that the nth order determinants containing a row of ones formed from Pascal's rectangular array still have value one if the kth row is shifted m spaces left, $k = 1, 2, \cdots, n$, $m \geq 0$, where m is arbitrary and can be a different value for each row!

<u>Corollary 1.4.1.</u> If the kth row of an $n \times n$ determinant contains n consecutive nonzero members of the $(k-1)^{st}$ row of Pascal's triangle written in rectangular form for each $k = 1, 2, \cdots, n$, the determinant has value one.

Since all arrays discussed in this section contain a row (or column) of ones, we also have infinitely many determinants that can be immediately written with arbitrary value c: merely add $(c-1)$ to each element of the matrix of any one of the unit determinant arrays! The proof is simple: since every element in the first row equals c, factor out c. Then subtract $(c-1)$ times the first row from each other row, returning to the original array which had a determinant value of one. Thus, the new determinant has value c. This is a special case of a wide variety of problems concerning the lambda number of a matrix [3], [4].

Eves' Theorem also applies to the arrays of convolution triangles and multinomial coefficients which follow, but the development using other methods of proof is more informative.

2. OTHER DETERMINANT VALUES FROM PASCAL'S TRIANGLE

Return again to matrix P of (1.1). Suppose that we remove the top row and left column, and then evaluate the $k \times k$ determinants containing the upper left corner. Then

$$\begin{vmatrix} 2 \end{vmatrix} = 2, \qquad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \qquad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4,$$

and the $k \times k$ determinant has value $(k+1)$.

Proof is by mathematical induction. Assume that the $(k-1) \times (k-1)$ determinant has value k. In the $k \times k$ determinant, subtract the preceding column from each column successively for $j = k, k-1, k-2, \cdots, 2$. Then subtract the preceding row from each row successively for $i = k, k-1, k-2, \cdots, 2$, leaving

$$\begin{vmatrix} 2 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 2 & 3 & 4 & \cdots \\ 0 & 3 & 6 & 10 & \cdots \\ 0 & 4 & 10 & 20 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 1 + k.$$

Returning to matrix P, take 2×2 determinants along the second and third rows:

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \qquad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \qquad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \cdots,$$

giving the values found in the second column of Pascal's left-justified triangle, for

$$\begin{vmatrix} \binom{j}{1} & \binom{j+1}{1} \\ \binom{j+1}{2} & \binom{j+2}{2} \end{vmatrix} = \binom{j+1}{2}$$

by simple algebra. Of course, 1×1 determinants along the second row of P yield the successive values found in the first column of Pascal's triangle. Taking 3×3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \qquad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \qquad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10,$$

the successive entries in the third column of Pascal's triangle. In fact, taking successive $k \times k$ determinants along the 2nd, 3rd, $\cdots$, and $(k+1)^{st}$ rows yields the successive entries of the kth column of Pascal's triangle.

To formalize our statement,

Theorem 2.1: The determinant of the $k \times k$ matrix $R(k,j)$ taken with its first column the jth column of P, the rectangular form of Pascal's triangle imbedded in a matrix, and its first row the second row of P, is the binomial coefficient

$$\binom{j-1+k}{k} .$$

To illustrate,

$$\det R(4,3) = \begin{vmatrix} 3 & 4 & 5 & 6 \\ 6 & 10 & 15 & 21 \\ 10 & 20 & 35 & 56 \\ 15 & 35 & 70 & 126 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 6 & 4 & 5 & 6 \\ 10 & 10 & 15 & 21 \\ 15 & 20 & 35 & 56 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 5 \\ 0 & 6 & 10 & 15 \\ 0 & 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 10 & 15 \\ 5 & 10 & 20 & 35 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 & 5 \\ 3 & 6 & 10 & 15 \\ 4 & 10 & 20 & 35 \\ 5 & 15 & 35 & 70 \end{vmatrix}$$

$$= \det R(3,3) + \det R(4,2)$$

$$= \binom{5}{3} + \binom{5}{4} = \binom{6}{4} = 15 .$$

First, the preceding column was subtracted from each column successively, $j = k, k-1, \cdots, 2$, and then the preceding row was subtracted from each row successively for $i = k, k-1, \cdots, 2$. Then the determinant was made the sum of two determinants, one bordering $R(3,3)$ and the other equal to $R(4,2)$ by adding the jth column to the $(j+1)^{st}$, $j = 1, 2, \cdots, k-1$.

By following the above procedure, we can make

$$\det R(k,j) = \det R(k-1, j) + \det R(k, j-1) .$$

We have already proved that

$$\det R(k,1) = 1 = \binom{k+0}{k}, \qquad \det R(k,2) = k+1 = \binom{k+1}{k} \text{ for all } k,$$

$$\det R(1,j) = j = \binom{j+0}{1}, \qquad \det R(2,j) = \binom{j+1}{2} \qquad \text{for all } j.$$

If

$$\det R(k-1, j) = \binom{j+k-2}{k-1} \quad \text{and} \quad \det R(k, j-1) = \binom{j+k-2}{k},$$

then

$$\det R(k,j) = \binom{j+k-2}{k-1} + \binom{j+k-2}{k} = \binom{j+k-1}{k}$$

for all k and all j by mathematical induction.

Since P is its own transpose, Theorem 2.1 is also true if the words "column" and "row" are everywhere exchanged.

Consider Pascal's triangle in the configuration of A^T, which is just Pascal's rectangular array P with the ith row moved $(i-1)$ spaces right, $i = 1, 2, 3, \cdots$. Form $k \times k$ matrices $R'(k,j)$ such that the first row of $R'(k,j)$ is the second row of A^T beginning with the jth column of A^T. Then $AR'(k, j-1) = R(k,j)$ as can be shown by considering their column generating functions, and since $\det A = 1$, $\det R'(k,j-1) = \det R(k,j)$, leading us to the following theorems.

Theorem 2.2. Let A^T be the $n \times n$ matrix containing Pascal's triangle on and above its main diagonal so that the rows of Pascal's triangle are placed vertically. Any $k \times k$ submatrix of A^T selected with its first row along the second row of A^T and its first column the jth column of A^T, has determinant value

$$\binom{k+j-2}{k} .$$

Since A is the transpose of A^T, wording Theorem 2.2 in terms of the usual Pascal triangle provides the following.

Theorem 2.3. If Pascal's triangle is written in left-justified form, any $k \times k$ matrix selected within the array with its first column the first column of Pascal's triangle and its first row the ith row has determinant value given by the binomial coefficient

$$\binom{k+i-1}{k} .$$

3. MULTINOMIAL COEFFICIENT ARRAYS

The trinomial coefficients arising in the expansions of $(1 + x + x^2)^n$, written in left-justified form, are

$$\begin{array}{lcccccccccccccc}
 & 1 & & & & & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & & \\
 & 1 & 2 & 3 & 2 & 1 & & & & & & & & \\
(3.1) & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & & & & & \\
 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 & & & & \\
 & 1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

where the entry t_{ij} in the ith row and j^{th} column is obtained by the relationship

$$t_{ij} = t_{i-1,j} + t_{i-1,j-1} + t_{i-1,j-2} .$$

The sums of elements on rising diagonals formed by beginning in the left-most column and counting up one and right one are the Tribonacci numbers $1, 1, 2, 4, 7, 13, \cdots, T_{n+3} = T_{n+2} + T_{n+1} + T_n$. (As in Pascal's triangle, the left-most column is the zeroth column and the top row the zeroth row.)

If the summands for the Fibonacci numbers from the rising diagonals of Pascal's triangle (1.2) in left-justified form are used in reverse order as the rows to form an $n \times n$ matrix

$$F_1 = (f_{ij}), \qquad f_{ij} = \binom{j-1}{i-j},$$

which has the rows of Pascal's triangle written in vertical position on and below the main diagonal, and the matrix A^T is written with Pascal's triangle on and above the main diagonal, then the matrix product F_1A^T is the trinomial coefficient array (3.1) but written vertically rather than in the horizontal arrangement just given. To illustrate, when $n = 7$,

$$F_1A^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 \\ 0 & 0 & 0 & 0 & 1 & 5 & 15 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 6 & 10 & 15 & 21 \\ 0 & 0 & 2 & 7 & 16 & 30 & 50 \\ 0 & 0 & 1 & 6 & 19 & 45 & 90 \\ 0 & 0 & 0 & 3 & 16 & 51 & 126 \\ 0 & 0 & 0 & 1 & 10 & 45 & 141 \end{bmatrix} .$$

Generating functions give an easy proof. The generating function for the jth column of F_1 is $[x(1+x)]^{j-1}$, $j = 1, 2, \cdots$. The generating function for the jth column of A^T is $(1+x)^{j-1}$, $j = 1, 2, \cdots$, so the generating function for the jth column of F_1P is $[1 + x(1+x)]^{j-1}$ or $(1 + x + x^2)^{j-1}$, so that F_1A^T has the rows of the trinomial triangle as its columns. Since $\det F_1 = \det A^T = 1$, $\det F_1A^T = 1$, and $n \times n$ determinants containing the upper left corner of the trinomial triangle have value one. Further, any $k \times k$ submatrix of F_1A^T containing a row of ones has a unit determinant, for any $k \times k$ submatrix of F_1A^T selected with a row of ones is always the product of a submatrix of F_1 and a submatrix

of A^T containing a row of ones, both of which here have unit determinants. Lastly, any $k \times k$ submatrix of F_1A^T having its first row along the second row of F_1A^T and its first column the jth column of F_1A^T is the product of a submatrix of F_1 with a unit determinant and a $k \times k$ submatrix of A^T satisfying Theorem 2.2, and thus has determinant value

$$\binom{k+j-2}{k} .$$

We can form matrices F_m analogous to F_1 using the rising diagonals of the triangle of multinomial coefficients arising from the expansions of $(1 + x + x^2 + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, as the rows of F_m, so that the rows of the multinomial triangle appear as the columns of F_m written on and below the main diagonal. (In each case, the row sums of F_m are $1 = r_1 = r_2 = \cdots = r_m$, $r_n = r_{n-1} + r_{n-2} + \cdots + r_{n-m}$, $n \geq m + 1$.) The matrix product F_mA^T will contain the coefficients arising from the expansions of

$$(1 + x + x^2 + \cdots + x^{m+1})^n$$

written in vertical position on and above the main diagonal. As proof, the generating functions of columns of F_m are $[x(1 + x + \cdots + x^m)]^{j-1}$, $j = 1, 2, \cdots, n$, while the generating function for the jth column of A^T is $(1 + x)^{j-1}$, $j = 1, 2, \cdots, n$, so that the generating function for the jth column of F_mA^T is

$$[1 + (x(1 + x + \cdots + x^m))]^{j-1} = [1 + x + x^2 + \cdots + x^{m+1}]^{j-1}, \quad j = 1, 2, \cdots, n .$$

If a submatrix of F_mA^T is the product of a submatrix of F_m with unit determinant and a submatrix of $F_{m-1}A^T$ taken in the corresponding position, it will thus have the same determinant as the submatrix of $F_{m-1}A^T$. Notice that this is true for submatrices taken across the first or second columns of F_mA^T. Since F_1A^T has its submatrices with the same determinant values as the submatrices of A^T taken in corresponding position, and these values are given in Theorem 2.2, we have proved the following theorems by mathematical induction if we look at the transpose of F_mA^T.

Theorem 3.1. If the multinomial coefficients arising from the expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, are written in left-justified form the determinant of the $k \times k$ matrix formed with its first column taken anywhere along the leftmost column of ones of the array is one.

Theorem 3.2. If the multinomial coefficients arising from the expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, are written in left-justified form the determinant of the $k \times k$ matrix formed with its first column the first column of the array (the column of successive whole numbers) and its first row the ith row of the multinomial coefficient array, has determinant value given by the binomial coefficient

$$\binom{k+i-1}{k} .$$

4. THE FIBONACCI CONVOLUTION ARRAY AND RELATED CONVOLUTION ARRAYS

If $\{a_i\}_{i=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ are two sequences, the convolution sequence $\{c_i\}_{i=1}^{\infty}$ is given by

$$c_1 = a_1b_1, \quad c_2 = a_2b_1 + a_1b_2, \quad c_3 = a_3b_1 + a_2b_2 + a_1b_3, \cdots,$$

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1} .$$

If $g(x)$ is the generating function for $\{a_i\}$, then $[g(x)]^{k+1}$ is the generating function for the kth convolution of $\{a_i\}$ with itself.

The Fibonacci sequence, when convoluted with itself $j - 1$ times, forms the sequence in the jth column of the matrix $C = (c_{ij})$ below [1], where the original sequence is in the leftmost column

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 5 & 9 & 14 & 20 & 27 & \cdots \\ 3 & 10 & 22 & 40 & 65 & 98 & \cdots \\ 5 & 20 & 51 & 105 & 190 & 315 & \cdots \\ 8 & 38 & 111 & 256 & 511 & 924 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n} .$$

For formation of the Fibonacci convolution array,

$$c_{ij} = c_{i-1,j} + c_{i-2,j} + c_{i,j-1} .$$

Let the $n \times n$ matrix F_1 be formed as in Section 3 with the rows of Pascal's triangle in vertical position on and below the main diagonal, and let the $n \times n$ matrix P be Pascal's rectangular array (1.1). That F_1P is the Fibonacci convolution array in rectangular form is easily proved. The generating functions for the columns of P_1 are $[x(1 + x)]^{j-1}$, while those for P are $1/(1 - x)^j$, $j = 1, 2, \cdots, n$, so that the generating functions for the columns of F_1P are $1/[1 - x(1 + x)]^j$ or $1/(1 - x - x^2)^j$, the generators of C, the Fibonacci convolution array. Since $\det F_1 = \det P = 1$, $\det F_1P = 1$, and any $n \times n$ matrix formed using the upper left corner of the Fibonacci convolution array has a unit determinant.

Further, since submatrices of C taken along either the first or second row of C are the product of submatrices of F_1 with unit determinants and similarly placed submatrices of P whose determinants are given in Theorems 1.3 and 2.1, we have the following theorem.

<u>Theorem 4.1.</u> Let the Fibonacci convolution triangle be written in rectangular form and imbedded in an $n \times n$ matrix C. Then the determinant of any $k \times k$ submatrix of C placed

to contain the row of ones is one. Also, the determinant of any $k \times k$ submatrix of C selected with its first row along the second row of C and its first column the jth column of C has determinant value given by the binomial coefficient $\binom{k+j-1}{k}$.

The generalization to convolution triangles for sequences which are found as sums of rising diagonals of multinomial coefficient triangles written in left-justified form is not difficult.

Form the matrix F_m as in Section 4 to have its rows the elements found on the rising diagonals of the left-justified multinomial coefficient triangle induced by expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$. Then $F_m P$ is the convolution triangle for the sequence of sums of the rising diagonals just described, for F_m has column generators $[x(1 + x + \cdots + x^m)]^{j-1}$ and P has column generators $1/(1 - x)^j$, making the column generators of $F_m P$ be $1/(1 - x - x^2 - \cdots - x^{m+1})^j$, $j = 1, 2, \cdots, n$, which for $j = 1$ is known to be the generating function for the sequence of sums found along the rising diagonals of the given multinomial coefficient triangle [5].

As before, since a submatrix of $F_m P$ is the product of a submatrix of F_m with unit determinant and a similarly placed submatrix of P with determinant given by Theorems 1.3 and 2.1 when we take submatrices along the first or second column of $F_m P$, we can write the following.

<u>Theorem 4.2.</u> Let the convolution triangle for the sequences of sums found along the rising diagonals of the left-justified multinomial coefficient array induced by expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$, be written in rectangular form and imbedded in an $n \times n$ matrix C^*. Then the determinant of any $k \times k$ submatrix of C^* placed to contain the row of ones is one. Also, the determinant of any $k \times k$ submatrix of C^* selected with its first row along the second row of C^* and its first column the jth column of C^* has determinant value given by the binomial coefficient $\binom{k+j-1}{k}$.

The convolution triangle imbedded in the matrix $C^* = (c_{ij})$ of Theorem 4.2 has $c_{ij} = c_{i-1,j} + c_{i-2,j} + \cdots + c_{i-m,j} + c_{i,j-1}$. If we form an array $B = (b_{ij})$ with rule of formation $b_{ij} = b_{i-1,j} + b_{i-2,j} + \cdots + b_{2,j} + b_{1,j} + b_{i,j-1}$, B becomes the powers of 2 convolution array.

$$\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
2 & 5 & 9 & 14 & 20 & 27 & \cdots \\
4 & 12 & 25 & 44 & 70 & 104 & \cdots \\
8 & 28 & 66 & 129 & 225 & 463 & \cdots \\
16 & 64 & 168 & 360 & 681 & 1291 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}.$$

Notice in passing that the rising diagonal sums are 1, 2, 5, 13, 34, $\cdots$, the Fibonacci numbers with odd subscripts. This convolution array has both unit determinant properties discussed in Section 1 because it fulfills Eves' Theorem. In fact, all the convolution arrays of this section fulfill Eves' Theorem, so that any $k \times k$ matrix placed to contain a row of ones in any one of these convolution arrays has determinant value one.

5. CONVOLUTION ARRAYS FOR DIAGONAL SEQUENCES FROM MULTINOMIAL COEFFICIENT ARRAYS

Let the $n \times n$ matrix D be

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 4 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{n \times n},$$

where the row sums of D are the rising diagonal sums from Pascal's triangle (1.2) found by beginning at the leftmost column and going up 2 and to the right one, namely, 1, 1, 1, 2, 3, 4, 6, 9, 13, $\cdots$. The column generators of D are $[x(1+x^2)]^j$, $j = 1, 2, \cdots$. Then, the matrix product DP, where P is the rectangular Pascal matrix (1.1), is the convolution triangle for the sequence 1, 1, 1, 2, 3, 4, 6, 9, 13, $\cdots$, $u_{n+1} = u_n + u_{n-2}$, for the column generators of DP are $1/[1 - x(1+x^2)]^j = 1/(1 - x - x^3)^j$, $j = 1, 2, \cdots, n$, where the generating function for $j = 1$ is given as the generating function for the sequence discussed in [5]. The results of Theorem 1.3 and 2.1 can be extended to cover the matrix DP as before.

Now, consider the sequence of elements in Pascal's triangle that lie on the diagonals found by beginning at the leftmost column of (1.2) and going up p and to the right 1 throughout the array (1.2). (The sum of the elements on these diagonals are the generalized Fibonacci numbers $u(n; p, 1)$ of Harris and Styles [6].) Place the elements from successive diagonals on rows of a matrix $D_1(p, 1)$ in reverse order such that the column of ones in A lies on the diagonal of $D_1(p, 1)$. Then the columns of $D_1(p, 1)$ are the rows of Pascal's triangle in left-justified form but with the entries separated by $(p - 1)$ zeroes. Notice that $\det D_1(p, 1) = 1$. The column generators of $D_1(p, 1)$ are $[x(1 + x^p)]^j$, $j = 1, 2, \cdots$, so that the column generators of $D_1(p, 1)P$ are $1/[1 - x(1 + x^p)]^j = 1/(1 - x - x^{p+1})^j$, $j = 1, 2, \cdots, n$, the generating functions for the convolution array for the numbers $u(n; p, 1)$ (see [5]). The same arguments hold for $D_1(p, 1)P$ as for $DP = D_1(2, 1)P$, so that $D_1(p, 1)P$ has the determinant properties which extend from Theorems 1.3 and 2.1.

Now, the same techniques can be applied to the elements on the rising diagonals in all the generalized Pascal triangles induced by expansions of $(1 + x + \cdots + x^m)^n$, $m \geq 1$, $n \geq 0$. Form the $n \times n$ matrix $D_m(p, 1)$ so that elements on the diagonals formed by beginning in the leftmost column and going up p and right one throughout the left-justified multinomial coefficient array lie in reverse order on its rows. $D_m(p, 1)$ will have a one for each element on its main diagonal and each column will contain the corresponding row of the multinomial array but with $(p - 1)$ zeroes between entries. Generating functions for the columns of

$D_m(p,1)$ are $[x(1 + x^p + x^{2p} + \cdots + x^{(m-1)p})]^j$, $j = 1, 2, \cdots, n$, while for P they are, of course, $1(1 - x)^j$. The generating functions for the columns of $D_m(p,1)P$ are then

$$1/[1 - x(1 + x^p + x^{2p} + \cdots + x^{(m-1)p})]^2$$

or

$$1/[1 - x - x^{p+1} - x^{2p+1} - \cdots - x^{(m-1)p+1}]^j, \quad j = 1, 2, \cdots, n,$$

which, for $j = 1$, is known to be the generating function for the diagonal sums here considered [5].

Since again the $k \times k$ submatrices of the product $D_m(p,1)P$ taken along either the first or second row of $D_m(p,1)P$ have the same determinants as the correspondingly placed $k \times k$ submatrices of P, we look again at Theorems 1.3 and 2.1 to write our final theorem.

Theorem 5.1. Write the convolution triangle in rectangular form imbedded in an $n \times n$ matrix C_m^* for the sequence of sums found on the rising diagonals formed by beginning in the leftmost column and moving up p and right one throughout any left-justified multinomial coefficient array. The $k \times k$ submatrix formed to include the first row of ones has determinant one. The $k \times k$ submatrix formed with its first row the second row of C_m^* and its first column the jth column of C_m^* has determinant given by the binomial coefficient

$$\binom{k + j - 1}{k} .$$

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THE DENSITY OF THE PRODUCT OF ARITHMETIC PROGRESSION

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This paper is devoted to the proof of the following theorem.

Theorem. Let a, b, c, and d be positive integers such that $(a,b) = 1 = (c,d)$, Then the density of numbers of the form

$$(ax + b)(cy + d) ,$$

where x and y range over the positive integers, is

$$\frac{1}{(a,c)} .$$

The question arose in the study of the density of products of sets of integers.

The proof is elementary except for the use of Dirichlet's theorem on primes in an arithmetical progression.

1. INTRODUCTION

Let A be a set of positive integers. For a positive integer n let $A(n)$ denote the number of elements in A that lie in the interval $[1,n]$. The upper density of A, $\overline{\delta}(A)$, is defined as

$$\lim_{n \to \infty} \sup A(n)/n .$$

Similarly, the lower density of A, $\underline{\delta}(A)$, is defined as

$$\lim_{n \to \infty} \inf A(n)/n .$$

If $\underline{\delta}(A) = \overline{\delta}(A)$, A is said to have a density, $\delta(A)$, which is the common value of its lower and upper densities. For instance, the arithmetic progression $ax + b$ has density $1/a$.

Consider, as another example, the set

$$S = \{(2x + 1)(2y + 1) \mid x,y \geq 1\} ,$$

which we will abbreviate to

$$\{(2x + 1)(2y + 1)\} .$$

S is clearly contained in the set of odd numbers, $\{2t + 1\}$, whose density is 1/2. However, S does not exhaust this progression, since no prime is in S. But since the set of primes has density 0, S has density 1/2, the same density as the set $\{2t + 1\}$. This sets the stage for the following concept, which will be used often in the proof of the main theorem.

Definition. Let a, b, c, and d be positive integers. The set

$$S = \{(ax + b)(cy + d)\}$$

is full if it has the same density as the arithmetic progression

$$\{(a,c)t + bd\} ,$$

namely $1/(a,c)$.

Since S lies in $\{(a,c)t + bd\}$, the definition simply asserts that S fills the progression except for a set of density 0.

The proof of the main theorem depends primarily on the following lemma.

Lemma 2.3. Let a, b, and d be positive integers such that $(a,b) = 1$. Then the set $\{(ax + b)(ay + d)\}$ is full, that is, has density $1/a$.

The general outline of the proof of the theorem is illustrated by the following example, which will be helpful as a reference point when following the proof.

Say that we wish to prove that $\{(2x + 1)(9y + 1)\}$ is full, that is, has density $1/(2,9) = 1$. We begin as follows. The progression $\{2x + 1\}$ is the disjoint union of the nine progressions

$$\{18x + 1\}, \ \{18x + 3\}, \cdots, \{18x + 1 + 2i\}, \cdots, \{18x + 17\} .$$

The progression $\{9y + 1\}$ is the disjoint union of the two progressions

$$\{18y + 1\} \qquad \text{and} \qquad \{18y + 10\} .$$

Consequently, $\{(2x + 1)(9y + 1)\}$ is the union of the eighteen sets

$$\{\{(18x + 1 + 2i)(18y + 1 + 9j)\} \mid 0 \le i \le 8, \quad 0 \le j \le 1\} .$$

It is not hard to show that these eighteen sets are disjoint. If we showed that each is full, that is, has density 1/18, we would be done, for then $\{(2x + 1)(9y + 1)\}$ would have density $18/18 = 1$. Lemma 2.3 shows that fifteen of the eighteen sets are full. Three cases remain, namely

(i) $\{(18x + 3)(18y + 10)\}$

(ii) $\{(18x + 9)(18y + 10)\}$

(iii) $\{(18x + 15)(18y + 10)\}$.

Let $\underline{\delta}$ and $\overline{\delta}$ denote the lower and upper densities of $\{(2x+1)(9y+1)\}$. Clearly $\underline{\delta} \le \overline{\delta} \le 1$. We know at this point that

$$1 - 3/18 \le \underline{\delta} .$$

This completes the first stage.

The second stage consists of repeating the argument of the first stage on each of the unsettled cases (i), (ii), and (iii).

Analysis for case (i): Write $(18x+3)(18y+10)$ as

$$6(6x+1)(9y+5) ,$$

and treat $\{(6x+1)(9y+5)\}$ by the method already illustrated. It turns out that $\{(6x+1)\times(9y+5)\}$ is the disjoint union of the six sets

$$\{\{18x+1+6i)(18y+5+9j)\} \mid 0 \le i \le 2,\ 0 \le j \le 1\} .$$

Each is covered by Lemma 2.3, hence has density 1/18. Thus the density of $\{(18x+3)\times(18y+10)\}$ is

$$\frac{1}{6}\cdot\frac{6}{18} = \frac{1}{18} .$$

(Note that $\{(18x+3)(18y+10)\}$ is full, even though not covered by Lemma 2.3.)

Analysis for case (ii): Write $(18x+9)(18y+10)$ as

$$18(2x+1)(9y+5) ,$$

and apply the argument of the first stage to

$$\{(2x+1)(9y+5)\} . \tag{1.1}$$

It turns out that (1.1) is the disjoint union of the eighteen sets

$$\{(18x+1+2i)(18y+5+9j) \mid 0 \le i \le 8,\ 0 \le j \le 1\} .$$

Fifteen are covered by Lemma 2.3 and have a total density of 15/18. Three cases remain,

$$\{(18x+3)(18y+14)\},\quad \{(18x+9)(18y+14)\},\quad \{(18x+15)(18y+14)\} .$$

Analysis for case (iii): Write $(18x+15)(18y+10)$ as

$$6(6x+5)(9y+5) .$$

It turns out that $\{(6x + 5)(9y + 5)\}$ splits into six cases, each covered by Lemma 2.3, and case (iii) has density $(1/6)(6/18) = 1/18$.

Combining these three cases, we find at the end of the second stage that

$$1 - 3/18^2 \le \underline{\delta} \le \overline{\delta} \le 1 .$$

Note that the sets removed at each stage by Lemma 2.3 are full.

It turns out that as the process is continued through the third and higher stages, Lemma 2.3 disposes of some sets, and they are full, while the amount left unsettled at the nth stage diminishes toward 0 as $n \to \infty$.

The proof of the theorem shows that the phenomena exhibited by this example occur in general.

2. LEMMAS

We shall make use of the following two basic number-theoretic lemmas. The notation is in the form in which it is used in the proof of the theorem.

Lemma 2.1. Let a, B, c, and D be positive integers such that $(a, B) = 1 = (c, D)$. Then

$$\left[\frac{[a,c]}{(c,B)}, \frac{[a,c]}{(a,D)}\right] = [a,c] .$$

To prove this, consider first the case where a and c are powers of the same prime p, $a = p^{a'}$ and $c = p^{c'}$. The general case follows immediately by using the prime factorization of a and c.

Lemma 2.2. Let a, b, c, and d be positive integers such that $(a,b) = 1 = (c,d)$. Then the sets

$$\{([a,c]\ x + b + ia)([a,c]\ y + d + jc)\} ,$$

$$0 \le i \le ([a,c]/a) - 1, \quad 0 \le j \le ([a,c]/c) - 1 ,$$

are disjoint.

Proof. Assume that

$$(b + ia)(d + jc) \equiv (b + i'a)(d + j'c) \pmod{[a,c]} ,$$

or equivalently,

$$(i - i')ad \equiv (j' - j)bc \pmod{[a,c]} .$$

Then

$$(i - i')ad \equiv (j' - j)bc \pmod{c/(a,c)} ,$$

hence

$$(i - i')ad \equiv 0 \pmod{c/(a,c)} .$$

Since ad and $c/(a,c)$ are relatively prime, it follows that

$$i - i' \equiv 0 \pmod{c/(a,c)} .$$

But

$$0 \le \left| i - i' \right| \le ([a,c]/a) - 1.$$

Since

$$[a,c]/a = c/(a,c) ,$$

it follows that $i = i'$. Similarly, $j = j'$, and the lemma is proved.

The next lemma depends on certain properties of primes that we now review. If $p_1, p_2, \cdots, p_k$ are distinct primes, then the set of positive integers divisible by none of them has density

$$\prod_{i=1}^{k} (1 - p_i^{-1}) .$$

Consequently, if $p_1, p_2, \cdots, p_i, \cdots$ is an infinite sequence of primes such that

$$\sum p_i^{-1} = \infty ,$$

then the set of positive integers divisible by at least one of the p_i's has density 1.

Dirichlet's theorem on primes in arithmetic progressions implies that if a and b are relatively prime positive integers, then the arithmetic progression $\{ax + b\}$ contains an infinite set of primes, $p_1, p_2, \cdots, p_i, \cdots$, and $\sum p_i^{-1} = \infty$.

With this background we are ready for the main lemma.

Lemma 2.3. Let a, b, and d be positive integers such that $(a,b) = 1$. Then the set

$$\{(ax + b)(ax + d)\}$$

is full, that is, has density $1/a$.

Proof. Consider the set $\{az + bd\}$. By Dirichlet's theorem, the density of the subset of $\{(ax + b)(ay + b)\}$ that is divisible by at least one prime p of the form $ax + b$ is $1/a$. If $n = az + bd$ is divisible by p, there is an integer q such that

$$n = az + bd = pq .$$

Taking congruences modulo a we have

$$bd \equiv pq \pmod{a} .$$

Hence

$$bd \equiv bq \pmod{a} ,$$

and since $(a,b) = 1$,

$$d \equiv q \pmod{a} .$$

Thus q has the form $ay + d$, and we conclude that $n = az + bd$ is an element of $\{(ax + b)\times(ay + d)\}$. This proves the lemma.

3. PROOF OF THE THEOREM

Let S be the set

$$\{(ax + b)(cy + d)\} \quad , \tag{3.1}$$

where $(a,b) = 1 = (c,d)$. We wish to prove that S is full. To begin, let $\underline{\delta}$ and $\overline{\delta}$ be the lower and upper densities of (3.1). Clearly $\overline{\delta} \le 1/(a,c)$. We will now show that $\underline{\delta} = 1/(a,c)$.

The progression $\{ax + b\}$ is the disjoint union of the $[a,c]/a$ progressions

$$\{[a,c]\ x + b + ia)\}, \quad 0 \le i \le ([a,c]/a) - 1.$$

Similarly, $\{ay + d\}$ is the disjoint union of the $[a,c]/c$ progressions

$$\{[a,c]\ y + d + jc\}, \quad 0 \le j \le ([a,c]/c) - 1 .$$

Thus, by Lemma 2.2, S is the disjoint union of the

$$\frac{[a,c]}{c} \cdot \frac{[a,c]}{a}$$

sets

$$\{([a,c]\ x + b + ia)([a,c]\ y + d + jc)\} , \tag{3.2}$$

$$0 \le i \le ([a,c]/a) - 1, \quad 0 \le j \le ([a,c]/c) - 1 .$$

For convenience, let $B = b + ia$ and $D = d + jc$ in (3.2). Note that $(a,B) = 1 = (c,D)$. Some of the sets (3.2) are covered by Lemma 2.3. Let us call them "good." Some may not be and will be called "bad." (It will turn out that all of them are full.)

If $(c,B) = 1$ or if $(a,D) = 1$ then the set (3.2) is good. Lemma 2.3 shows that it is full.

If $(c,B) > 1$ and $(a,D) > 1$ the set (3.2) is bad. In order to have a reasonable bound on the upper density of the finite union of these bad sets (3.2), it is necessary to determine how many there are of them.

First compute the number of $B = b + ia$, $0 \le i \le ([a,c]/a) - 1$ that are not relatively prime to $[a,c]$, or, since $(a,b) = 1$, not relatively prime to c. To do so, let $p_1, p_2, \cdots, p_k$ be the distinct primes that divide c but not a. (There may be no such primes.) As i runs through $p_1p_2 \cdots p_k$ consecutive integers, $B = b + ia$ runs through a complete residue system modulo $p_1p_2 \cdots p_k$, of which

$$p_1p_2 \cdots p_k - \phi(p_1p_2 \cdots p_k)$$

are not relatively prime to c. Similarly, let $q_1, q_2, \cdots, q_m$ be the primes that divide a but not c. The number of bad sets of the form (3.2) is then

$$\frac{[a,c]}{a} \cdot \frac{[a,c]}{c} \left(1 - \prod_{p_i} (1 - p_i^{-1})\right)\left(1 - \prod_{q_j} (1 - q_j^{-1})\right) .$$

There may be no bad sets, in which case the proof is already complete.

Each good set of type (3.2) is full, by Lemma 2.3. Each bad set of type (3.2) has upper density at most $1/[a,c]$. Hence, the upper density of the union of the bad sets is at most

$$\frac{1}{[a,c]} \cdot \frac{[a,c]}{a} \cdot \frac{[a,c]}{c} \left(\prod_{p_i | ac} (1 - p_i^{-1})\right)^2 ,$$

which equals

$$(1/(a,c))\left(\prod_{p_i | ac} (1 - p_i^{-1})\right)^2 .$$

Let

$$k = \left(\prod_{p_i | ac} (1 - p_i^{-1})\right)^2 .$$

At the end of the first stage it is known that

$$\underline{\delta} \geq 1/(a,c) - k .$$

Each of the bad sets in stage 1 is then treated as in the example. That is, a bad set

$$\{([a,c]\, x + B)([a,c]\, y + D)\}$$

is written as

$$\left\{(c,B)(a,D) \left(\frac{[a,c]}{(c,B)} x + \frac{B}{(c,B)}\right)\left(\frac{[a,c]}{(a,D)} y + \frac{D}{(a,D)}\right)\right\}$$

(Keep in mind that $([a,c],B) = (c,B)$ and $([a,c],D) = (a,D)$.)

The set

$$\left\{\left(\frac{[a,c]}{(a,B)} x + \frac{B}{(c,B)}\right)\left(\frac{[a,c]}{(a,D)} y + \frac{D}{(a,D)}\right)\right\}$$

is then decomposed into products of progressions with equal moduli, as S was. The common modulus is

$$\left[\frac{[a,c]}{(a,B)}, \frac{[a,c]}{(c,D)}\right] , \tag{3.3}$$

whose prime divisors are clearly among the primes that divide $[a,c]$. (In fact, by Lemma 2.1, (3.3) equals $[a,c]$, but this fact plays no role in the proof.)

After completing stage 2, we then have

$$\underline{\delta} \geq 1/(a,c) - k^2 .$$

Similarly, the nth stage shows that

$$\underline{\delta} \geq 1/(a,c) - k^n .$$

Consequently, $\underline{\delta} \geq 1/(a,c)$, and S has density $1/(a,c)$. This concludes the proof.

4. REPRESENTATION OF INTEGERS BY THE FORM $axy + bx + cy$

The theorem has as an immediate consequence the following information about the representation of integers by a certain polynomial form.

Theorem 4.1. Let a be a positive integer, and let b and c be non-negative integers. Then the set of integers expressible in the form

$$axy + bx + cy$$

for some positive integers x and y has a density equal to

$$\frac{a}{(a,b)(a,c)\left(\frac{a}{(a,c)}, \frac{a}{(a,b)}\right)} .$$

Proof. The equation

$$z = axy + bx + cy$$

is equivalent to the equation

$$\begin{aligned} az &= (ax + c)(ay + b) - bc \\ &= (a,c)(a,b)\left(\frac{a}{(a,c)}x + \frac{c}{(a,c)}\right)\left(\frac{a}{(a,b)}y + \frac{b}{(a,b)}\right) - bc . \end{aligned}$$

By the theorem, the set of integers z of the form $axy + bx + cy$ thus has density

$$a \cdot \frac{1}{(a,c)} \cdot \frac{1}{(a,b)} \cdot \frac{1}{\left(\frac{a}{(a,c)}, \frac{a}{(a,b)}\right)} .$$

This completes the proof.

In particular, Theorem 4.1 shows that if $(a,b) = 1$ or if $(a,c) = 1$, then the set $\{axy + bx + cy\}$ has density 1.

n - FIBONACCI PRODUCTS

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1. NOTATION

Let ϕ^n be the n-dimensional vector space, i.e., for

$$X = [x_1, x_2, \cdots, x_n] \in \phi^n, \qquad x_1, x_2, \cdots, x_n \in \phi^1 .$$

In addition, let I be the set of positive integers, J the set of non-negative integers, $I(n) \subset I$, be such that if $k \in 1(n)$ then $k \leq n$, $J(n) \subset J$, be such that if $k \in J(n)$ then $k \leq n$. $W(n) \subset \phi^n$ is such that if $K \in W(n)$, where $K = [k_1, k_2, \cdots, k_n]$, then $k_m \in J$, for $m \in 1(n)$. In particular, $U = [1, 1, \cdots, 1] \in W(n)$.

With $K \in W(n)$ and $X \in \phi^n$, we write

$$X^K = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} = \prod_{m=1}^{n} x_m^{k_m} , \tag{1}$$

and in particular

$$X^U = x_1 x_2 \cdots x_n . \tag{2}$$

Also

$$|X| = \sum_{m=1}^{n} x_m , \tag{3}$$

and

$$\sum_{K=0}^{P} f(K) \tag{4}$$

is the sum of all elements of the form $f(K)$ where the component of K, i.e., k_m, $m \in I(n)$, take all integer values such that $0 \leq k_m \leq p_m$, where $P = [p_1, p_2, \cdots, p_n] \in W(n)$.

Let $E(m)$ be the partial translation operator for the variable x_m, i.e.,

$$E(m) f(x_k) = \delta_m^k f(x_k + 1) , \qquad k, m \in I(n) , \tag{5}$$

where δ_k^m is the Kronecker delta. In addition, let $\Delta(m) = E(m) - \mathrm{Id}$, Id being the identity operator. Using the vector notation introduced earlier, we have

(6) $$E = [E(1), E(2), \cdots, E(n)]$$

(7) $$\Delta = [\Delta(1), \Delta(2), \cdots, \Delta(n)]$$

and

$$E^U = E(1)E(2)\cdots E(n), \qquad \Delta^U = \Delta(1)\Delta(2)\cdots\Delta(n) .$$

2. FIBONACCI AND LUCAS PRODUCTS

Let $F(m)$ be the general term of the Fibonacci sequence, $L(m)$ the general term of the Lucas sequence as defined in [1] and $H(m)$ the general term of the generalized Fibonacci sequence. Using the notation introduced in Section 1, we have with

$$K = [k_1, k_2, \cdots, k_n] \in W(n)$$

(8) $$F(K) = [F(k_1), F(k_2), \cdots, F(k_n)]$$

(9) $$L(K) = [L(k_1), L(k_2), \cdots, L(k_n)]$$

(10) $$H(K) = [H(k_1), H(k_2), \cdots, H(k_n)]$$

and

(11) $$f(K) = [F(K)]^U = \prod_{m=1}^{n} F(k_m)$$

(12) $$\lambda(K) = [L(K)]^U = \prod_{m=1}^{n} L(k_m)$$

(13) $$h(K) = [H(K)]^U = \prod_{m=1}^{n} H(k_m) .$$

The numbers $f(K)$, $\lambda(K)$, and $h(K)$ are called the n-Fibonacci, Lucas and generalized Fibonacci products.

3. RECURRENCE RELATIONS

According to [1] we have for the three sequences considered

(14) $$F(k_m + 2) = F(k_m + 1) + F(k_m)$$

which we can write

(15) $$E(m)\Delta(m)F(k_m) = F(k_m)$$

or

(16) $$[E(m)\Delta(m) - Id]F(k_m) = 0 .$$

Starting from (15) we can write

$$\prod_{m=1}^{n} F(m)\Delta(m)F(k_m) = \prod_{m=1}^{n} F(k_m) ,$$

or

$$E^U \Delta^U f(K) = f(K) ,$$

or again

$$(E^U \Delta^U - Id)f(K) = 0 . \tag{17}$$

Thus the Fibonacci products satisfy a recurrence relation similar to the one dimension, i.e., (16). The same applies to the Lucas and generalized Fibonacci products, i.e.,

$$(E^U \Delta^U - Id)\lambda(K) = 0 , \tag{18}$$

$$(E^U \Delta^U - Id)h(K) = 0 . \tag{19}$$

4. OTHER RELATIONS

The relations given in [1, pp. 59-60] can be generalized for n-Fibonacci and Lucas products. We illustrate by two examples:

(i) Relation (I 14) reads: $L(m) = F(m + 2) - F(m - 2)$, or

$$L(m + 2) = F(m + 4) - F(m)$$

or, on operator form

$$E^2(m)L(m) = [E^4(m) - Id]F(m) . \tag{20}$$

But

$$E^4(m) - Id = [E(m) - Id][E(m) + Id][E^2(m) + Id] ,$$

where $E(m) - Id = \Delta(m)$.

$$E(m) + Id = 2M(m)$$

where $M(m)$ is the partial mean operator. We define correspondingly

$$M = [M(1), M(2), \cdots, M(n)] ,$$

and

$$M^U = \prod_{m=1}^{n} M(m) .$$

In addition let

$$P(m) = E^2(m) + Id, \qquad P = [P(1), P(2), \cdots, P(n)] ,$$

and

$$P^U = P(1)P(2)\cdots P(n) = \prod_{m=1}^{n} P(m) .$$

We take now the product of both sides of (20) which we rewrite

$$\prod_{m=1}^{n} E^2(m)L(k_m) = \prod_{m=1}^{n} 2\Delta(m)M(m)P(m)F(k_m) \tag{21}$$

or

$$E^{2U}\lambda(K) = 2^n\Delta^U M^U P^U f(K) , \tag{22}$$

which is the relation corresponding to (I 14) of [1] for n-Fibonacci and Lucas products.

(ii) Relation (I 41) can be written

$$\sum_{k=0}^{2q} \binom{2q}{k} F(2k + p) = 5^q F(2q + p) ,$$

or, introducing the variable m and the usual operators

$$\sum_{k_m=0}^{2q_m} \binom{2q_m}{k_m} E^{2k_m}(m)F(p_m) = 5^{q_k} E^{2q_m}(m)F(p_m) . \tag{23}$$

Taking the product over m from $m = 1$ to $m = n$ and using the notation

$$\prod_{m=1}^{n} \binom{2q_m}{k_m} = \binom{2Q}{K} ,$$

where

$$K = [k_1, k_2, \cdots, k_n] \in W(n) , \qquad Q, P \in W(n) ,$$

we obtain the formula corresponding to (I 41), i.e.,

$$\left[\sum_{K=0}^{2Q} \binom{2Q}{K} E^{2K} - 5^{|Q|} E^{2Q}\right] f(P) = 0 . \tag{24}$$

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SOME CORRECTIONS TO CARLSON'S "DETERMINATION OF HERONIAN TRIANGLES"

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In [1], Carlson presents a determination of all Heronian triangles, i.e., triangles with integral sides and area. He correctly shows that every such triangle, or a multiple thereof, can be split into two Pythagorean triangles, i.e., right triangles with integer sides. Unfortunately, he then makes a common error in incorrectly assuming that all Pythagoreal triangles are of the form:

(1) $$u^2 + v^2, \quad u^2 - v^2, \quad 2uv ,$$

rather than the correct form:

(2) $$a(u^2 + v^2), \quad a(u^2 - v^2), \quad 2auv ,$$

(One can easily verify that 15, 9, 12 is a Pythagorean triangle which cannot be expressed by (1). The same error is also made in [2].)

Using the form (2) with Carlson's main theorem, we have the following correct form of his

<u>Corollary 1.</u> A triangle is Heronian if and only if its sides can be represented as

(i) $$a(u^2 + v^2), \quad b(r^2 + s^2), \quad a(u^2 - v^2) + b(r^2 - s^2) ,$$

where $auv = brs$:

(ii) $$a(u^2 + v^2), \quad b(r^2 + s^2), \quad a(u^2 - v^2) + 2brs ,$$

where $2auv = b(r^2 - s^2)$:

(iii) $$a(u^2 + v^2), \quad b(r^2 + s^2), \quad 2auv + 2brs ,$$

where $a(u^2 - v^2) = b(r^2 - s^2)$; or:

(iv) a reduction by a common factor of a triangle given by (i), (ii), or (iii).

Carlson's incorrect form of Corollary 1 had three conditions numbered (3), (4) and (5) corresponding to our (i), (iii) and (iv) without the parameters a and b. It appears that the original Corollary 1 neglected to consider that the common side of the two Pythagorean triangles might be of the form $u^2 - v^2$ in one and of the form $2rs$ in the other. If one constructs a short table of Pythagorean triples from (1), one has:

u	v	$u^2 + v^2$	$u^2 - v^2$	$2uv$
2	1	5	3	4
3	1	10	8	6
3	2	13	5	12
4	1	17	15	8
4	2	20	12	16
4	3	25	7	24

One immediately wants to construct a Heronian triangle from 10, 8, 6 and 17, 15, 8, obtaining 10, 17, 21. This construction is of form (ii) of the corrected Corollary 1, but does not fit into Carlson's (3), (4) or (5). To see that (5) does not apply, suppose that it did. Then we must put together either:

10c, 6c, 8c and 17c, 15c, 8c, which gives $16c = 2u^2$ and $32c = 2r^2$, which is impossible; or:

10c, 8c, 6c and 17c, 8c, 15c, which gives $18c = 2u^2$ and $25c = 2r^2$, which is impossible.

(Possibly Corollary 1 may apply in its original form to the splitting by one of the other altitudes.)

Carlson's Lemma 2 is unclear. (The following remarks assume the reader has 1 at hand.) What he has proven, but not clearly stated, is that an isosceles Heronian triangle is obtained by putting together two equal Pythagorean triangles. The first step of the proof, that a primitive Heronian triangle has only one even side, is not proven until four pages later, on p. 505. The fact that the side of the isosceles triangle is odd is not used, only the fact that the base is even is needed and this holds for any isosceles Heronian triangle since it holds for primitive ones. Carlson's parameter Q is simply the altitude on the base and his result $A = nQ$ is direct from the ordinary area formula. (There may be some historical interest in using Hero's formula for the area, but I would not consider the added interest to be worth the added complexity.) Further, to obtain primitiveness, one must make assumptions on GCD (u, v) and the parity of u and v. We give the following clearer and correct form of Carlson's

Lemma 2. A triangle is an isosceles Heronian triangle if and only if its sides can be represented as:

(i) $$a(u^2 + v^2), \quad a(u^2 + v^2), \quad 4auv\,;$$

or:

(ii) $$a(u^2 + v^2), \quad a(u^2 + v^2), \quad 2a(u^2 - v^2)\,.$$

The triangle is then primitive if and only if

$$a = 1, \quad \mathrm{GCD}(u,v) = 1 \text{ and } u \not\equiv v \pmod 2.$$

Incidentally, it is possible to obtain different representations of isosceles Heronian triangles. Consider the Pythagorean triangles 30, 18, 24 (obtained from $u = 2$, $v = 1$, $a = 6$) and 25, 7, 24 (obtained from $u = 4$, $v = 3$, $a = 1$). These fit together to form the isosceles Heronian triangle 25, 25, 30 which reduces to 5, 5, 6.

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AN OBSERVATION ON FIBONACCI PRIMITIVE ROOTS

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1. OBSERVATION

A prime p has a Fibonacci Primitive Root g if g is a primitive root of p that satisfies

$$g^2 \equiv 1 + g \pmod{p} . \tag{1}$$

Some properties of the F. P. R.'s are proven or conjectured in [1]. Another property that was not noticed then is given in the following.

Theorem. If $p \equiv 3 \pmod{4}$ has g as a F. P. R., then $g - 1$ and $g - 2$ are also primitive roots of p.

Examples. From [1, Table 1, p. 164].

$p = 11$ has 8, 7, 6 as primitive roots;
$p = 19$ has 15, 14, 13 as primitive roots;
$p = 31$ has 13, 12, 11 as primitive roots.

Proof. Since

$$g(g - 1) \equiv 1 \pmod{p} ,$$

$g - 1$ is the inverse of $g \pmod{p}$ and therefore is a primitive root of p if g is. Next,

$$(g - 1)^2 = g^2 - 2g + 1 \equiv -g + 2 \pmod{p}$$

from (1) and, since $p = 4k + 3$,

$$(g - 1)^{2k+1} \equiv -1 \pmod{p} .$$

Therefore,

$$(g - 1)^{2k+3} \equiv g - 2 \pmod{p} ,$$

and since $2k + 3$ is prime to $4k + 2$, $g - 2$ is also a primitive root of p.

2. ASYMPTOTIC DENSITY

What ratio r of all primes $p \equiv 3 \pmod{4}$, asymptotically speaking, have such a triple of primitive roots? By [1, p. 167] it is immediate that the proper conjecture is

$$r \overset{?}{=} \frac{18}{19} A = 0.35427\ 39286\ 91876 \tag{2}$$

where A is Artin's constant. By the discussion in [1] there is little doubt that (2) is true even though it is not now provable.

3. OTHER TRIPLES

Another criterion, entirely different, for three consecutive primitive roots is this, cf. [2, p. 80, Ex. 61]. If

$$p = 8k + 7 \quad \text{and} \quad q = 4k + 3$$

are both prime, then

(3) $$p - 2, \quad p - 3, \quad p - 4$$

are primitive roots of p.

This is easily proven. As an example, $p = 23$ (having $q = 11$) has 21, 20, 19 as primitive roots.

Now, what primes p simultaneously satisfy both sufficient conditions, and thereby have both triples, that in (3) and the

(4) $$g, \quad g - 1, \quad g - 2$$

triple above? It is easily seen that any such p must satisfy $p \equiv 119 \pmod{120}$ and therefore that the run (3) extends, at least, to

(5) $$p - 2, \quad p - 3, \quad p - 4, \quad p - 5, \quad p - 6.$$

The smallest example is $p = 359$ with primitive roots

(4a) 106, 105, 104, also 103,

and

(5a) 357, 356, 355, 354, 353 .

The next example is $p = 479$ with

(4b) 229, 228, 227, also 226,

and the powerful run

(5b) 477, 476, 475, 474, 473, also
472, 471, 470, 469, 468, 467.

The run of 11 in (5b) is due to the fact that 479 is a "negative square." See [3, Table II, p. 436] and the discussion there for an explanation of this last point.

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A GROUP-THEORETICAL PROOF OF A THEOREM IN ELEMENTARY NUMBER THEORY

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It is well known that if

$$N = 2^{\ell} p_1^{\ell_1} p_2^{\ell_2} \cdots p_s^{\ell_s}$$

then the number of solutions to the congruence $x^2 \equiv 1 \pmod N$ is 2^s if $\ell = 0$ or 1, 2^{s+1} if $\ell = 2$, 2^{s+2} if $\ell \geq 3$ [2, p. 191]. In this note, we give a group-theoretical proof of this fact. To fix the idea, let

$$N = 2^{\ell} p_1^{\ell_1} p_2^{\ell_2} \cdots p_s^{\ell_s} = 2^{\ell} N_0 = p_1^{\ell_1} N_1 = \cdots = p_s^{\ell_s} N_s .$$

Hence $N_i = N/p_i^{\ell_i}$, and the N_i are relatively prime, identifying 2 with p_0.

Lemma. Let $k_0, k_1, \cdots, k_s$ be integers such that $k_0N_0 + k_1N_1 + \cdots + k_sN_s = 1$, let $e_0 = \pm 1$, or ± 1 + some power of 2, $e_i = \pm 1$ for $1 \leq i \leq s$, and let

$$M = e_0k_0N_0 + e_1k_1N_1 + \cdots + e_sk_sN_s .$$

Then for any choice of e_i, $0 \leq i \leq s$, $(M, N) = 1$.

Proof. Since $p_i | N_j$ for $i \neq j$ and $p_i \nmid N_i$, p_i must not divide k_i, otherwise p_i would divide 1. Suppose $(M, N) \neq 1$, then some $p_i | M$, but this p_i must then divide $e_ik_iN_i$, which is impossible.

Theorem. The number of solutions to the congruence $x^2 \equiv 1 \pmod N$ is 2^s if $\ell = 0$ or 1, 2^{s+1} if $\ell = 2$, and 2^{s+2} if $\ell \geq 3$.

Proof. Let $\langle c \rangle$ be a cyclic group of order N. First notice that a nontrivial automorphism λ of $\langle c \rangle$ takes c to c^x, where $(x, N) = 1$; if λ is of order 2, then $x^2 \equiv 1 \pmod N$. Moreover, since every solution x_0 of $x^2 \equiv 1 \pmod N$ is prime to N, $\lambda(c) = c^{x_0}$ is an automorphism of order 2. Since the automorphism group of a cyclic group is abelian, the set of automorphisms of order 2 form a subgroup. The order of this subgroup is the number of solutions to the congruence $x^2 \equiv 1 \pmod N$.

Each Sylow p_i-subgroup is generated by c^{N_i} and is characteristic in $\langle c \rangle$. An automorphism λ of order 2 must take c^{N_i} to c^{N_i} or c^{-N_i} for $1 \leq i \leq s$ since $x^2 \equiv 1 \pmod{p^n}$ has only two solutions ± 1 for an odd prime p. As for the 2-Sylow subgroup $\langle c^{N_0} \rangle$, if its order is 2, it admits only the identity automorphism; if its order is 4, it admits 2 automorphisms, namely $c^{N_0} \to c^{N_0}$ and $c^{N_0} \to c^{-N_0}$; if its order is 2^{ℓ}, $\ell \geq 3$, it admits 4 automorphisms, with the other two being

$$c^{N_0} \longrightarrow c^{N_0(2^{\ell-1}+1)} \qquad \text{and} \qquad c^{N_0} \longrightarrow c^{N_0(2^{\ell-1}-1)} .$$

We have thus seen that an automorphism λ of order 2 either leaves a Sylow p_i-subgroup elementwise fixed or takes its elements to their inverses or, in case of the Sylow 2-subgroup of order $2^{\ell} \geq 8$, takes the elements to their $2^{\ell-1} \pm 1$ powers.

Conversely, mappings that act on one Sylow subgroup as above and leave all others elementwise fixed are automorphisms of order 2 and so are their compositions. In fact, let λ be such a mapping,

$$\lambda(c) = \lambda(c^{k_0N_0+\cdots+k_sN_s}) = \lambda(c^{k_0N_0})\lambda(c^{k_1N_1}) \cdots \lambda(c^{k_sN_s}) = (c^{e_0k_0N_0})(c^{e_1k_1N_1}) \cdots (c^{e_sk_sN_s}) = c^{M} ,$$

clearly $(M, N) = 1$ by the lemma and λ is an automorphism of order 2.

Since the group of automorphisms of order 2 is a direct product of the groups of automorphisms of order 2 of its Sylow subgroups, the conclusion of the theorem is established.

REFERENCES

1. W. Burnside, Theory of Groups of Finite Order, Dover, 1955.
2. I. M. Vinogradov, Elements of Number Theory, Dover, 1954.

ERRATA

Please make the following corrections on errors occurring in "The Autobiography of Leonardo Pisano," appearing on page 99, Volume 11, No. 1, February 1973:

Page 100, line 13 — The fourth word in this line should be "quedam," not "quedem."

Page 101, line 11 — Please underline "per qualche giorno."

line 5 from bottom — Please underline the last word, "in."

Page 102, line 6 — Please change the last underscored word from "posta" to "postea."

line 21 — Please underline the words "disputationis conflictum."

Page 103, line 1 — Please change the word "reconing" to read "reckoning."

line 20 — Please change the last word on this line to read "ur2."

line 33 — Please change the next to last word to read "a Tq3."

line 5 from bottom — Please read the sixth from last word as "ed."

Page 104, line 1 — Please underline the word "algorismum."

[Continued on page 168.]

PERIODICITY OF SECOND - AND THIRD - ORDER RECURRING SEQUENCES

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Define a sequence of generalized Fibonacci numbers

(1) $$\{w_n\}_0^\infty = \{w_n(b,c;\ P,Q)\}_0^\infty$$

by

(2) $$w_n = bw_{n-1} + cw_{n-2}\ ,$$

where n denotes an integer ≥ 2, $w_0 = P$ and $w_1 = Q$. Considering a special form of this sequence

$$\{w_n^{(1)}\}_0^\infty = \{w_n(1,1;\ 0,1)\}_0^\infty\ ,$$

D. D. Wall [1] has shown that

$$\{w_n^{(1)} \pmod{m}\}_0^\infty$$

(where m denotes a positive integer) is simply periodic. Our objective is to point out a rigorous proof of the same and extend it to the sequence of Tribonacci numbers

(3) $$\{T_n\}_0^\infty = \{T_n(b,c,d;\ P,Q,R)\}_0^\infty\ .$$

This sequence of numbers is defined by

(4) $$T_n = bT_{n-1} + cT_{n-2} + dT_{n-3}\ ,$$

where n denotes an integer ≥ 3, $T_0 = P$, $T_1 = Q$ and $T_2 = R$.

<u>Theorem a.</u>

$$\{w_n^{(1)} \pmod{m}\}_0^\infty$$

is simply periodic.

<u>Proof.</u> Let

$$m = \Pi p_j^{a_j} ,$$

where $j = 1, 2, \cdots, i$ and p_j represents a prime. Since

$$\{ w_n^{(1)} \pmod{p_j^{a_j}} \}_0^\infty$$

is known to be periodic [1], we denote the length of the period

$$\{ w_n^{(1)} \pmod{p_j^{a_j}} \}_0^\infty$$

by k_j and write

$$w_{k_j}^{(1)} \equiv 0 \pmod{p_j^{a_j}} , \qquad w_{k_j+1}^{(1)} \equiv 1 \pmod{p_j^{a_j}} . \tag{5}$$

Then it is easy to show that

$$w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{p_1^{a_1}}, \qquad w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{p_2^{a_2}} , \cdots, \qquad w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{p_i^{a_i}}$$

(6) and

$$w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{p_1^{a_1}}, \qquad w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{p_2^{a_2}} , \cdots, \qquad w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{p_i^{a_i}} .$$

Therefore, it follows that

$$w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{m}$$

(7) and

$$w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{m}$$

and

$$\{ w_n^{(1)} \pmod{m} \}_0^\infty$$

becomes simply periodic.

Theorem b. If $(b, c, P, Q, m) = 1$, then $\{w_n \pmod m\}_0^\infty$ is simply periodic.

Proof. Let

$$\{w_n^{(2)}\}_0^\infty = \{w_n(b, c;\ 0, 1)\}_0^\infty .$$

For p denoting a prime, if $(b, c, p) = 1$, then it has been shown in [3], that $\{w_n^{(2)} \pmod p\}_0^\infty$ is simply periodic. Also, since

$$w_n = pw_n^{(2)} + cQw_{n-1}^{(2)} ,$$

it follows that if $(b, c, P, Q, p) = 1$, then $\{w_n \pmod p\}_0^\infty$ is simply periodic, and the technique of Theorem a renders that $\{w_n \pmod m\}_0^\infty$ is simply periodic.

Theorem c. Let

$$\{T_n^{(9)}\}_0^\infty = \{T_n(1, 1, 1;\ 0, 0, 1)\}_0^\infty .$$

Then

$$\{T_n^{(9)} \pmod m\}_0^\infty$$

is simply periodic.

Proof. We have shown in [2], that $\{T_n^{(9)} \pmod p\}_0^\infty$ is simply periodic and the proof that $\{T_n^{(9)} \pmod m\}_0^\infty$ is simply periodic follows from the technique of Theorem a.

Theorem d. If $(b, c, d, P, Q, R, m) = 1$, then $\{T_n \pmod m\}_0^\infty$ is simply periodic. The proof of this theorem is similar to that of Theorem c and is left to the reader.

ACKNOWLEDGEMENT

I am grateful to Dr. Joseph Arkin for drawing my attention to the problem of this note in connection with an earlier paper.

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1. D. D. Wall, "Fibonacci Series Modulo m," Amer. Math. Monthly, 67 (1960), pp. 525-532.
2. C. C. Yalavigi, "Properties of Tribonacci Numbers," submitted to the Fibonacci Quarterly.
3. C. C. Yalavigi, "A Further Generalization of Fibonacci Sequence," to appear, Fibonacci Quarterly.
4. C. C. Yalavigi, "On the Periodic Lengths of Fibonacci Sequence, modulo p," to appear, Fibonacci Quarterly.
5. C. C. Yalavigi and H. V. Krishna, "On the Periodic Lengths of Generalized Fibonacci Sequence Modulo p," to appear, Fibonacci Quarterly.

ON SOLVING NON - HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

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In a recent paper, Weinshenk and Hoggatt [1] gave two methods for obtaining the general solution of the difference equation

$$C_{n+2} = C_{n+1} + C_n + n^m \quad . \tag{1}$$

One method is by expansion and the other by operators. However, in the latter method there are still some open convergence questions. Here we give another method which is equivalent to one of the operator methods but which avoids the convergence question. It will be valid for any linear difference equation with constant coefficients and with any non-homogeneous term on the right-hand side. The solution will be given in terms of the solutions of the corresponding homogeneous equation.

We consider the equation

$$L(E)A_n = G_n \quad , \tag{2}$$

where the linear operator is given by

$$L(E) = E^r + a_1E^{r-1} + a_2E^{r^2} + \cdots + a_r$$

and the a_i's are constants. The corresponding homogeneous equation, $L(E)A_n = 0$, can be solved in the standard way in terms of the roots of $L(x) = 0$. We will denote a solution of the homogeneous equation by the sequence $\{B_i\}$ and for simplicity we will assume that the initial conditions on the B_i's are such that

$$\frac{1}{1 + a_1x + a_2x^2 + \cdots + a_rx^r} = B_0 + B_1x + B_2x^2 + \cdots \quad . \tag{3}$$

If we had chosen arbitrary initial conditions for the B_i's, then the numerator (1) on the left-hand side would have been replaced by some polynomial entailing a further calculation subsequently. This procedure is analogous to solving linear non-homogeneous differential equations. One first solves the homogeneous equation subject to quiescent conditions and then obtains the general solution by a convolution in terms of the non-homogeneous term and the latter solution.

To solve (2), we first write down a generating function of the solution, i.e.,

$$A(x) = A_0 + A_1x + A_2x^2 + \cdots + A_rx^r + \cdots \quad .$$

Then

$$\begin{aligned} a_1xA(x) &= a_1A_0x + a_1A_1x^2 + \cdots + a_1A_{r-1}x^r + \cdots , \\ a_2x^2A(x) &= \qquad a_2A_0x^2 + \cdots + a_2A_{r-2}x^r + \cdots , \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ a_rx^rA(x) &= \qquad\qquad\qquad a_rA_0x^r + \cdots . \end{aligned}$$

Adding:

$$A(x)(1 + a_1x + a_2x^2 + \cdots + a_rx^r) = S_0 + S_1x + S_2x^2 + \cdots + S_{r-1}x^{r-1} + G_0x^r + G_1x^{r+2} + G_2x^{r+2} + \cdots ,$$

where

$$S_i = a_iA_0 + a_{i-1}A_1 + A_{i-2}A_2 + \cdots + a_0A_i \qquad (a_0 = 1) .$$

Now using (3) and carrying out the multiplication, we obtain the convolution

$$\begin{aligned} A_n = &\{S_0B_n + S_1B_{n-1} + S_2B_{n-2} + \cdots + S_{r-1}B_{n-r-1}\} \\ &+ \{G_0B_{n-r} + G_1B_{n-r-1} + G_2B_{n-r-2} + \cdots + G_{n-r}B_0\} \qquad (n \geq r) . \end{aligned} \tag{4}$$

The top part of the right-hand side of (4) corresponds to the complementary (homogeneous) solution of (2) whereas the bottom part corresponds to the particular solution. It is to be noted that the method is valid even if the non-homogeneous right-hand side of (2) is part of the complementary solution (i.e., if $L(E)G_n = 0$) .

We now apply this technique to (1). One complementary solution of (1) is of course the Fibonacci sequence 1, 1, 2, 3, $\cdots$. Thus,

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^3 + \cdots .$$

Solution (4) now becomes

$$C_n = C_0F_{n+1} + (C_0 + C_1)F_n + F_1(n - 2)^m + F_2(n - 3)^m + \cdots + F_{n-2}(1)^m$$

or

$$C_n = C_0F_{n-1} + C_1F_n + \sum_{i=1}^{n-1} F_i(n - i - 1)^m \qquad (n \geq 2) .$$

This corresponds to the solution in [1] provided a stopping rule is used there.

In their concluding remarks, the authors of [1] raise the question of determining conditions under which their operational methods for obtaining a particular solution are valid. They point out the example of D. Lind that if $C_{n+1} - C_n + n$ were to be solved by their operational method, one would obtain

$$C_n = \frac{-1}{1 - E}\{n\} = -\sum_{k=0}^{\infty} E^k n , \tag{5}$$

which diverges unless some stopping rule is involved. However, the divergent solution can be justified if one considers its analytic continuation. First replace n by n^s where $R_e(s) > 1$.

Then in terms of the Riemann Zeta function,

$$-C_n = \frac{1}{n^s} + \frac{1}{n^{s+1}} + \frac{1}{n^{s+2}} + \cdots$$

or

$$C_n = \left\{\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{(n-1)^s}\right\} - \zeta(s) .$$

However, the zeta function can be analytically continued for $R_e(s) < 1$ and for negative integers it is given by [2]

$$\zeta(-2m) = 0, \qquad \zeta(1-2m) = (-1)^m B_m/(2m), \qquad m = 1, 2, 3, \cdots ,$$

$$\zeta(0) = -1/2 \qquad (B_m \text{ are the Bernoulli numbers}).$$

Now letting $s = -1$ above, gives the valid particular solution

$$C_n = (1 + 2 + 3 + \cdots + n - 1) - \zeta(-1) .$$

Since the constant $\zeta(-1)$ satisfies the homogeneous equation, it can be deleted.

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1. R. J. Weinshenk and V. E. Hoggatt, Jr., "On Solving $C_{n+2} = C_{n+1} + C_n + n^m$ by Expansions and Operators," Fibonacci Quarterly, Vol. 8, No. 1, 1970, pp. 39-48.
2. C. N. Watson, A Course in Modern Analysis, Cambridge University Press, Cambridge, 1946, pp. 267-268.

[Continued from page 162.]

ERRATA

Please make the following correction to "A New Greatest Common Divisor Property of the Binomial Coefficients," appearing on p. 579, Vol. 10, No. 6, Dec. 1972:

On page 584, last equation, for

$$\binom{n+n}{k+a} \qquad \text{read} \qquad \binom{n+a}{k+a}.$$

In "Some Combinatorial Identities of Bruckman," appearing on page 613 of the same issue, please make the following correction.

On the right-hand side of Eq. (12), p. 615, for

$$\frac{2k}{2k+1} \qquad \text{read} \qquad \frac{2^k}{2k+1} .$$

A RELIABLITY PROBLEM

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ABSTRACT

An $m \times n$ array of elements is considered in which each element has a probability p of being reliable. The array as a whole is considered reliable if there does not exist in the array any polydominoe of a given form in any orientation having all of its elements unreliable. A method is given for determining the probability of reliability for the array and solutions are worked out explicitly for several special cases.

1. INTRODUCTION

We are given an $m \times n$ array

	1	2	...	n
1				
2				
⋮				
m				

in which each of the mn elements has a given probability p of being reliable (and a probability q of being unreliable where $p + q = 1$). The $m \times n$ matrix as a whole will be considered reliable, if and only if, there does not exist in the array, any polydominoe of a given form in any orientation having all of its elements unreliable. The problem then is to calculate the probability of reliability of the array. The special case where $m = 2$ and the given polydominoe is a 2×1 arose in the design of a low-altitude detection antenna. If any 2×1 polydominoe had both its elements unreliable, then the antenna could not fulfill its detection mission.

The specific cases to be considered here explicitly are the following:

	Array size	Failure polydominoe
(C_1)	$2 \times n$	$\boxed{q \mid q}$

	Array size	Failure polydominoe
(C_2)	$2 \times n$	$\begin{array}{\|c\|c\|}\hline q & \\ \hline q & q \\ \hline \end{array}$
(C_3)	$2 \times n$	$\begin{array}{\|c\|c\|}\hline q & q \\ \hline q & q \\ \hline \end{array}$
(C_4)	$3 \times n$	$\begin{array}{\|c\|c\|}\hline q & q \\ \hline \end{array}$
(C_5)	$3 \times n$	$\begin{array}{\|c\|c\|}\hline q & q \\ \hline q & q \\ \hline \end{array}$

2. PROBABILITY FOR RELIABILITY

For all the cases, we will let P_n denote the probability of the $2 \times n$ or $3 \times n$ array being reliable. For the $2 \times n$ array, A_n, B_n, C_n, D_n will denote the respective probabilities of reliability of the array if the end 2×1 polydominoe has the form

$$\begin{array}{|c|}\hline p \\ \hline p \\ \hline \end{array}\ ,\quad \begin{array}{|c|}\hline p \\ \hline q \\ \hline \end{array}\ ,\quad \begin{array}{|c|}\hline q \\ \hline p \\ \hline \end{array}\ ,\quad \begin{array}{|c|}\hline q \\ \hline q \\ \hline \end{array}$$

and then

(1) $$P_n = A_n + B_n + C_n + D_n .$$

For the case (C_1), $D_n = 0$ and $B_n = C_n$. Here, for an A_{n+1} array, the end 2×2 polydominoe must have one of the three following forms:

$$\begin{array}{|c|c|}\hline p & p \\ \hline p & p \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|}\hline p & p \\ \hline q & p \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|}\hline q & p \\ \hline p & p \\ \hline \end{array}\ .$$

Thus,

(2) $$A_{n+1} = p^2(A_n + B_n + C_n) .$$

For a B_{n+1} array, the end 2×2 polydominoe must have one of the two forms

$$\begin{array}{|c|c|}\hline p & p \\ \hline p & q \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|}\hline q & p \\ \hline p & q \\ \hline \end{array}$$

and thus

(3) $$B_{n+1} = pq(A_n + C_n)$$

and similarly

(4) $$C_{n+1} = pq(A_n + B_n) .$$

On eliminating B_n and C_n, we obtain

(5) $$A_{n+1} = p^2 P_n ,$$

(6) $$P_{n+1} = pP_n + pqA_n ,$$

and then

(7) $$P_{n+1} = pP_n + p^3 q P_{n-1} .$$

For initial conditions, we have

(8) $$A_1 = p^2, \quad B_1 = pq = C_1, \quad D_1 = 0.$$

Whence,

(9) $$P_1 = 1 - q^2, \quad P_2 = 2p^2 - p^4 .$$

The solution of (7) is then given by

$$P_n = k_1 r_1^n + k_2 r_2^n ,$$

where r_1, r_2 are the roots of $x^2 = px + p^3 q$ and constants k_1, k_2 are determined so as to satisfy (9). This gives

(10) $$P_n = \frac{1 - q^2}{a}\left\{\left(\frac{p + a}{2}\right)^n - \left(\frac{p - a}{2}\right)^n\right\} + \frac{p^3 q}{a}\left\{\left(\frac{p + a}{2}\right)^{n-1} - \left(\frac{p - a}{2}\right)^{n-1}\right\} ,$$

where $a = \sqrt{p^2 + 4p^3 q}$.

For (C_2), it then follows as before that

(11) $$A_{n+1} = p^2(A_n + B_n + C_n + D_n) ,$$

(12) $$B_{n+1} = C_{n+1} = pq(A_n + B_n + C_n) ,$$

(13) $$D_{n+1} = q^2 A_n$$

subject to the initial conditions,

(14) $$A_1 = p^2, \quad B_1 = pq = C_1, \quad D_1 = q^2 .$$

Eliminating B_n, C_n, D_n, we obtain

(15) $$A_{n+2} = (p^2 + 2pq)A_{n+1} + p^2 q^2 A_n - 2p^3 q^3 A_{n-1} .$$

Whence,

$$A_n = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n ,$$

where r_1, r_2, r_3 are the roots of

$$x^3 = (p^2 + 2pq)x^2 + p^2 q^2 x - 2p^3 q^3$$

and the constants k_1, k_2, k_3 are determined from the initial conditions (note that here $A_1 = A_2 = p^2$, $A_3 = p^4[1 + 2pq + 5q^2]$). Then B_n, C_n, D_n and P_n are easily determined.

For (C_3), we have (11) and

(16) $$B_{n+1} = C_{n+1} = pq(A_n + B_n + C_n + D_n) ,$$

(17) $$D_{n+1} = q^2(A_n + B_n + C_n)$$

(again all subject to conditions (14)). On eliminating D_n, we obtain

(18) $$A_{n+1} = p^2 A_n + B_n + C_n + q^2(A_{n-1} + B_{n-1} + C_{n-1}) ,$$

(19) $$pB_{n+1} = pC_{n+1} = qA_{n+1} .$$

Whence,

(20) $$A_{n+1} = p(p + 2q)(A_n + q^2A_{n-1}) .$$

Then,

$$A_n = k_1r_1^n + k_2r_2^n ,$$

where r_i are the roots of

$$x^2 = p(p + 2q)(x + q3)$$

and the k_i's are determined from the initial conditions.

Then B_n, C_n, D_n and P_n are found from (14), (17) and (1).

For the $3 \times n$ arrays, we let A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n denote the respective probabilities of reliability of the array if the end 3×1 polydominoe has the form

$$\begin{array}{|c|}\hline p\\\hline p\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline p\\\hline p\\\hline q\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline p\\\hline q\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline p\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline q\\\hline p\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline p\\\hline q\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline p\\\hline q\\\hline q\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline q\\\hline q\\\hline q\\\hline\end{array}$$

and

(21) $$P_n = A_n + B_n + C_n + D_n + E_n + F_n + G_n + H_n .$$

For (C_4),

$$E_n = G_n = H_n = 0, \quad B_n = D_n ,$$

(22) $$A_1 = p^3, \quad B_1 = C_1 = D_1 = p^2q, \quad F_1 = pq^2 ,$$

(23) $$A_{n+1} = p^3(A_n + B_n + C_n + D_n + F_n) ,$$

(24) $$B_{n+1} = D_{n+1} = p^2q(A_n + C_n + D_n) ,$$

(25) $$C_{n+1} = p^2q(A_n + B_n + D_n + F_n) ,$$

(26) $$F_{n+1} = pq^2(A_n + C_n) .$$

For (C_5),

$$B_n = C_n = D_n, \quad E_n = G_n,$$

(27) $$A_1 = p^3, \quad B_1 = C_1 = D_1 = p^2q, \quad E_1 = F_1 = G_1 = pq^2, \quad H_1 = q^3 ,$$

(28) $$A_{n+1} = p^3 P_n ,$$

(29) $$B_{n+1} = p^2 q P_n ,$$

(30) $$E_{n+1} = pq^2(A_n + B_n + C_n + D_n + F_n + G_n) ,$$

(31) $$F_{n+1} = pq^2 P_n ,$$

(32) $$H_{n+1} = q^3(A_n + B_n + C_n + D_n + F_n) .$$

Although we can carry out the elimination process for (C_4) and (C_5) by means of the operator E and then determine P_n in terms of the roots of a higher order polynomial, it is not worthwhile. In these cases (and even some of the prior ones), one can just use a computer on the recurrence relations to determine the P_n's.

3. HIGHER ORDER POLYDOMINOES

The previous methods, with some adaptation, will also apply when the failure polydominoe is of higher order than the previous ones. As in the last two cases, it will suffice to just get the appropriate recurrence equations. If the failure polydominoe is of the type

q	q	q

in a $3 \times n$ array, then we would need terms A_n, B_n, $\cdots$, corresponding to a reliable $3 \times n$ array whose end 2×3 polydominoe has the forms

p	p
p	p
p	p

,

p	p
p	p
p	q

,

p	p
p	q
p	p

etc.

This will, of course, lead to an increased number of recurrence relations. Other arrays which can be solved similarly are cylindrical and torodial ones as well as higher dimensional ones.

THE FIRST SOLUTION OF THE CLASSICAL EULERIAN MAGIC CUBE PROBLEM OF ORDER TEN

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In this paper for the first time three Latin cubes of the tenth order have been superimposed to form an Eulerian cube. A Latin cube of the tenth order is defined as a cube of 1000 cells (in ten rows, ten columns, and ten files) in which 1000 numbers consisting of 100 zeros, 100 ones, $\cdots$, 100 nines, are arranged in the cells so that the ten numbers in each row, each column, and each file are different.

In this paper, we actually solved two problems, since in addition to having solved the Eulerian cube of order ten, we have also made the cube magic (for the first time). A magic cube is such that the ten cells in each diagonal (or "diameter") and in every row, every file, and every column is the same — namely, 4995 (see [1]).

In what follows, it will be noted that each of the ten SQUARES contain 100 cells and each cell contains a three-digit number. Now, if we delete the third digit on the right side in each and every cell, it is easily verified that each of the ten SQUARES has become pairwise orthogonal.

In 1779, Euler conjectured that no pair of orthogonal squares exist for $n \equiv 2 \pmod 4$. Then in 1959, the Euler conjecture was shown to be incorrect by the remarkable mathematics of Bose, Shrikande and Parker [2]. Recently (in 1972) Hoggatt and this author extended Bose, Shrikande and Parker's work by finding a way to make the 10×10 square pairwise orthogonal as well as magic. For a square to be magic, each of the two diagonals must have the same sum as in every row and in every column — namely (since we are considering the sum of ten cells with two digits in each cell), 495 (see [3]).

Let us label the cells in each square as follows: (row, column, square number) = (r, c, s) = some number in a cell. For example, the number 763 in Square Number 0 reads 763 = (0,0,0), or say we wish to consider the number 338 in Square Number 1: we then write 338 = (6, 2, 1).

THEN THE SUM OF EACH DIAGONAL (OR "DIAMETER") IN THE FOLLOWING FOUR-DIAMETER MAGIC CUBE IS, RESPECTIVELY,

$$\sum_{r,c,s=0}^{9} (r,c,s) = \sum_{r,c,s=0}^{9} (9-r,c,s) = \sum_{r,c,s=0}^{9} (r,9-c,s) = \sum_{r,c,s=0}^{9} (9-r,9-c,s) = 4995.$$

Now, let us define a magic route as that path which goes through ten different squares and passes through one cell in each square and no two cells that the route traverses are in

the same file, and the sum total of the numbers in the ten cells that make up this magic route equals 4995.

Then it may be easily shown that any cell in the cube begins a magic route. For example:

$$(4,2,0) + (8,4,1) + (6,0,2) + (0,7,3) + (5,8,4) + (9,5,5) + (1,3,6) + (3,1,7) + (2,6,8) + (7,9,9) = 4995.$$

For the convenience of the reader, we list, respectively, the numbers represented by notation above — 754, 321, 737, 575, 762, 003, 480, 396, 648, and 319.)

Note: The general method of how to find magic routes in singly-even magic cubes (except 2 and 6) will be given in the forthcoming paper mentioned above.

SQUARE NUMBER 0

763	886	540	979	015	428	601	354	232	197
279	963	097	654	832	301	728	186	440	515
897	340	463	201	579	632	154	915	028	786
140	454	901	063	628	715	879	297	586	332
932	228	754	815	163	086	597	401	379	640
328	697	132	740	486	563	215	079	954	801
554	032	286	128	701	997	363	840	615	479
415	779	828	532	397	240	986	663	101	054
686	501	315	497	254	179	040	732	863	928
001	115	679	386	940	854	432	528	797	263

SQUARE NUMBER 1

472	138	264	085	793	616	947	821	359	500
385	072	700	921	159	847	416	538	664	293
100	864	672	347	285	959	521	093	716	438
564	621	047	772	916	493	185	300	238	859
059	316	421	193	572	738	200	647	885	964
816	900	559	464	638	272	393	785	021	147
221	759	338	516	447	000	872	164	993	685
693	485	116	259	800	364	038	972	547	721
938	247	893	600	321	585	764	459	172	016
747	593	985	838	064	121	659	216	400	372

SQUARE NUMBER 2

190	924	771	313	808	565	289	637	446	052
413	390	852	237	946	689	165	024	571	708
952	671	590	489	713	246	037	308	865	124
071	537	389	890	265	108	913	452	724	646
346	465	137	908	090	824	752	589	613	271
665	252	046	171	524	790	408	813	337	989
737	846	424	065	189	352	690	971	208	513
508	113	965	746	652	471	324	290	089	837
224	789	608	552	437	013	871	146	990	365
889	008	213	624	371	937	546	765	152	490

SQUARE NUMBER 3

987	250	823	431	649	002	794	575	118	366
131	487	666	775	218	594	902	350	023	849
266	523	087	194	831	718	375	449	602	950
323	075	494	687	702	949	231	166	850	518
418	102	975	249	387	650	866	094	531	723
502	766	318	923	050	887	149	631	475	294
875	618	150	302	994	466	587	223	749	031
049	931	202	818	566	123	450	787	394	675
750	894	549	066	175	331	623	918	287	402
694	349	731	550	423	275	018	802	966	187

SQUARE NUMBER 4

606	541	355	727	434	999	010	162	883	278
827	706	478	062	583	110	699	241	955	334
578	155	906	810	327	083	262	734	499	641
255	962	710	406	099	634	527	878	341	183
783	899	662	534	206	441	378	910	127	055
199	078	283	655	941	306	834	427	762	510
362	483	841	299	610	778	106	555	034	927
934	627	599	383	178	855	741	006	210	462
041	310	134	978	862	227	455	683	506	799
410	234	027	141	755	562	983	399	678	806

SQUARE NUMBER 5

525	069	488	842	157	230	376	903	691	714
642	825	114	303	091	976	530	769	288	457
014	988	225	676	442	391	703	857	130	569
788	203	876	125	330	557	042	614	469	991
891	630	503	057	725	169	414	276	942	388
930	314	791	588	269	425	657	142	803	076
403	191	669	730	576	814	925	088	357	242
257	542	030	491	914	688	869	325	776	103
369	476	957	214	603	742	188	591	025	830
176	757	342	969	888	003	291	430	514	625

SQUARE NUMBER 6

044	312	136	698	961	777	453	280	505	829
598	644	929	480	305	253	077	812	736	161
329	236	744	553	198	405	880	661	977	012
836	780	653	944	477	061	398	529	112	205
605	577	080	361	844	912	129	753	298	436
277	429	805	036	712	144	561	998	680	353
180	905	512	877	053	629	244	336	461	798
761	098	377	105	229	536	612	444	853	980
412	153	261	729	580	898	936	005	344	677
953	861	498	212	636	380	705	177	029	544

SQUARE NUMBER 7

239	773	617	104	582	351	868	096	920	445
904	139	545	896	720	068	251	473	317	682
745	017	339	968	604	820	496	182	551	273
417	396	168	539	851	282	704	945	673	020
120	951	296	782	439	573	645	368	004	817
051	845	420	217	373	639	982	504	196	768
696	520	973	451	268	145	039	717	882	304
382	204	751	620	045	917	173	839	468	596
873	668	082	345	996	404	517	220	739	151
568	482	804	073	117	796	320	651	245	939

SQUARE NUMBER 8

311	495	909	556	270	884	122	748	067	633
056	511	233	148	467	722	384	695	809	970
433	709	811	022	956	167	648	570	284	395
609	848	522	211	184	370	456	033	995	767
567	084	348	470	611	295	933	822	756	109
784	133	667	309	895	911	070	256	548	422
948	267	095	684	322	533	711	409	170	856
870	356	484	967	733	009	595	111	622	248
195	922	770	833	048	656	209	367	411	584
222	670	156	795	509	448	867	984	333	011

SQUARE NUMBER 9

858	607	092	260	326	143	535	419	774	981
760	258	381	519	674	435	843	907	192	026
681	492	158	735	060	574	919	226	343	807
992	119	235	358	543	826	660	781	007	474
274	743	819	626	958	307	081	135	460	592
443	581	974	892	107	058	726	360	219	635
019	374	707	943	835	281	458	692	526	160
126	860	643	074	481	792	207	558	935	319
507	035	426	181	719	960	392	874	658	243
335	926	560	407	292	619	174	043	881	758

REFERENCES

1. Ball and Coxeter, Mathematical Recreations and Essays, Macmillan, 1962. Refer to Chapters 6 and 7, for some history on these classical problems, especially page 217, where Professor Coxeter points out that there are no known rules for constructing magic cubes of singly-even order.
2. This author learned of the remarkable results of Bose, Shrikande and Parker from two sources:
   a. Ball and Coxeter, Mathematical Recreations and Essays, Macmillan, 1962, p. 191.
   b. On August 29, 1972, in the lobby of the mathematics building at Dartmouth College, Hanover, New Hampshire, I had the privilege of viewing the magnificent 10×10 pairwise orthogonal mosaic square greeting you as you enter. The mosaic square was placed in this prominent position in honor of their genius.
3. Arkin and Hoggatt, "The Arkin-Hoggatt Game," presented in person at the Seventy-Seventh Summer Meeting, Dartmouth College, Hanover, New Hampshire, Aug. 29 - Sept. 1, 1972; appeared in the Notices of the Amer. Math. Soc., Vol. 19, No. 5, Issue 139, Aug. 1972, p. A-619, under the number 695-05-8.

NOTE: This paper was presented in person at Brown University, Providence, Rhode Island, 10/28/72; appeared in the AMS Notices, Vol. 19, No. 6, Issue 140, 10/72, p. A-728, under the number 697-A2.

REMARK. No computing machine of any type was used to get the results of this paper.

ON K-NUMBERS

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1. INTRODUCTION

We call K-numbers the numbers defined by

$$K(k,n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} m^k . \tag{1}$$

In [1, p. 249], the following results are given: $K(k,n) = 0$ for $k < n$ and $K(1,1) = (-1)^k k!$. We shall study general K-numbers and shall complete the definition by writing $K(k,n) = 0$ for $k,n \le 0$. We shall use two results given in [1]:

$$\sum_{s=0}^{t} \binom{s}{p} = \binom{t+1}{p+1} \tag{2}$$

cf. p. 246, No. 3, and

$$t^\alpha - \binom{p}{1}(t+1)^\alpha + \binom{p}{2}(t+2)^\alpha + \cdots + (-1)^p \binom{p}{p}(t+p)^\alpha = \sum_{q=0}^{p} (-1)^q (t+q)^\alpha \binom{p}{q} = 0 , \tag{3}$$

cf. p. 249.

The K-numbers are met in certain problems in combinatorics.

2. RECURRENCE RELATION

It will be observed in (1) that the term for $m = 1$ can be omitted since it is zero. Consider

$$K(k,n+1) = \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} m^k$$

and the difference

$$S = K(k,n+1) - K(k,n) = \sum_{m=0}^{n+1} (-1)^m \left[\binom{n+1}{m} - \binom{n}{m}\right] = \sum_{m=0}^{n+1} (-1)^m \binom{n}{m-1} m^k,$$

where use has been made of the relation

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}.$$

But

$$\binom{a}{b} = \frac{b+1}{a+1}\binom{a+1}{b+1},$$

so that

$$S = \frac{1}{n+1}\sum_{m=0}^{n+1}(-1)^m\binom{n+1}{m}m^{k+1} = K(k+1, n+1)/(n+1),$$

thus the K-numbers satisfy the recurrence relation,

$$K(k+1, n+1) = (n+1)[K(k, n+1) - K(k,n)] \tag{4}$$

or

$$K(k,n) = n[K(k-1, n) - K(k-1, n-1)]. \tag{4a}$$

3. NUMERICAL RESULTS

We observe that

$$K(k,1) = \sum_{m=0}^{1}(-1)^m\binom{1}{m}m^k = -1. \tag{5}$$

Using the results of Section 1 and (4), we obtain the following table of $K(k,n)$:

k \ n	1	2	3	4	5	6	7
1	-1						
2	-1	2					
3	-1	6	-6				
4	-1	14	-36	24			
5	-1	30	-150	240	-120		
6	-1	62	-540	1560	-1800	720	
7	-1	126	-1806	8400	-16800	15120	-5040

4. HORIZONTAL SUMS

Consider the "horizontal sum"

$$S = \sum_{n=0}^{k}K(k,n) = \sum_{n=0}^{k}\sum_{m=0}^{n}(-1)^m\binom{n}{m}m^k = \sum_{m=0}^{k}(-1)^m m^k\sum_{n=0}^{k}\binom{n}{m},$$

and using (2)

$$S = \sum_{m=0}^{k} (-1)^m m^k \binom{k+1}{m+1}.$$

Let in (3)

$$q = m + 1, \quad t = -1, \quad \alpha = k, \quad p = k + 1,$$

then (3) becomes

$$\sum_{m=-1}^{k+1} (-1)^{m+1} m^k \binom{k+1}{m+1} = 0,$$

or

$$(-1)^k - \sum_{m=0}^{k} (-1)^m m^k \binom{k+1}{m+1} = 0,$$

thus

$$S = \sum_{n=0}^{k} K(k,n) = (-1)^k. \tag{6}$$

5. GENERATING FUNCTION FOR THE K-NUMBERS

To find the generating function of the K-numbers, we use a technique given in [2]. We have

$$GK(k,n) = \sum_{k=0}^{\infty} K(k,n)t^k = u(n,t), \tag{7}$$

and

$$\sum_{k=n-1}^{\infty} K(k+1, n)t^k = GK(k,n)/t = u(n,t)/t$$

$$\sum_{k=n}^{\infty} K(k+1, n+1)t^k = GK(k, n+1)/t = u(n+1, t)/t = GK(k+1, n+1).$$

According to (4) it follows that

$$GK(k+1, n+1) = (n+1)\left[GK(k, n+1) - GK(k,n)\right]$$

or, substituting,

$$u(n+1, t)/t = (n+1)u(n+1, t) - (n+1)u(n,t),$$

which shows that $u(n,t)$ is a solution of the difference equation

(8) $$[1 - t(n + 1)]u(n + 1, t) + t(n + 1)u(n, t) = 0 .$$

We solve (8) using the classical technique given in [2] and obtain

$$u(n, t) = \frac{(2t)(3t)\cdots(nt)}{(2t - 1)(3t - 1)\cdots(nt - 1)} u(1, t) = n!\Gamma\left(2 - \frac{1}{t}\right)u(1, t)\Big/\Gamma\left(n + 1 - \frac{1}{t}\right).$$

According to (5), $K(k, 1) = -1$, thus

$$u(1, t) = \sum_{k=1}^{\infty} K(k, 1)t^k = -\sum_{k=1}^{\infty} t^k = t/(t - 1), \quad |t| < 1 ,$$

thus substituting into $u(n, t)$

(9) $$u(n, t) = GK(k, n) = n!\,\Gamma\left(1 - \frac{1}{t}\right)\Big/\Gamma\left(n + 1 - \frac{1}{t}\right) ,$$

which is the generating function for the K-numbers.

6. QUASI-ORTHOGONAL NUMBERS OF THE K-NUMBERS

According to [3] and correcting an error committed there, since the K-numbers satisfy a relation of the form

$$B_k^n = \frac{M(n + 1)}{N(n + 1)} B_{k-1}^n + \frac{1}{N(n)} B_{k-1}^{n-1} ,$$

where clearly (cf. (4a)), $N(n) = -1/n$, $M(n) = (n - 1)/n$, so that, still according to [3] the quasi-orthogonal numbers satisfy the relation

$$A_k^n = M(k)A_{k-1}^n + N(k)A_{k-1}^{n-1} ;$$

calling $L(k, n)$ the numbers quasi-orthogonal to the numbers $K(k, n)$, we have

(10) $$L(k, n) = \frac{(k - 1)}{k} L(k - 1, n) - \frac{1}{k} L(k - 1, n - 1) .$$

Through the quasi-orthogonality condition we get $L(k, k) = (-1)^k/k!$, and since $K(k, 1) = -1$ it follows that for $k > 1$,

$$\sum_{n=1}^{k} K(k, n) = 0 .$$

It will also be easily verified that $L(k,1) = -1/k$. We thus obtain the following table of values of $L(k,n)$:

	n: 1	2	3	4	5
k=					
1	-1				
2	-1/2	1/2			
3	-1/3	1/2	-1/6		
4	-1/4	11/24	-1/4	1/24	
5	-1/5	5/12	-7/24	1/12	-1/120

7. RELATIONS TO STIRLING NUMBERS

We consider the numbers

$$\omega(k,n) = (-1)^k k!\, L(k,n) ,$$

i.e.,

$$L(k,n) = (-1)^k \omega(k,n)/k! .$$

By substituting into (10) we obtain

$$\omega(k,n) = -(k-1)\omega(k-1, n) + \omega(k-1, n-1) ,$$

which is the recurrence relation for Stirling numbers of the first kind (cf. [2, p. 143]). Since $\omega(1,1) = 1 = St(1,1)$, $\omega(2,1) = -1 = St(2,1)$, etc., it follows that $\omega(k,n) = St(k,n)$, the Stirling numbers of the first kind, thus

$$L(k,n) = (-1)^k St(k,n)/k! . \tag{12}$$

Similarly it can be easily checked that the K-numbers are related to the Stirling numbers of the second kind $st(k,n)$ by the relation

$$K(k,n) = (-1)^n n!\, st(k,n) . \tag{12a}$$

REFERENCES

1. E. Netto, Lehrbuch der Kombinatorik, Chelsea, N. Y., Reprint of the second edition of 1927.
2. Ch. Jordan, Calculus of Finite Differences, Chelsea, N. Y., 1950.
3. S. Tauber, "On Quasi-Orthogonal Numbers," Amer. Math. Monthly, 72 (1962), pp. 365-372.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-215 Proposed by Ralph Fecke, North Texas State University, Denton, Texas

a. Prove

$$\sum_{i=n}^{n+2} 2^i P_i \equiv 0 \pmod 5$$

for all positive integers, n; P_i is the ith term of the Pell sequence, $P_1 = 1$, $P_2 = 2$, $P_{n+1} = 2P_n + P_{n-1}$ $(n \geq 2)$.

b. Prove $2^n L_n \equiv 2 \pmod{10}$ for all positive integers n; L_n is the nth term of the Lucas sequence.

H-216 Proposed by Guy A. R. Guillotte, 229 St. Joseph Blvd., Cowansville, Quebec, Canada.

Let G_m be a set of rational integers and

$$\sum_{n=1}^{\infty}\left[\log_e\left(\sum_{m=0}^{\infty}\frac{G_m}{(m)!\,(F_{2n+1})^m}\right)\right] = \frac{\pi}{4}\,.$$

Find a formula for G_m.

H-217 Proposed by S. Krishna, Orissa, India.

A. Show that

$$2^{4n-4x-4}\binom{2x+2}{x+1} \equiv \binom{4n-2x-2}{2n-x-1} \pmod{4n+1}\,,$$

where n is a positive integer and $-1 \le x \le 2n - 1$ and x is an integer, also.

B. Show that

$$2^{4n-4x-6}\binom{2x+4}{x+2} + \binom{4n-2x-2}{2n-x-1} \equiv 0 \pmod{4n+3},$$

where n is a positive integer and $-2 \le x \le 2n - 1$ and x is an integer, also.

H-218 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & & \cdots \\ 0 & 1 & 0 & & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ \cdots & 2 & 1 & & \cdots \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the staggered Pascal Triangle and

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the Pascal Binomial Array. Finally, let

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \end{pmatrix}_{n\times n}$$

represent the matrix corresponding to the Fibonacci Convolution Array. Prove $AB = C$.

H-219 Proposed by Paul Bruckman, University of Illinois, Urbana, Illinois.

Prove the identity

$$(-1)^n\binom{x}{n}\sum_{i=0}^{n}\binom{n}{i}(-2)^i \cdot \frac{x-n}{x-i} = \sum_{i=0}^{n}\binom{x}{i},$$

where

$$\binom{x}{i} = \frac{x(x-1)(x-2)\cdots(x-i+1)}{i!}$$

(x not necessarily an integer).

H-220 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{k=0}^{\infty} \frac{a^k z^k}{(z)_{k+1}} = \sum_{r=0}^{\infty} \frac{a^r q^{r^2} z^{2r}}{(z)_{r+1}(az)_{r+1}} ,$$

where

$$(z)_n = (1 - z)(1 - qz) \cdots (1 - q^{n-1}z), \qquad (z)_0 = 1 .$$

SOLUTIONS

ANOTHER PIECE

H-125 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define a sequence of positive integers to be left-normal if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be right-normal if there exists a member of the sequence ending with the string of digits.

Show that the sequence whose n^{th} terms are given by the following are left-normal but not right-normal.

a. $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.

b. P_n, where P_n is the nth prime.

c. $n!$

d. F_n, where F_n is the nth Fibonacci number.

Partial solution by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Using a theorem of R. S. Bird*, one may show that each of the above is left-normal. If

$$\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = \theta ,$$

where $\theta = 1$ or θ is not a rational power of 10 or if

$$\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{S_n S_{n+2}}{S_{n+1}^2} = 1 ,$$

then $\{S_n\}_{n=1}^{\infty}$ is left-normal (extendable in base 10).

a.
$$\lim_{n \to \infty} \frac{P(n + 1)}{P(n)} = 1 ,$$

hence $\{P(n)\}_{n=1}^{\infty}$ is left normal.

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*R. S. Bird, "Integers with Given Initial Digits," Amer. Math. Monthly, Apr. 1972, pp. 367-370.

c. $$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n!(n+2)!}{((n+1)!)^2} = 1 ,$$

thus $\{n!\}_{n=1}^{\infty}$ is left-normal.

d. $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} ,$$

thus $\{F_n\}_{n=1}^{\infty}$ is left-normal, also. The only question which remains is the demonstration that the sequences are not right-normal.

(c) is easy, since $n!$ is divisible by 4 for $n \geq 4$. Clearly no factorial, then, ends in 21, in particular. The final problem which remains is the question of right-normality for (a) and (d).

COMMENT ON H-174

H-174 Proposed by Daniel W. Burns, Chicago, Illinois.

Let k be any non-zero integer and $\{S_n\}_{n=1}^{\infty}$ be the sequence defined by $S_n = nk$.

Define the Burns Function, $B(k)$, as follows: $B(k)$ is the minimal value of n for which each of the ten digits, $0, 1, \cdots, 9$ have occurred in at least one S_m where $1 \leq m \leq n$. For example, $B(1) = 10$, $B(2) = 45$. Does $B(k)$ exist for all k? If so, find an effective formula or algorithm for calculating it.

Comment by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Using the theorem by Bird, referred to in the above H-125, we have

$$\lim_{n\to\infty} \frac{(n+1)k}{nk} = 1 \qquad \text{and} \qquad \{nk\}_{n=1}^{\infty}$$

is left-normal, or extendable in base 10. Thus, in particular, the sequence $123\cdots 90$ occurs at $\{nk\}_{n=1}^{\infty}$. The existence of $B(k)$ now follows by well ordering. One can show that $B(2k) > B(k)$ and other assorted inequalities.

ANOTHER REMARK

H-182 Proposed by S. Krishna, Orissa, India.

Prove or disprove:

(i) $$\sum_{k=1}^{m} \frac{1}{k^2} \equiv 0 \pmod{2m+1} ,$$

and

(ii) $$\sum_{k=1}^{m} \frac{1}{(2k-1)^2} \equiv 0 \pmod{2m+1}$$

when $2m + 1$ is prime and larger than 3.

Comment by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

It is well known* that

$$\text{(iii)} \qquad \sum_{k=1}^{2m} \frac{1}{k^2} \equiv 0 \pmod{2m + 1} ,$$

when $2m + 1$ is prime and larger than 3.

Set

$$\sigma_1 = \sum_{k=1}^{m} \frac{1}{k^2} \qquad \text{and} \qquad \sigma_2 = \sum_{k=1}^{m} \frac{1}{(2k - 1)^2} .$$

Using (iii), we have

$$1/4\sigma_1 + \sigma_2 \equiv 0 \pmod{2m + 1}$$

or

$$\sigma_1 + 4\sigma_2 \equiv 0 \pmod{2m + 1} .$$

From the above, it follows that (i) and (ii) are equivalent.

NOT THIS TIME

H-193 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Prove or disprove: If $x + y + z = 2^{2n+1} - 1$ and $x^3 + y^3 + z^3 = 2^{6n+1} - 1$, then $6n + 1$ and $2^{6n+1} - 1$ are primes.

Solution by Paul Bruckman, University of Illinois, Urbana, Illinois.

The following particular solution is sufficient to prove that the conjecture is false. If $(x, y, z) = (1, 2^{2n} - 2^n - 1, 2^{2n} + 2^n - 1)$, it is easily verified that this solution satisfies the requirements (a) $x + y + z = 2^{2n+1} - 1$, and (b) $x^3 + y^3 + z^3 = 2^{6n+1} - 1$. Moreover, this is true for all non-negative integers n, in particular when $n = 4$, i.e., $6n + 1 = 25$, which is <u>not</u> prime. It might be of interest to determine if any other solutions, not necessarily Diophantine, exist, although this was not attempted here.

Also solved by T. Carroll, D. Finkel, and D. Zeitlin.

The editor would like to acknowledge solutions to the following Problems:

<u>H-173, H-176</u> Clyde Bridger; <u>H-187</u> K. Wayland and D. Priest, E. Just, G. Wulczyn, and J. Ire; <u>H-190</u> L. Frohman, R. Fecke, L. Carlitz, P. Smith; <u>H-191</u> L. Carlitz; <u>H-192</u> L. Carlitz, D. Zeitlin, and P. Bruckman.

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*Hardy and Wright, <u>The Theory of Numbers</u>, Oxford University Press, London, 1962, p. 90.

THROUGH THE OTHER END OF THE TELESCOPE

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Base two has this interesting property that all integers may be represented uniquely by a sequence of zeros and ones. If instead of starting with base two, we had started with the sequence of ones and zeros and correlated the integers with them, then we would have seen that it is powers of two that correspond to a representation one followed by a number of zeros. This is what is meant in the title by looking through the other end of the telescope.

Table 1
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS

Representations	Integers	Representations	Integers
1	1	1001	9
10	2	1010	10
11	3	1011	11
100	4	1100	12
101	5	1101	13
110	6	1110	14
111	7	1111	15
1000	8	10000	16

If we continue this sequence of ones and zeros, will a one followed by zeros always be a power of two? Yes it will. For example, the four zeros in the representation of 16 will take on all the changes from 0001 to 1111 and bring us to 31 so that 100000 will be 32. In general, if there is a one followed by r zeros representing 2^r the last number that can be represented before increasing the number of digits will be:

$$2^r + (2^r - 1) = 2^{r+1} - 1 .$$

Thus, the next representation which is a 1 followed by $r + 1$ zeros will represent 2^{r+1}.

But is there anything particularly sacred about the way our sequence of ones and zeros has been chosen? Must it even be that the ones in various positions must represent the power of a number?

Suppose we change the rules for creating our succession of representations by insisting that no two ones be adjacent to each other.

Table 2
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS,
NO TWO ONES ADJACENT TO EACH OTHER

Representations	Integers	Representations	Integers	Representations	Integers
1	1	101010	20	10001000	39
10	2	1000000	21	10001001	40
100	3	1000001	22	10001010	41
101	4	1000010	23	10010000	42
1000	5	1000100	24	10010001	43
1001	6	1000101	25	10010010	44
1010	7	1001000	26	10010100	45
10000	8	1001001	27	10010101	46
10001	9	1001010	28	10100000	47
10010	10	1010000	29	10100001	48
10100	11	1010001	30	10100010	49
10101	12	1010010	31	10100100	50
100000	13	1010100	32	10100101	51
100001	14	1010101	33	10101000	52
100010	15	10000000	34	10101001	53
100100	16	10000001	35	10101010	54
100101	17	10000010	36	100000000	55
101000	18	10000100	37		
101001	19	10000101	38		

It is a matter of observation from this table that one followed by zeros is a Fibonacci number. If we take the series as $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, $\cdots$ then the one in the rth place from the right represents F_{r+1}.

Will this continue? Consider one followed by nine zeros or F_{10}. Since there may not be a one next to the first one, the numbers added to F_{10} in the succeeding representations are all the numbers up to and including 33, so that the final sum can be represented with ten digits is $55 + 34 - 1 = 89 - 1$. Thus one followed by ten zeros is 89 or F_{11}. A similar argument can be applied in general.

What happens if we insist that no two ones have less than two zeros between them? Again we can form a table. (See Table 3.) The sequence of integers that correspond to one followed by zeros is: 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, $\cdots$. Is there a law of formation of the sequence? It appears that

$$\begin{aligned} 9 &= 6 + 3 \\ 13 &= 9 + 4 \\ 19 &= 13 + 6 \\ 28 &= 19 + 9 \\ 41 &= 28 + 13 \end{aligned}$$

Table 3
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS,
NO TWO ONES SEPARATED BY LESS THAN TWO ZEROS

Representations	Integers	Representations	Integers	Representations	Integers
1	1	1000010	15	100000001	29
10	2	1000100	16	100000010	30
100	3	1001000	17	100000100	31
1000	4	1001001	18	100001000	32
1001	5	10000000	19	100001001	33
10000	6	10000001	20	100010000	34
10001	7	10000010	21	100010001	35
10010	8	10000100	22	100010010	36
100000	9	10001000	23	100100000	37
100001	10	10001001	24	100100001	38
100010	11	10010000	25	100100010	39
100100	12	10010001	26	100100100	40
1000000	13	10010010	27	1000000000	41
1000001	14	100000000	28		

or if the terms of the sequence are denoted by T_n,

$$T_{n+1} = T_n + T_{n-2} .$$

Will this continue? If we go beyond 41 the largest number that can be represented before increasing the number of digits is 1000000000 plus 1001001. Since this puts a 1 three places beyond the first 1 and is the largest number that can be represented of this type. Hence one followed by 10 zeros is 41 + 19 or 60. Evidently the argument can be applied in general.

Going one step further, we set the condition that two ones may not have less than three zeros between them.

Table 4
CORRELATION OF INTEGERS WITH 1-0 REPRESENTATIONS,
NO TWO ONES SEPARATED BY LESS THAN THREE ZEROS

Representations	Integers	Representations	Integers	Representations	Integers
1	1	100001	8	10000001	15
10	2	100010	9	10000010	16
100	3	1000000	10	10000100	17
1000	4	1000001	11	10001000	18
10000	5	1000010	12	100000000	19
10001	6	1000100	13	100000001	20
100000	7	10000000	14	100000010	21

(Table continues on the following page.)

Table 4 (Continued)

Representations	Integers	Representations	Integers	Representations	Integers
100000100	22	1000000001	27	1000010001	32
100001000	23	1000000010	28	1000100000	33
100010000	24	1000000100	29	1000100001	34
100010001	25	1000001000	30	1000100010	35
1000000000	26	1000010000	31	10000000000	36

We note that

$$14 = 10 + 4$$
$$19 = 14 + 5$$
$$26 = 19 + 7$$
$$36 = 26 + 10$$

suggesting the relation

$$T_{n+1} = T_n + T_{n-3} .$$

The following table summarizes the situation out to the case in which two ones may not have less than six zeros between them (system denoted S_6).

Table 5

NUMBERS REPRESENTED BY A UNIT IN THE nth PLACE FROM THE LEFT FOR VARIOUS ZERO SPACINGS

n	S_0	S_1	S_2	S_3	S_4	S_5	S_6
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	4	3	3	3	3	3	3
4	8	5	4	4	4	4	4
5	16	8	6	5	5	5	5
6	32	13	9	7	6	6	6
7	64	21	13	10	8	7	7
8	128	34	19	14	11	9	8
9	256	55	28	19	15	12	10
10	512	89	41	26	20	16	13
11	1024	144	60	36	26	21	17
12	2048	233	88	50	34	27	22
13	4096	377	129	69	45	34	28
14	8192	610	189	95	60	43	35
15	16384	987	277	131	80	55	43
16	32768	1597	406	181	106	71	53

(Table continues on following page.)

Table 5 (Continued)

n	S_0	S_1	S_2	S_3	S_4	S_5	S_6
17	65536	2584	595	250	140	92	66
18	131072	4181	872	345	185	119	83
19	262144	6765	1278	476	245	153	105
20	524288	10946	1873	657	325	196	133

To represent a given number in any one of these systems it is simply necessary to keep subtracting out the largest number less than or equal to the remainder. Thus to represent 342 (base 10) in S_4, we proceed as follows:

$$\begin{aligned} 342 - 325 &= 17 \\ 17 - 15 &= 2 . \end{aligned}$$

The representation is 10000000000100000010. Representations of 342 in all the systems are as follows.

S_0	101010110
S_1	101000101010
S_2	100010000001001
S_3	10001000100001000
S_4	10000000000100000010
S_5	100000000000001000001000
S_6	1000000000000000100000010

GENERATING FUNCTIONS OF THESE SYSTEMS

The following are somewhat more advanced considerations for the benefit of those who can pursue them. A generating function as employed here is an algebraic expression which on being developed into an infinite power series has for coefficients the terms of a given sequence. Thus for S_0, it can be found by a straight process of division that formally:

$$\frac{1}{1 - 2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots .$$

For S_1, the Fibonacci sequence, it is known that:

$$\frac{1 + x}{1 - x - x^2} = F_2 + F_3x + F_4x^2 + F_5x^3 + \cdots .$$

The process of determining such coefficients may be illustrated by this case. Set

$$\frac{1 + x}{1 - x - x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots ,$$

so that on multiplying through by $1 - x - x^2$,

$$1 + x = (1 - x - x^2)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) \quad .$$

This must be an identity so that the coefficients of the powers of x on the left-hand side must equal the coefficients of the corresponding powers of x on the right-hand side. Thus:

$$\begin{aligned}
&a_0 = 1 \\
&a_1 - a_0 = 1, && \text{so that} \quad a_1 = 2 \\
&a_2 - a_1 - a_0 = 0, && \text{so that} \quad a_2 = 3 \\
&a_3 - a_2 - a_1 = 0, && \text{so that} \quad a_3 = 5
\end{aligned}$$

and since in general $a_n - a_{n-1} - a_{n-2} = 0$, it is clear that the Fibonacci relation holds for successive sets of terms of the sequence, so that the Fibonacci numbers must continue to appear in order with $a_n = F_{n+2}$.

On the basis of the initial terms of the sequence and the type of recursion relation involved, the generating function for S_2 should be:

$$\frac{1 + x + x^2}{1 - x - x^3} ,$$

which can be verified in the same way as for S_1.

In general for S_k, the generating function would be:

$$\frac{1 + x + x^2 + \cdots + x^k}{1 - x - x^{k+1}} \quad .$$

CONCLUSION

There is an endless sequence of number representations involving only ones and zeros with the following properties:

1. In each system, every number has a unique representation.
2. In the system S_k (two ones separated by not less than k zeros), the recursion relation connecting the numbers represented by units in the various positions is:

$$T_{n+1} = T_n + T_{n-k} .$$

3. The well known unique representations in base 2 and by means of non-adjacent Fibonacci numbers (Zeckendorf's Theorem) are the first two of these number representations, namely, S_0 and S_1.

THE GOLDEN RATIO AND A GREEK CRISIS*

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The story of the discovery of irrational numbers by the school of Pythagoras around 500 B.C., and the devastating effect of that discovery on the Pythagorean philosophy is well known. On the one hand there was an undermining of the Pythagorean dictum "All is number," the conviction that everything in our world is expressible in terms of integers or ratios of integers. On the other hand, many geometric arguments were invalidated. Namely, those proofs requiring the existence of a common unit of measurement for any given pair of line segments were seen to be incomplete. Credit for the discovery of incommensurables is generally accorded to Hippasus of Metapontum. One may read, for example, in the excellent treatise of Van der Waerden [5], the legends of the fate that befell Hippasus for publicizing this and other secrets of the Pythagoreans. A brief and very readable account of these matters may be found in Meschkowski [4].

This note concerns itself with the question of how incommensurables might plausibly have been discovered. In particular, it will be seen how a study of the Golden Ratio could lead one to stumble onto the existence of incommensurable segments. The basic idea presented here is certainly not new and represents only a slight variant of ideas suggested in Meschkowski [4] and a definitive article by Heller [2]. It is hoped that the presentation given here might be of pedagogical value. In particular, a development along the lines given here might serve as a suitable vehicle for a classroom investigation of topics dealing with the history of irrational numbers or topics involving early Greek geometry and the Golden Ratio.

We begin by recalling that two line segments are <u>commensurable</u>, that is, have a common unit of measure, if each can be subdivided into smaller segments of equal length u (the length u being the same for both segments). In this case, if the two given segments have lengths a and b, respectively, we have

$$a = mu \qquad \text{and} \qquad b = nu \tag{1}$$

for some positive integers m and n. Thus, for commensurable segments, we have the ratio $a/b = m/n$ is a rational number. Conversely, if we are given two line segments of lengths a and b such that the ratio a/b is equal to m/n, where m and n are positive integers, then the number $u \equiv a/m = b/n$ will serve as a common unit of measure, so the segments are commensurable. Thus commensurable pairs of line segments are precisely those for

*Revised version of a lecture given before the Fibonacci Association in San Francisco on April 22, 1972.

which the ratio of the lengths is a rational number, and incommensurable pairs are those for which the ratio is an irrational number.

The best known example of an incommensurable pair of segments is given by a side and diagonal of a square. In a square, the ratio of diagonal length to side length is $\sqrt{2}$, which an easy number theoretic argument (as given in Book X of Euclid's Elements) shows to be irrational. But historical evidence indicates that the discovery of incommensurables came about in a purely geometric fashion, and the known geometric proofs that diagonal and side of a square are incommensurable seem to have the nature of being concocted after the initial discovery was well known. The reader will find the standard geometric argument in Eves [1, p. 60]. One would like to see a pair of line segments whose incommensurability can be more intuitively grapsed in a purely geometric manner. This is where the Golden Ratio engers the scene.

The Pythagoreans were much taken with the properties of the regular pentagon, whose vertices are also the vertices of the Pythagorean symbol of health, the regular five-pointed star.

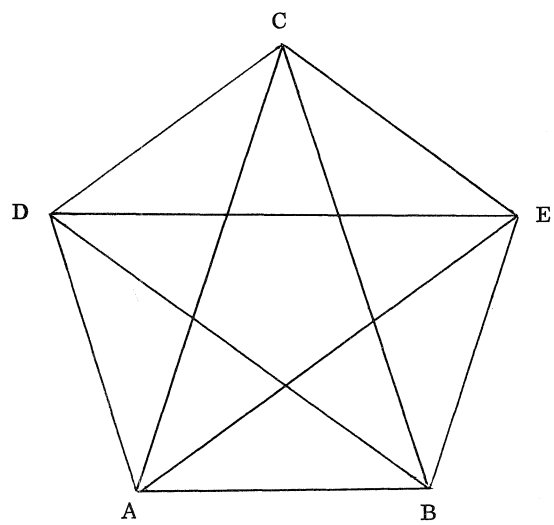

Figure 1

The Golden Ratio is the ratio of the diagonal length of a regular pentagon to the side length. Designating this ratio by the symbol ϕ, we have from Fig. 1,

$$\phi = \frac{AC}{AB} . \tag{2}$$

Some simple geometry shows that in Fig. 1, triangle ACB is an isosceles triangle with apex angle 36° and base angles 72° each. Such a triangle we shall call a Golden Triangle. Then the Golden Ratio is the ratio of side to base in any Golden Triangle. A property of the Golden

Triangle that undoubtedly intrigued the Pythagoreans is that when one draws the bisector of a base angle, there appears another smaller Golden Triangle. Thus in Fig. 2, if triangle ACB is a Golden Triangle and AD bisects the angle at A, then triangle BAD is also a Golden Triangle.

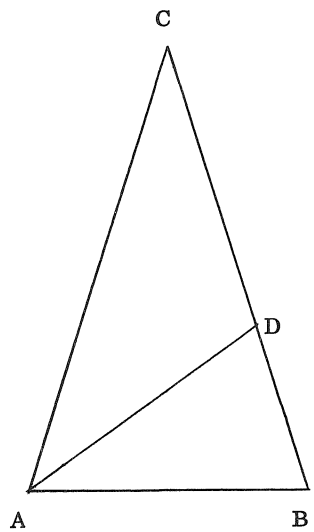

Figure 2

To see why this is true, observe in Fig. 2, that $\measuredangle$ BAD = $\measuredangle$ CAD = 36° and $\measuredangle$ ABD = 72°. It follows that $\measuredangle$ ADB = 72°, so triangle BAD is indeed a Golden Triangle. In this self-replicating property of the Golden Triangle lies the key to the incommensurability of its side and base. If one next draws the bisector of the angle at D to a point D' on AB, then draws the bisector of the angle at D' to a point D'' on BD, and continues this process indefinitely, one obtains an infinite sequence of smaller and smaller Golden Triangles. We shall see in a moment how the existence of this sequence contradicts the possibility that the side and base of the triangle might be commensurable.

It will be crucial to our argument to observe that in Fig. 2, AD = CD, which follows from the fact that $\measuredangle$ DAC = $\measuredangle$ DCA = 36°.

How then does one see geometrically that the side and base of a Golden Triangle are not commensurable? We might place ourselves in the sandals of an ancient Greek philosopher ruminating over a Golden Triangle ACB sketched in the sand. Wondering about a common unit of measure of AC and AB, we imagine it is possible to subdivide AC and AB into smaller segments all of the same length, say u. Subdividing BC into segments of the same length u we obtain an "evenly subdivided" triangle that might look something like triangle ACB in Fig. 3, where all the little segments are supposed to have the same length u. Now comes a crucial observation. Suppose we draw the bisector of the base angle at A, intersecting the opposite side in a point D. What can we say about D? The crucial observation

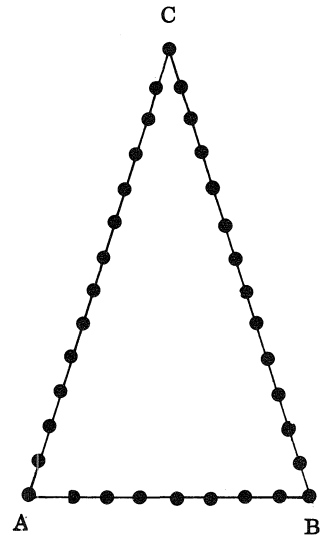

Figure 3

is that D must be one of the subdivision points! The reason is simple. Referring to Fig. 2, recall that AB = AD = CD. Thus CD, being equal to AB, must be an integral multiple of u, hence D must be a subdivision point. Thus appears a basic revelation: <u>If we have any evenly subdivided Golden Triangle, then the bisector of a base angle must strike the opposite side in a subdivision point.</u> Figure 4 illustrates this, with the bisector AD also subdivided.

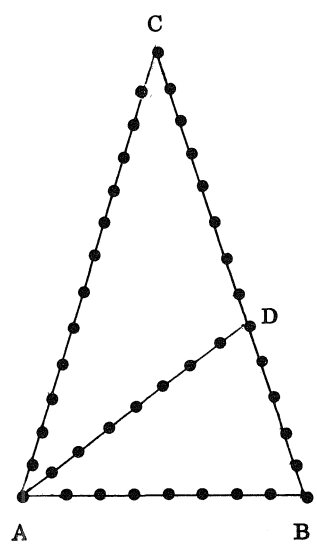

Figure 4

But triangle BAD is also an evenly subdivided Golden Triangle; hence if the bisector DD' of $\measuredangle$ ADB is drawn, the point D' where it strikes side AB must also be one of the original subdivision points, as indicated in Fig. 5.

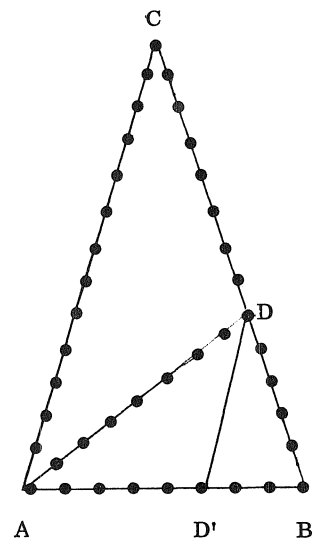

Figure 5

Repeating the process on the evenly subdivided Golden Triangle D'DB, we next see that the bisector of $\measuredangle$ BD'D must strike BD in one of the original subdivision points D''. It now becomes clear that we can repeat this procedure endlessly, drawing successively angle bisectors DD', D'D'', D''D''', $\cdots$, striking at each step the different subdivision points D', D'', D''', $\cdots$. Since at each step of this procedure we strike one of our original subdivision points, we have arrived at a contradiction, there being only finitely many such points. Thus we see that an evenly subdivided Golden Triangle is impossible, and hence the side and base are not commensurable.

It is of interest to examine an algebraic proof of the irrationality of the Golden Ratio ϕ that parallels the preceding geometric argument. We begin by deriving an important equation satisfied by ϕ. Since ACB and BAD are similar Golden Triangles in Fig. 2, we have

$$\phi = \frac{BC}{AB} = \frac{AB}{BD} = \frac{AB}{BC - DC} = \frac{AB}{BC - AB} = \frac{1}{\phi - 1} , \tag{3}$$

where we also used the fact that DC = AB and made some minor algebraic adjustments. If now we have $\phi = m/n$, with m, n positive integers, then Eq. (3) implies

$$\frac{m}{n} = \phi = \frac{1}{\phi - 1} = \frac{n}{m - n} . \tag{4}$$

Since $\phi > 1$, we automatically have $m > n$, and defining $m' = n$ and $n' = m - n$, we obtain $\phi = m'/n'$, with m', n' positive integers and $m > m'$. Repeating the process we obtain positive integers m'', n'' with $\phi = m''/n''$ and $m' > m''$. Repeating the procedure endlessly we obtain an infinite decreasing sequence of positive integers $m > m' > m'' > \cdots$, a contradiction. Hence there do not exist positive integers m, n such that $\phi = m/n$, and we have proved that ϕ is not rational.

In both the preceding proofs we may avoid the construction of infinite sequences by appealing to the fact that any nonempty set of positive integers contains a smallest element. In the case of our geometric proof, suppose there existed evenly subdivided Golden Triangles. With each such subdivided triangle associate the total number of subdivision points. Then there is a smallest such integer N and corresponding evenly subdivided Golden Triangle. But then by bisecting a base angle of this triangle we produce an evenly subdivided Golden Triangle with less than N total subdivision points. This contradiction shows that there exist no evenly subdivided Golden Triangles. In the case of our algebraic proof of the irrationality of ϕ, suppose there existed positive integers m and n such that $\phi = m/n$. With each such representation associate the numerator m. Then there is a smallest such integer m for which $\phi = m/n$. But then Eq. (4) gives $\phi = n/(m - n)$, which is a representation with still smaller numerator. The contradiction shows that there is no representation $\phi = m/n$ with positive integers m and n. Hence ϕ is not rational.

No discussion of these matters would be complete without mentioning how from Eq. (3), or from its equivalent

$$\phi = 1 + \frac{1}{\phi}, \tag{5}$$

one may obtain rational approximations to ϕ by ratios of successive Fibonacci numbers, with the analogous geometric approximations to a Golden Triangle by integer-sided triangles. Having mentioned it, we now leave it, hoping that any reader unfamiliar with these matters will, with whetted appetite, consult the fine book of Hoggatt [3] for a detailed exposition of the relationship between geometry and the Fibonacci numbers.

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INTERSECTIONS OF LINES CONNECTING TWO PARALLEL LINES

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The purpose of this note is to show that the geometrical method used by the author [1] in proving that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$ also can be used in proving the following result. Given two parallel lines with p points on the first, and q points on the second. Suppose each of the p points is joined by a straight line to each of the q points. Assume that between the parallel lines, no more than two lines intersect at any point. Then the lines joining the points have $\frac{1}{4}pq(p-1)(q-1)$ intersections between the parallel lines.

A proof of this result is as follows:

Label the p points $a_1, a_2, \cdots, a_p$ so that if the index j is greater than the index i, the directed line segment from a_i to a_j is in the same direction for each choice of i and j, $i, j = 1, 2, \cdots, p$ and $i < j$. (Thus the labeling is, for example, from left to right or bottom to top.) See Fig. 1. Label the q points $b_1, b_2, \cdots, b_q$ in a similar manner and so that for $i < j$, the directed line segment from b_i to b_j is in the opposite direction as that from a_1 to a_p. Denote by (a_i, b_j), $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$ the line between a_i and b_j.

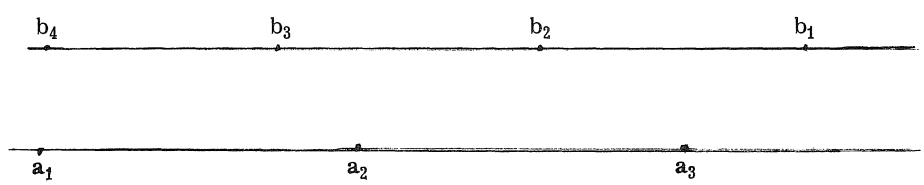

Fig. 1 $(p = 3, \ q = 4)$

Generally we shall place the pq lines sequentially in a certain order, to be specified, and count the number of intersections which arise. The order of placement is lexicographic:

$$(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_q), (a_2, b_1), \cdots, (a_2, b_q), \cdots, (a_k, b_1), \cdots, (a_k, b_q), \cdots, (a_p, b_1), \cdots, (a_p, b_q).$$

The first set of lines $(a_1, b_1), \cdots, (a_1, b_q)$ contributes no intersections. See Fig. 2.

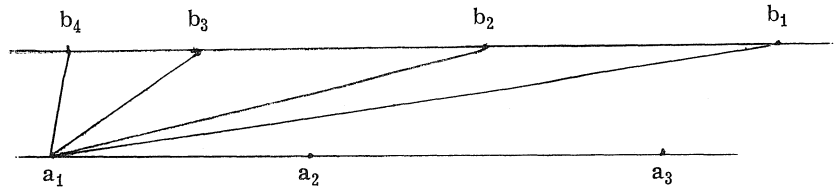

Fig. 2 $(p = 3, \ q = 4)$

Considering the second set of q lines $(a_2, b_1), \cdots, (a_2, b_k), \cdots, (a_2, b_q)$, none of them intersect with each other and (a_2, b_k) intersects with $(k - 1)$ previously placed lines:

$$(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_{k-1}) .$$

Thus these q lines contribute

$$(1 + 2 + \cdots + q - 1) = \frac{q(q - 1)}{2}$$

intersections. See Fig. 3.

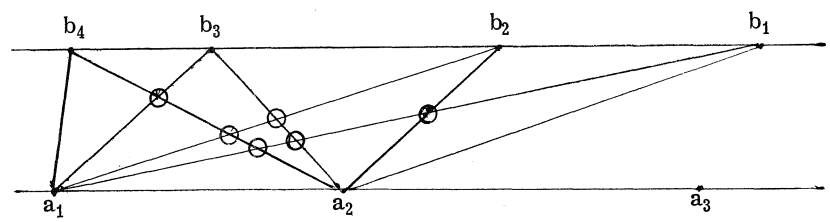

Fig. 3 $(p = 3, q = 4)$

The third set of q lines $(a_2, b_1), \cdots, (a_3, b_q)$ do not intersect with one another. The line (a_3, b_k) does intersect with the lines (a_1, b_j) and the lines (a_2, b_j) for $j = 1, 2, \cdots, (k - 1)$. Since here k may be equal to any of the integers $1, 2, \cdots,$ or q, the third set of lines contributes $2(1 + 2 + \cdots + q - 1)$ intersections. See Fig. 4.

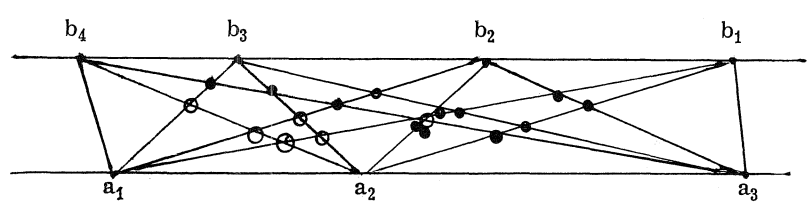

Fig. 4 $(p = 3, q = 4)$

Similarly, the rth set of q lines $(a_r, b_1), \cdots, (a_r, b_q)$, $r = 1, 2, \cdots, p$ do not intersect with each other. The line (a_r, b_k) intersects with the lines (a_i, b_j), $i = 1, 2, \cdots, r - 1$ and $j = 1, 2, \cdots, (k - 1)$. Thus placement of the line (a_r, b_1) contributes no intersections, placement of (a_r, b_2) contributes $(r - 1)(1)$ intersections, placement of (a_r, b_3) contributes $(r - 1)(2)$ intersections, placement of (a_r, b_k) contributes $(r - 1)(k - 1)$ intersections and finally, placement of (a_r, b_q) contributes $(r - 1)(q - 1)$ intersections. In total, the rth set of lines contributes $(r - 1)(1 + 2 + \cdots + q - 1)$ intersections.

Since the rth set contributes $(r - 1)(1 + 2 + \cdots + q - 1)$ intersections and the index r may be $1, 2, \cdots,$ or p we have that the total number of intersections is

$$\sum_{r=1}^{p} (r - 1)(1 + 2 + \cdots + q - 1) = (1 + 2 + \cdots + p - 1)(1 + 2 + \cdots + q - 1) ,$$

which is, as shown in [1] by the same method of sequential line placement, also equal to

$$\frac{p(p - 1)}{2} \frac{q(q - 1)}{2} = \frac{1}{4} pq(p - 1)(q - 1) .$$

Proof by G. Polya. In a private communication, Professor Polya has given the following shorter proof: Consider the trapezium of which the intersecting line segments are the diagonals. (See Fig. 5. The trapezium consists of (b_1, b_2), (b_3, a_2), (a_2, a_3), (a_3, b_1).)

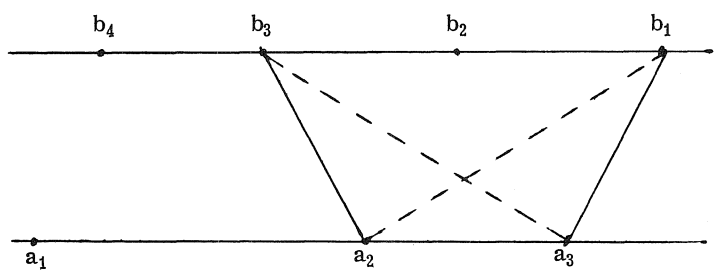

Fig. 5 $(p = 3, q = 4)$

Each trapezium is determined if a pair of points on each line is chosen and each different trapezium determines a different one of the intersections. Since there are

$$\binom{p}{2}\binom{q}{2} = \frac{p(p - 1)}{2} \cdot \frac{q(q - 1)}{2}$$

such choices, the result follows. This latter method of proof and the result are quite similar to the solution of the problem of finding the number of intersections of the diagonals of a convex polygon of n sides as discussed in [2].

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1. F. Stern, "The Sum of the First n Positive Integers — Geometrically," Fibonacci Quarterly, Vol. 9, No. 5, December, 1971, p. 526.
2. G. Polya, Mathematical Discovery, Vol. 1, Wiley, 1962, Problem 3.27, p. 83 and p. 178.

A GEOMETRIC TREATMENT OF SOME OF THE ALGEBRAIC PROPERTIES OF THE GOLDEN SECTION

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Our object is to present a single geometric setting in which it is possible to deduce some of the more familiar algebraic properties of the golden section, ϕ. In this setting, we also uncover some less familiar properties of ϕ and some extensions and generalizations of the "golden" sequence:

$$1, \phi, \phi^2, \phi^3, \cdots, \phi^n, \cdots .$$

The setting in which we will work is motivated by consideration of the following problem: construct a semi-circle on a given setment $\overline{AB}$. Locate a point P on the semi-circle so that the length of the projection of $\overline{PA}$ on $\overline{AB}$ is equal to PB. (See Fig. 1.) Since in right triangle APB, PB is the mean proportional between AB and GB, and since AG = PB, we conclude that G is the golden section of $\overline{AB}$. (For a more familiar construction of G, cf. [1].)

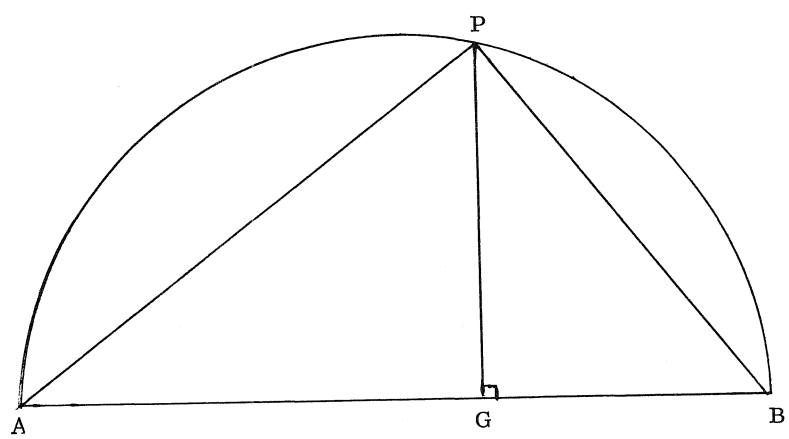

Figure 1

The right triangle APB is not "golden." In fact, if we normalize by taking GB = 1, then AG = PB = $(1 + \sqrt{5})/2$, which as usual, we denote by ϕ. Since $\overline{PG}$ is easily seen to have length $\sqrt{\phi}$, we deduce from right triangle PGB the property that $\phi^2 = \phi + 1$. Using this property in conjunction with right triangle APB, we conclude that PA = $\phi^{3/2}$. So, we

see that the ratio of the legs of right triangle APB is $\sqrt{\phi}$. Nevertheless, this normalized right triangle is the one with which we shall work.

At B construct a perpendicular on the same side of $\overline{AB}$ as P and extend $\overline{AP}$ until it meets the perpendicular at P_2. At P_2 construct a perpendicular meeting the extension of $\overline{AB}$ at B_2. Set $P_1 = P$, $B_0 = G$, and $B_1 = B$. (See Fig. 2.)

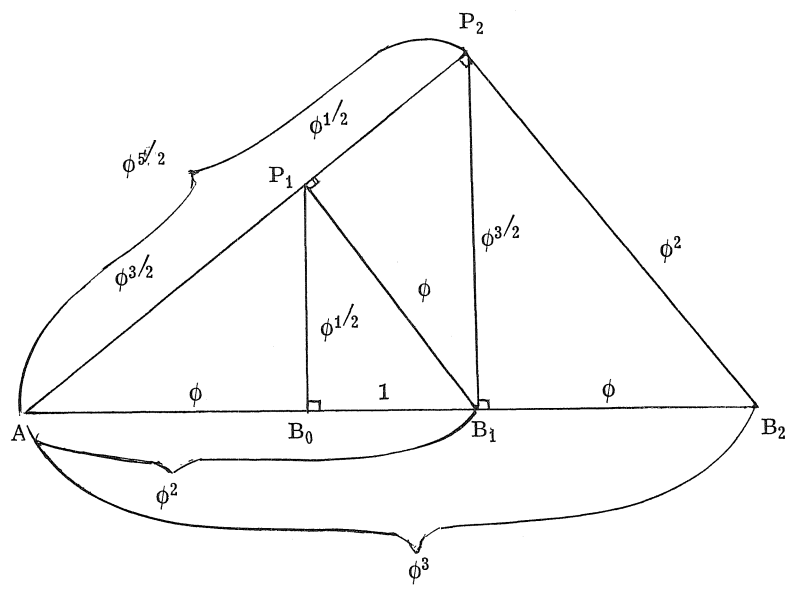

Figure 2

From above, we have that $AB_0 = \phi$, $B_0B_1 = 1$, $AB_1 = \phi^2$, $P_1B_1 = \phi$, $AP_1 = \phi^{3/2}$. The following lengths are easily deduced from these, the Pythagorean theorem, and similarity arguments: $P_1P_2 = \phi^{1/2}$, $P_2B_1 = \phi^{3/2}$, $AP_2 = \phi^{5/2}$, $B_1B_2 = \phi$, $P_2B_2 = \phi^2$. $AB_2 = AB_1 + B_1B_2 = \phi^2 + \phi = \phi(\phi + 1) = \phi^3$. (See Fig. 2.)

Since $AP_2 = AP_1 + P_1P_2$, and $AB_2 = AB_0 + B_0B_1 + B_1B_2$, we deduce (purely geometrically) two more properties: $\phi^{5/2} = \phi^{3/2} + \phi^{1/2}$, $\phi^3 = 2\phi + 1$.

The procedure for continuing now is clear: in the same manner as we constructed triangle AP_2B_2 from triangle AB_1P_1, we construct a new triangle AP_3B_2 from AP_2B_2, and so on. That is, we can generate a sequence of right triangles AP_1B_1, AP_2B_2, $\cdots$, AP_nB_n, $\cdots$ having the following characteristics:

(i) $$P_nB_n = \phi^n, \quad n = 1, 2, 3, \cdots$$

(ii) $$AB_n = \phi^{n+1}, \quad n = 0, 1, 2, \cdots$$

(iii) $$AP_n = \phi^{(2n+1)/2}, \quad n = 1, 2, 3, \cdots$$

(iv) $$P_{n+1}B_n = \phi^{(2n+1)/2}, \quad n = 0, 1, 2, \cdots$$

(v) $$B_nB_{n+1} = \phi^n, \quad n = 0, 1, 2, \cdots$$

(vi) $$P_nP_{n+1} = \phi^{(2n-1)/2}, \quad n = 1, 2, 3, \cdots$$

(vii) $$AB_n = AB_{n-1} + B_{n-1}B_n, \quad n = 1, 2, 3, \cdots,$$

implying $$\phi^{n+1} = \phi^n + \phi^{n-1}$$

(viii) $$AP_{n+1} = AP_n + P_nP_{n+1}, \quad n = 1, 2, 3, \cdots,$$

implying $$\phi^{(2n+3)/2} = \phi^{(2n+1)/2} + \phi^{(2n-1)/2}.$$

(ix)
$$\begin{aligned}
AB_1 &= AB_0 + B_0B_1, & \text{implying } \phi^2 &= \phi + 1\\
AB_2 &= AB_1 + B_1B_2, & \text{implying } \phi^3 &= 2\phi + 1\\
AB_3 &= AB_2 + B_2B_3, & \text{implying } \phi^4 &= 3\phi + 2\\
AB_4 &= AB_3 + B_3B_4, & \text{implying } \phi^5 &= 5\phi + 3\\
&\vdots & &\vdots
\end{aligned}$$

We note that the geometric result in (ix) demonstrates the equivalence of the geometric sequence $1, \phi, \phi^2, \phi^3, \cdots$ and the Fibonacci sequence $1, \phi, \phi + 1, 2\phi + 1, \cdots$, implying that the sequence $1, \phi, \phi^2, \phi^3, \cdots$ is a "golden" sequence.

Having constructed a sequence of triangles "on the right" of the golden cut G in Fig. 1, we now construct a sequence "on the left." Drop a perpendicular from G to $\overline{AP}$, intersecting $\overline{AP}$ at P_0. From P_0 drop a perpendicular to $\overline{AB}$ intersecting $\overline{AB}$ at B_1'. Repeat this procedure once more, obtaining P_1' on $\overline{AP}$ and B_2' on $\overline{AB}$. Set $B_0 = G$, $P_1 = P$, and $B_1 = B$. (See Figure 3.) From above, we know that $AB_0 = \phi = P_1B_1$, $B_0B_1 = 1$, $AP_1 = \phi^{3/2}$, and $P_1B_0 = \phi^{1/2}$. Our object is to compute the lengths of the remaining segments in Fig. 3. The same kinds of arguments as above result in the following:
$P_0P_1 = \phi^{-1/2}$, $AP_0 = \phi^{1/2}$, $P_0B_0 = 1$, $B_1'B_0 = \phi^{-1}$, $AB_1' = 1$, $P_0B_1' = \phi^{-1/2}$, $P_1'P_0 = \phi^{-3/2}$, $AP_1' = \phi^{-1/2}$, $P_1'B_1' = \phi^{-1}$, $B_2'B_1' = \phi^{-2}$, $AB_2' = \phi^{-1}$, and $P_1'B_2' = \phi^{-3/2}$. (See Figure 3.)

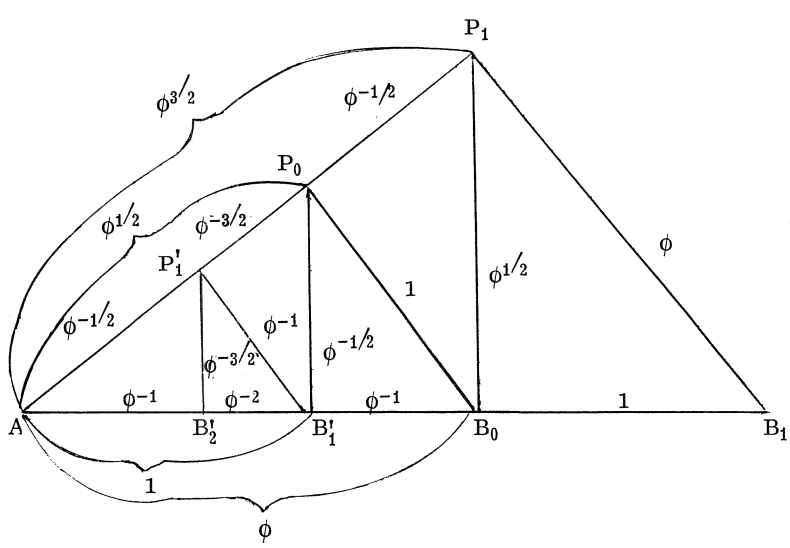

Figure 3

Again, the continuation procedure is clear: we can generate a sequence of right triangles AP_0B_0, $AP_1'B_1'$, $AP_2'B_2'$, $\cdots$, $AP_n'B_n'$, $\cdots$, with the following characteristics:

(i) $P_n'B_n' = \phi^{-n}$, $n = 0, 1, 2, \cdots$ (where $P_0'B_0' = P_0B_0$).

(ii) $AB_n' = \phi^{-n+1}$, $n = 0, 1, 2, \cdots$ (where $AB_0' = AB_0$).

(iii) $AP_n' = \phi^{-(2n-1)/2}$, $n = 0,1,2,\cdots$ (where $AP_0' = AP_0$).

(iv) $P_n'B_{n+1}' = \phi^{-(2n+1)/2}$, $n = 0,1,2,\cdots$ (where $P_0'B_1' = P_0B_1'$).

(v) $B_{n+1}'B_n' = \phi^{-(n+1)}$, $n = 0,1,2,\cdots$ (where $B_1'B_0' = B_1'B_0$).

(vi) $P_{n+1}'P_n' = \phi^{-(2n+3)/2}$, $n = 0,1,2,\cdots$ (where $P_1'P_0' = P_1'P_0$).

(vii) $AB_n' = AB_{n+1}' + B_{n+1}'B_n'$, $n = 0,1,2,\cdots$, implying

$$\phi^{-n+1} = \phi^{-n} + \phi^{-n-1}.$$

(viii) $AP_n' = AP_{n+1}' + P_{n+1}'P_n'$, $n = 0,1,2,\cdots$, implying

$$\phi^{-(2n-1)/2} = \phi^{-(2n+1)/2} + \phi^{-(2n+3)/2}.$$

(ix) $AB_0 = B_1'B_0 + B_2'B_1' + B_3'B_2' + \cdots + B_{n+1}'B_n' + \cdots$, implying

$$\phi = 1/\phi + 1/\phi^2 + 1/\phi^3 + \cdots + 1/\phi^{+(n+1)} + \cdots.$$

(x) $AP_0 = P_1'P_0 + P_2'P_1' + P_3'P_2' + \cdots + P_{n+1}'P_n' + \cdots$, implying

$$\phi^{1/2} = \phi^{-3/2} + \phi^{-5/2} + \phi^{-7/2} + \cdots + \phi^{-(2n+3)/2} + \cdots$$

<u>Remark 1:</u> A more familiar geometric setting for property (ix) is a rectangular spiral (cf. [1]) the length of whose n^{th} side is ϕ^{-n}. (See Fig. 4.) Figure 3 suggests that if one were to "unfold" the rectangular spiral onto a straight line, the union of the sides, $\overline{B_{n+1}'B_n'}$, would be a segment with length equal to AB_0. (An analogous remark can be made for property (x).)

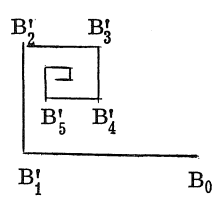

Figure 4

<u>Remark 2:</u> Figure 3 suggests the following generalization of properties (ix) and (x):

$$\phi^k = \phi^{k-2} + \phi^{k-3} + \cdots + \phi^{-1} + 1 + \phi + \phi^2 + \cdots,$$

for $k = 0, \pm 1/2, \pm 2/2, \pm 3/2, \cdots$.

In Figure 2, we see that the "golden sequence," $1, \phi, \phi^2, \cdots, \phi^n, \cdots$ has its geometric analogue in the sequence of altitudes P_0B_0, P_1B_1, $P_2B_2, \cdots, P_nB_n, \cdots$. In that same figure, the sequence of altitudes P_1B_0, P_2B_1, $P_3B_2, \cdots, P_{n+1}B_n, \cdots$ suggests a second sequence which is also golden: $\phi^{1/2}$, $\phi^{3/2}$, $\phi^{5/2}, \cdots$. (That this sequence is geometric is clear; property (viii) demonstrates that it is also a Fibonacci sequence.) The following are additional extensions of the golden sequence suggested by the appropriate sequences of altitudes in Figs. 2 and 3 (we include the above two for completeness):

(1) $1, \phi, \phi^2, \phi^3, \cdots, \phi^{n-1}, \cdots$ (Golden Sequence)

(2) $1/\phi, 1/\phi^2, 1/\phi^3, \cdots, 1/\phi^n, \cdots$

(3) $\cdots, 1/\phi^n, 1/\phi^{n-1}, \cdots, 1/\phi, 1, \phi, \phi^2, \phi^3, \cdots, \phi^{n-1}, \cdots$

(4) $\phi^{1/2}, \phi^{3/2}, \phi^{5/2}, \cdots, \phi^{(2n-1)/2}, \cdots$

(5) $\phi^{-1/2}, \phi^{-3/2}, \phi^{-5/2}, \cdots, \phi^{-(2n-1)/2}, \cdots$

(6) $\cdots \phi^{-(2n-1)/2}, \cdots, \phi^{-1/2}, \phi^{1/2}, \phi^{3/2}, \phi^{5/2}, \cdots, \phi^{(2n-1)/2}, \cdots$

As a final remark, we consider the sequence suggested by the complete sequence of altitudes in Fig. 3:

$$\cdots, P'_nB'_{n+1}, P'_nB'_n, P'_{n-1}B'_n, P'_{n-1}B'_{n-1}, \cdots, P'_1B'_2, P'_1B'_1, P_0B'_1, P_0B_0,$$
$$P_1B_0, P_1B_1, \cdots, P_nB_{n-1}, P_nB_n, \cdots.$$

This geometric sequence, with ratio $\phi^{1/2}$, is evidently

$$\cdots, \phi^{-(2n+1)/2}, \phi^{-n}, \phi^{-(2n-1)/2}, \phi^{-(n-1)}, \cdots, \phi^{-3/2}, \phi^{-1}, \phi^{-1/2}, 1,$$
$$\phi^{1/2}, \phi, \cdots, \phi^{(2n-1)/2}, \phi^n, \cdots.$$

Although this is not a Fibonacci sequence (and Hence, not golden), it contains each of the golden sequences, (1)-(6), as subsequences, and has the easily verified property that any subsequence consisting of alternate terms of the sequence, is in fact, a golden sequence.

REFERENCE

1. H. E. Huntley, The Divine Proportion, Dover, New York, 1970.

ERRATA

In "Ye Olde Fibonacci Curiosity Shoppe," appearing in Vol. 10, No. 4, October, 1972, please make the following changes:

Page 443: In the first line of the second paragraph, insert the word "ten" between "of" and "consecutive," so that it reads "... the sum of the squares of ten consecutive Fibonacci numbers is always divisible by $F_{10} = 55$."

GENERALIZED FIBONACCI SHIFT FORMULAS

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A Fibonacci sequence is one which is governed by the relation

$$T_{n+1} = T_n + T_{n-1} \tag{1}$$

among successive terms. From this simple beginning we arrive at numerous recursion relations for related sequences of various types. These form the starting points for the generalized Fibonacci shift formulas to be treated in this paper.

Given any number of Fibonacci sequences such as:

$$\begin{aligned} F_n&: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \cdots \\ L_n&: 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \cdots \\ T_n&: 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, 1741, 2817, 4558, \cdots \end{aligned} \tag{2}$$

We can consider terms such as F_{n-3}, L_{n+4}, T_{n-1} and combine these to form a Fibonacci expression of some degree. For example

$$F^2_{n-3} L_{n+4} T_{n-1} + L^4_{n+4} - T^3_{n-1} F_{n-3}$$

would be spoken of as a homogeneous Fibonacci expression of the fourth degree. But we might also have subsets of terms of these sequences in which only every other term is involved in successive values of the given expression. Such terms would be of the form F_{2n+a}, L_{2n+b}, T_{2n+c}, where a, b, and c are fixed constants. In general, if terms are of the form F_{mn+a}, L_{mn+b}, T_{mn+c}, speak of such terms as being of class m.

Now the recursion relations for homogeneous expressions of any degree for a given class are Fibonacci coefficients which are built up from certain subsequences of the Fibonacci sequence (F_n). For class 1 $(m = 1)$, the Fibonomial coefficients are formed from F_1, F_2, F_3, F_4, $\cdots$. In analogy to the binomial coefficients

$$\binom{r}{j} = \frac{r!}{j!\,(r-j)!} = r_j / j! \quad , \tag{3}$$

where $r_j = r(r-1)(r-2)\cdots(r-j+1)$, we have for the Fibonomial coefficients

(4)
$$F[r,j] = \frac{[F_r]!}{[F_j]!\,[F_{r-j}]!}$$
$$= \frac{F_r F_{r-1} F_{r-2} \cdots F_2 F_1}{F_j F_{j-1} F_{j-2} \cdots F_2 F_1 F_{r-j} F_{r-j-1} \cdots F_2 F_1}$$
$$= [F_r]_j / [F_j]! \quad ,$$

where

$$[F_r]_j = F_r F_{r-1} \cdots F_{r-j+1} \ .$$

For example,

$$F[10,4] = \frac{F_{10} F_9 F_8 F_7}{F_1 F_2 F_3 F_4} = 85085 \ .$$

For class 2 $(m = 2)$, the Fibonomial coefficients are formed from F_2, F_4, F_6, F_8, F_{10}, $\cdots$. And in general for class m, the Fibonomial coefficients are formed from F_m, F_{2m}, F_{3m}, F_{4m}, $\cdots$. (For background on the Fibonomial coefficients and their relation to Fibonacci recursion formulas see references 1, 2, 3, 4, 5, and 6.)

To distinguish the various types of Fibonomial coefficients, we shall introduce the following symbolism.

$$F[\text{class, order, index}] = F[m,r,n] \ .$$

Thus the previously given $F[10,4]$ would be written $F[1,10,4]$. As another example

$$F[2,8,3] = (F_{16} F_{14} F_{12})/(F_2 F_4 F_6) = 2232594 \ .$$

It may be noted that $F[m,r,0] = 1$ by definition.

We need as well symbolism for homogeneous Fibonacci expressions of class m, degree d and running subscript n. These will be denoted

$$H[\text{class, degree, subscript}] = H[m,d,n] \ .$$

Thus we might have

$$H[3,4,n] = F_{3n} L_{3n+1}^2 T_{3n-2} - F_{3n+2}^4 \ .$$

The starting point of our shift formulas can be given as follows. For m even,

(5)
$$H[m, d, n+1] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m, d+1, i] H[m, d, n+1-i] \ .$$

For m odd,

$$H[m, d, n+1] = \sum_{i=1}^{d+1} (-1)^{[(i-1)/2]} F[m, d+1, i] H[m, d, n+1-i] \quad , \tag{6}$$

where the square brackets in the exponent signify the greatest integer function. Upward shift formulas can be derived from (5) and (6). For m even, we have

$$H[m, d, n-1] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m, d+1, i] H[m, d, n+i-1] \tag{7}$$

For m odd and d odd,

$$H[m, d, n-1] = \sum_{i=1}^{d+1} (-1)^{[(i+2)/2]} F[m, d+1, i] H[m, d, n+i-1] \tag{8a}$$

For m odd and d even,

$$H[m, d, n-1] = \sum_{i=1}^{d+1} (-1)^{[(i-1)/2]} F[m, d+1, i] H[m, d, n+i-1] \quad . \tag{8b}$$

It may be noted that in all the formulas (5) through (8) the coefficients are numerically the same differing only in sign.

The shift formulas considered in this article give $H[m, d, n+k]$ in terms of $H[m, d, n]$, $H[m, d, n-1]$, $\cdots$ for a downward shift of k. A second set of formulas give $H[m, d, n-k]$ in terms of $H[m, d, n]$, $H[m, d, n+1]$, $\cdots$ for an upward shift of k.

We shall proceed by examining the linear case for various classes, then expressions of the second degree, and so on until formulas applying to all degrees and classes emerge.

FIRST DEGREE. $m = 1$

$$H[1, 1, n+1] = H[1, 1, n] + H[1, 1, n-1] \quad .$$

We substitute $H[1, 1, n] = H[1, 1, n-1] + H[1, 1, n-2]$. To avoid a great deal of writing the following scheme showing only coefficients will be employed.

$k = 1$	1	1				
		1	1			Substitution line
$k = 2$		2	1			
			2	2		
$k = 3$			3	2		
				3	3	
$k = 4$				5	3	

It appears that $H[1, 1, n+1] = F_{k+1} H[1, 1, n] + F_k H[1, 1, n-1]$ a well known Fibonacci shift formula.

In the other direction $H[1, 1, n-1] = -H[1, 1, n] + H[1, 1, n+1]$. Schematically

$$\begin{array}{lcccc}
k = 1 & -1 & 1 & & \\
 & & \underline{+1} & \underline{-1} & \\
k = 2 & & 2 & -1 & \\
 & & & \underline{-2} & \underline{2} \\
k = 3 & & & -3 & 2 \\
 & & & & \underline{3} \quad \underline{-3} \\
k = 4 & & & & 5 \quad -3
\end{array}$$

leading to the relation

$$H[1, 1, n-k] = (-1)^k\{F_{k+1}H[1, 1, n] - F_kH[1, 1, n+1]\}$$

<u>m = 2</u>

The initial relation is

$$H[2, 1, n+1] = 3H[2, 1, n] - H[2, 1, n-1] \quad .$$

In terms of coefficients

$$\begin{array}{lcccc}
k = 1 & 3 & -1 & & \\
 & & \underline{9} & \underline{-3} & \\
k = 2 & & 8 & -3 & \\
 & & & \underline{24} & \underline{-8} \\
k = 3 & & & 21 & -8 \\
 & & & & \underline{63} \quad \underline{-21} \\
k = 4 & & & & 55 \quad -21
\end{array}$$

Generalizing

$$H[2, 1, n+k] = F_{2k+2}H[2, 1, n] - F_{2k}H[2, 1, n-1] \ .$$

For $m = 3$, coefficients are as follows:

$k = 1$	4	1
$k = 2$	17	4
$k = 3$	72	17
$k = 4$	305	72

To identify these quantities, consult a table of Fibonomial coefficients (see [7], for example) for $m = 2$. Then it can be seen that

$$H[3, 1, n+k] = F[3, k+1, 1]H[3, 1, n] + F[3, k, 1]H[3, 1, n-1] \quad .$$

To conclude, the formulas that apply in the first degree case are as follows.

<u>m odd</u>

$$H[m, 1, n+k] = F[m, k+1, 1]H[m, 1, n] + F[m, k, 1]H[m, 1, n-1]$$

$$H[m, 1, n-k] = (-1)^k\{F[m, k+1, 1]H[m, 1, n] - F[m, k, 1]H[m, 1, n+1]\}$$

<u>m even</u>

$$H[m, 1, n+k] = F[m, k+1, 1]H[m, 1, n] - F[m, k, 1]H[m, 1, n-1]$$

$$H[m, 1, n-k] = F[m, k+1, 1]H[m, 1, n] - F[m, k, 1]H[m, 1, n+1]$$

SECOND DEGREE

As an example of the way in which the coefficients are calculated consider the case $m = 3$ and downward shift.

$k = 1$	17	17	-1			
		289	289	-17		
$k = 2$		306	288	-17		
			5202	5202	-306	
$k = 3$			5490	5185	-306	
				93330	93330	-5490
$k = 4$				98515	93024	-5490

The leading quantity identifies as $F[3, k+2, 2]$, the final quantity as $F[3, k+1, 2]$. The middle quantity is $F[3, k+2, 1]\ F[3, k, 1]$.

The general formulas for the second degree are as follows:

m odd

$$H[m, 2, n+k] = F[m, k+2, 2]H[m,2,n] + F[m, k+2, 1]F[m,k,1]H[m, 2, n-1] - F[m, k+1, 2]H[m, 2, n-2]$$

$$H[m, 2, n-k] = F[m, k+2, 2]H[m,2,n] + F[m, k+2, 1]F[m,k,1]H[m, 2, n+1] - F[m, k+1, 2]H[m, 2, n+2].$$

m even

$$H[m, 2, n+k] = F[m, k+2, 2]H[m, 2, n] - F[m, k+2, 1]F[m,k,1]H[m, 2, n-1] + F[m, k+1, 2]H[m, 2, n-2]$$

$$H[m, 2, n-k] = F[m, k+2, 2]H[m,2,n] - F[m, k+2, 1]F[m,k,1]H[m, 2, n+1] + F[m, k+1, 2]H[m, 2, n+2].$$

For the third and fourth degrees, the results are given only for the downward shift case since the coefficients in the upward shift case are numerically the same.

THIRD DEGREE

m odd

$$H[m, 3, n+k] = F[m, k+3, 3]H[m, 3, n] + F[m, k+3, 2]F[m, k, 1]H[m, 3, n-1] - F[m, k+3, 1]F[m, k+1, 2]H[m, 3, n-2] - F[m, k+2, 3]H[m, 3, n-3].$$

m even

Same coefficients and functions with signs $+ - + -$.

FOURTH DEGREE

m odd

$$H[m, 4, n+k] = F[m, k+4, 4]H[m,4,n] + F[m, k+4, 3]F[m,k,1]H[m, 4, n-1] - F[m, k+4, 2]F[m, k+1, 2]H[m, 4, n-2] - F[m, k+4, 1]F[m, k+2, 3]H[m, 4, n-3] + F[m, k+3, 4]H[m, 4, n-4].$$

m even

Same expression with alternating signs.

The situation can be summarized as follows as a working hypothesis.

1. The coefficients for the downward and upward formulas are the same in absolute value for corresponding terms.
2. These coefficients for class m, degree d and shift k are as follows:

$$F[m, d+k, d]F[m, k-1, 0]$$
$$F[m, d+k, d-1]F[m, k, 1]$$
$$F[m, d+k, d-2]F[m, k+1, 2]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F[m, d+k, 1]F[m, k+d-2, d-1]$$
$$F[m, d+k, 0]F[m, k+d-1, d] .$$

3. The sign patterns for the various cases are:

Odd degree	Down	Up
m odd	+ + - -	$(-1)^k$[+ - - +]
m even	+ - + -	+ - + -
Even degree		
m odd	+ + - -	+ + - -
m even	+ - + -	+ - + -

Formulas covering all cases are as follows:

m odd

$$(9)\quad H[m,d,n+k] = \sum_{i=1}^{d+1} (-1)^{[(i-1)/2]} F[m,k+d, d-i+1]F[m,k+i-2, i-1]H[m,d, n-i+1] .$$

(10a) for d odd

$$H[m,d, n-k] = (-1)^k \sum_{i=1}^{d+1} (-1)^{[i/2]} F[m, k+d, d-i+1]F[m, k+i-2, i-1]H[m,d,n+i-1] .$$

(10b) for d even

$$H[m, d, n-k] = \sum_{i=1}^{d+1} (-1)^{[(i-1)/2]} F[m, k+d, d-i+1]F[m, k+i-2, i-1]H[m, d, n+i-1] .$$

m even

$$(11)\quad H[m,d, n+k] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m, k+d, d-i+1]F[m, k+i-2, i-1]H[m, d, n-i+1] .$$

$$(12)\quad H[m,d,\ n-k] = \sum_{i=1}^{d+1} (-1)^{i-1} F[m,\ k+d,\ d-i+1]F[m, k+i-2, i-1]H[m,\ d,\ n+i-1]\ .$$

PROOF OF THESE FORMULAS

The formulas are true for $k = 1$ since formulas (9) through (12) reduce to the respective formulas in (5) through (8). Assume then that these formulas are true for k. To go to $k + 1$, substitute for $H[m,\ d,\ n]$ either up or down as the case may be, using formulas (5) through (8). Consider first the case of downward shift. For m odd, the sign pattern for terms after substitution is:

$$\begin{array}{c} + - - + + - - + \\ + + - - + + - - \end{array}$$

where the second line represents the sign pattern for the substituted quantity $H[m,\ d,\ n]$. For m even, the sign pattern for terms after substitution is

$$\begin{array}{c} - + - + - + \\ + - + - + - \end{array}$$

In either case the pattern varies modulo 4.

<u>m odd</u>

By (5),

$$H[m,\ d,\ n] = F[m,\ d+1,\ 1]H[m,\ d,\ n-1] + F[m,\ d+1,\ 2]H[m,\ d,\ n-2]$$
$$- F[M,\ d+1,\ 3]H[m,\ d,\ n-3] - F[m,\ d+1,\ 4]H[m,\ d,\ n-4]\cdots\ .$$

Substitution into (9) gives:

$$\begin{aligned} H[m,\ d,\ n+k] = {} & \{F[m, d+k,\ d-1]F[m,k,1] + F[m,\ d+1,1]F[m,\ d+k,d]\}H[m,\ d,\ n-1] \\ & + \{-F[m,\ d+k,\ d-2]F[m,\ k+1,2] + F[m,\ d+1,2]F[m,\ d+k,d]\}H[m,d,n-2] \\ & + \{-F[m,\ d+k,d-3]F[m,\ k+2,3] - F[m,\ d+1,3]F[m,\ d+k,d]\}H[m,d,n-3] \\ & + \{F[m,\ d+k,\ d-4]F[m,\ k+3,4] - F[m,\ d+1,4]F[m,\ d+k,\ d]\}H[m,d,\ n-4] \\ & \quad \cdots \end{aligned}$$

COEFFICIENT OF $H[m,\ d,\ n-4j-1]$

This is given by:

$$F[m,\ d+k,\ d-4j-1]F[m,\ k+4j,\ 4j+1] + F[m,\ d+1,\ 4j+1]F[m,\ d+k,\ d]$$

$$= \frac{F_{m(d+k)}F_{m(d+k-1)}\cdots F_{m(k+4j+2)}}{F_mF_{2m}\cdots F_{m(d-4j-1)}}\ \frac{F_{m(k+4j)}F_{m(k+4j-1)}\cdots F_{mk}}{F_mF_{2m}\cdots F_{m(4j+1)}}$$

$$+ \frac{F_{m(d+1)}F_{md}\cdots F_{m(d-4j+1)}}{F_mF_{2m}\cdots F_{m(4j+1)}}\ \frac{F_{m(d+k)}F_{m(d+k-1)}\cdots F_{m(k+1)}}{F_mF_{2m}\cdots F_{dm}} =$$

$$= \frac{F_{m(d+k)}F_{m(d+k-1)} \cdots F_{m(k+4j+2)}}{F_m F_{2m} \cdots F_{m(d-4j-1)}} \quad \frac{F_{m(k+4j)}F_{m(k+4j-1)} \cdots F_{m(k+1)}}{F_m F_{2m} \cdots F_{m(4j+1)}}$$

$$\times \left\{ F_{mk} + \frac{F_{m(d+1)}F_{m(k+4j+1)}}{F_{m(d-4j)}} \right\} .$$

The expression

$$F_{mk}F_{m(d-4j)} + F_{m(d+1)}F_{m(k+4j+1)}$$

can be modified using the relation

(13) $$F_a F_b + F_{a+r}F_{b+r} = F_r F_{a+b+r} ,$$

where r is odd, giving the result

$$F_{m(4j+1)}F_{m(d+k+1)} ,$$

which leads to the final value of the coefficient of $H[m, d, n - 4j - 1]$ after substitution $F[m, d + k + 1, d - 4j]\ F[m, k + 4j, 4j]$.

That this is the correct value is seen from the following considerations. The quantity is a coefficient in the expansion for $H[m, d, n + k]$. If the subscripts of all the H's are raised by 1 to give an expression for $H[m, d, n + k + 1]$, it is seen that this quantity is the coefficient of $H[m, d, n - 4j]$. By formula (9) with k replaced by $k + 1$ and $i = 4j + 1$, we obtain for the coefficient of $H[m, d, n - 4j]$ the value

$$F[m, k + d + 1, d - 4j - 1 + 1]F[m, k + 1 + 4j + 1 - 2, 4j + 1 - 1]$$
$$= F[m, k + d + 1, d - 4j]F[m, d + 4j, 4j]$$

the result obtained by our analysis.

COEFFICIENT OF $H[m, d, n - 4j - 2]$

The development proceeds as before, ending with the following quantity in brackets

$$-F_{mk} + \frac{F_{m(d+1)}F_{m(k+4j+2)}}{F_{m(d-4j-1)}} .$$

The expression

$$-F_{mk}F_{m(d-4j-1)} + F_{m(d+1)}F_{m(k+4j+2)} = F_{m(4j+2)}F_{m(d+k+1)}$$

using the relation

(14) $$F_{a+c}F_{b+c} - F_a F_b = F_c F_{a+b+c} \quad \text{where } c \text{ is even} .$$

The final result is

$$F[m, d + k + 1, d - 4j - 1]F[m, k + 4j + 1, 4j + 1] .$$

This is the coefficient of $H[m, d, n - 4j - 2]$ in the expansion of $H[m, d, n + k]$. Shifting the H subscripts up one, this should be the coefficient of $H[m, d, n - 4j - 1]$ in the expansion of $H[m, d, n + k + 1]$. Replacing k by $k + 1$ and i by $4j + 2$ in (9) one finds for the coefficient of $H[m, d, n - 4j - 1]$ in the expansion of $H[m, d, n + k + 1]$

$$F[m, d + k + 1, d - 4j - 2 + 1]F[m, k + 1 + 4j + 2 - 2, 4j + 2 - 1]$$
$$= F[m, d + k + 1, d - 4j - 1]F[m, k + 4j + 1, 4j + 1]$$

in agreement with our analysis.

For the coefficient of $H[m, d, n - 4j - 3]$, the signs of the quantities in brackets are $-$ and $-$ with an odd subscript difference so that the formula (13) applies giving a result with a negative sign. For the coefficient of $H[m, d, n - 4j - 4]$ the signs are $+$ and $-$ with the latter predominating to give a negative sign.

The final term in the expansion after substitution would be

$$(-1)^{[d/2]}F[m, d + k, d]$$

This is to be compared with the final term in the expansion of $H[m, d, n + k + 1]$. Using formula (9) this should be

$$(-1)^{[(d+1-1)/2]}F[m, d + k + 1, d - d - 1 + 1]F[m, k + 1 + d + 1 - 2, d + 1 - 1]$$

or

$$(-1)^{[d/2]} F[m, d + k, d] \quad ,$$

which agrees with the result obtained by the substitution.

<u>m even. Downward shift</u>

The coefficients are the same so that we need simply consider the sign pattern.

	$4j + 1$	$4j + 2$	$4j + 3$	$4j + 4$
(1)	$-$	$+$	$-$	$+$
(2)	$+$	$-$	$+$	$-$
(3)	$+$	$-$	$+$	$-$

(1) is the sign pattern of terms that remain in the original expression.

(2) is the sign pattern of the substituted terms.

(3) is the sign pattern of the resulting expression.

Since the subscript differences are all even, formula (14) will apply in all these cases.

<u>m odd, d odd, k odd, Upward shift</u>

	$4j + 1$	$4j + 2$	$4j + 3$	$4j + 4$
(1)	$+$	$+$	$-$	$-$
(2)	$+$	$-$	$-$	$+$
(3)	$+$	$-$	$-$	$+$

Note that for $4j + 1$ and $4j + 3$ the subscript difference is odd and formula (13) applies. The final set of signs is correct since $k + 1$ is even.

m odd, d odd, k even. Upward shift

	$4j + 1$	$4j + 2$	$4j + 3$	$4j + 4$
(1)	-	-	+	+
(2)	-	+	+	-
(3)	-	+	+	-

The final set of signs is correct since $k + 1$ is odd.

m odd, d even, Upward shift

	$4j + 1$	$4j + 2$	$4j + 3$	$4j + 4$
(1)	+	-	-	+
(2)	+	+	-	-
(3)	+	+	-	-

the signs being correct for this case also.

For m even and upward shift, the sign pattern is the same as for m even and downward shift.

To conclude, two examples are given to show the generality of these shift formulas and the manner of using them.

Example 1. Let

$$H[3,4,n] = L_{3n-4}^2 F_{3n+2} T_{3n-1}$$

We wish to express $H[3, 4, n + 4]$ in terms of $H[3, 4, n]$ and quantities with lower subscript. By our shift formulas we have:

$$\begin{aligned} H[3, 4, n + 4] = {} & F[3,8,4]H[3,4,n] + F[3,8,3]F[3,4,1]H[3, 4, n - 1] \\ & - F[3,8,2]F[3,5,2]H[3,4, n - 2] - F[3,8,1]F[3,6,3]H[3,4, n - 3] \\ & + F[3,8,0]F[3,7,4]H[3, 4, n - 4] . \end{aligned}$$

Let $n = 3$ for purposes of checking. Then

$$\begin{aligned} L_{17}^2 F_{23} T_{20} = {} & 10212563270\, L_5^2 F_{11} T_8 + 2410834608 \cdot 72\, L_2^2 F_8 T_5 \\ & - 31721508 \cdot 5490\, L_{-1}^2 F_5 T_2 - 23184\ 417240\, L_{-4}^2 F_2 T_{-1} \\ & + 31716035\, L_{-7}^2 F_{-1} T_{-4} \end{aligned}$$

or

$$7055823593395596$$

should equal

$$10212563270 \cdot 646140 + 173580091776 * 2646 - 174151078920 * 20$$
$$- 9673292160(-98) + 31716035 \cdot 10092 \quad ,$$

which checks out.

Example 2. Let

$$H[3,\ 4,\ n] = T^4_{3n-7}$$
$$H[3,\ 4,\ n-3] = F[3,7,4]H[3,4,n] + F[3,7,3]F[3,3,1]H[3,\ 4,\ n+1]$$
$$- F[3,7,2]F[3,4,2]H[3,4,\ n+2] - F[3,7,1]F[3,5,3]H[3,4,\ n+3]$$
$$+ F[3,6,4]H[3,\ 6,\ n+4] \ .$$

Let $n = 2$.

$$T^4_{-10} = 31716035\,T^4_{-1} + 31716035 * 17\,T^4_2 - 1767779 \cdot 306\,T^4_5$$
$$- 5473 \cdot 5490\,T^4_8 + 98515\,T^4_{11}$$
$$T^4_{-10} = 212^4 = 2019963136 \ .$$

The right-hand side equals:

$$31716035 * 16 + 539172595 * 256 - 540940374 * 38416 - 30046770 * 12960000 + 98515 \cdot 4162314256$$

which checks.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Each proposed problem or solution should be submitted on a separate sheet or sheets, preferably typed in double spacing, in the format used below, to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106.

Solutions should be received within four months of the publication date of the proposed problem. Contributors in the United States who desire acknowledgement of receipt of contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n; \quad L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-256 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2n} - 3(-1)^n$ is the product of two Lucas numbers.

B-257 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $[L_{2n} + 3(-1)^n]/5$ is the product of two Fibonacci numbers.

B-258 Proposed by Paul Bruckman, University of Illinois, Chicago, Illinois.

Let $[x]$ denote the greatest integer in x, $a = (1 + \sqrt{5})/2$, and $e_n = \{1 + (-1)^n\}/2$. Prove that for all positive integers m and n:

a. $$nF_{n+1} = [naF_n] + e_n,$$

b. $$nF_{m+n} = F_m\{[naF_n] + e_n\} + nF_{m-1}F_n .$$

B-259 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Characterize the infinite sequence of ordered pairs of integers (m, r), with $4 \le 2r \le m$, for which the three binomial coefficients

$$\binom{m-2}{r-2}, \quad \binom{m-2}{r-1}, \quad \binom{m-2}{r}$$

are in arithmetic progression.

B-260 Proposed by John L. Hunsucker and Jack Nebb, University of Georgia, Athens, Georgia.

Let $\sigma(n)$ denote the sum of the positive integral divisors of n. Show that $\sigma(mn) > \sigma(m) + \sigma(n)$ for all integers $m > 1$ and $n > 1$.

B-261 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let d be a positive integer and let S be the set of all nonnegative integers n such that $2^n - 1$ is an integral multiple of d. Show that either $S = \{0\}$ or the integers in S form an infinite arithmetic progression.

SOLUTIONS

SLIGHT MISPRINT, OTHERWISE FINE

B-232 Proposed by Guy A. R. Guillotte, Quebec, Canada.

In the following multiplication alphametic, the five letters F, Q, I, N, and E represent distinct digits. The dashes denote not necessarily distinct digits. What are the digits of FINE FQ?

```
   FQ
   FQ
  ---
 ---
 ----
 FINE
```

(The number of dashes has been corrected.)

Solution by Ralph Garfield, The College of Insurance, New York, New York.

We first observe that F must be 9. Furthermore, Q must be 5 or more since $94^2 = 8836$, which does not give a first digit of 9. Clearly, since E and Q are distinct, Q cannot be 5 or 6. Also, since F and Q are different, Q cannot be 9. Hence Q must be 8. Then $98^2 = 9604$ gives 9604 98 as FINE FQ.

Also solved by Sister Marion Beiter, Ashok K. Chandra, J. A. H. Hunter, John W. Milsom, Charles W. Trigg, David Zeitlin, and the Proposer.

A FIBONACCI QUADRATIC

B-233 Proposed by Harlan L. Umansky, Emerson High School, Union City, New Jersey.

Show that the roots of $F_{n-1}x^2 - F_n x - F_{n+1} = 0$ are $x = -1$ and $x = F_{n+1}/F_{n-1}$. Generalize to show a similar result for all sequences formed in the same manner as the Fibonacci sequence.

Solution by Graham Lord, S. U. N. Y., Binghamton, New York.

Let a_n be any sequence satisfying $a_n = a_{n-1} + a_{n-2}$, for all n. Then

$$P(x) = a_{n-1}x^2 - a_n x - a_{n+1} = a_{n-1}x - (a_{n+1} - a_{n-1})x - a_{n+1}$$
$$= (a_{n-1}x - a_{n+1})(x + 1) .$$

Hence the roots of $P(x) = 0$ are -1 and a_{n+1}/a_{n-1}. In particular, if $a_n = F_n$ the first half of the question is also solved.

Also solved by Sister Marion Beiter, Paul S. Bruckman, Ashok K. Chandra, Herta T. Freitag, Ralph Garfield, J. A. H. Hunter, Edgar Karst, Peter A. Lindstrom, Graham Lord, John W. Milsom, Paul Salomaa, David Zeitlin, and the Proposer.

DUPLICATING A CUBE?

B-234 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California

Prove that

$$L_n^3 = 2F_{n-1}^3 + F_n^3 + 6F_{n-1}F_{n+1}^2 .$$

Solution by Paul S. Bruckman, University of Illinois, Chicago, Illinois.

Since $F_n = F_{n+1} - F_{n-1}$, we may cube both sides and obtain

$$F_n^3 = F_{n+1}^3 - 3F_{n+1}^2 F_{n-1} + 3F_{n+1}F_{n-1}^2 - F_{n-1}^3 .$$

Adding $2F_{n-1}^3 + 6F_{n-1}F_{n+1}^2$ to both sides of this expression, we obtain

$$2F_{n-1}^3 + F_n^3 + 6F_{n-1}F_{n+1}^2 = F_{n+1}^3 + 3F_{n+1}^2 F_{n-1} + 3F_{n+1}F_{n-1}^2 + F_{n-1}^3 = (F_{n+1} + F_{n-1})^3 = L_n^3.$$

Also solved by James D. Bryant, Ashok K. Chandra, Warren Cheves, Herta T. Freitag, Ralph Garfield, J. A. H. Hunter, Edgar Karst, Peter A. Lindstrom, Graham Lord, John W. Milsom, Paul Salomaa, David Zeitlin, and the Proposer.

A PROPERTY OF F_{16}

B-235 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the largest positive integer n such that F_n is smaller than the sum of the cubes of the digits of F_n.

Solution by Ashok K. Chandra, Graduate Student, Stanford University, California.

We need only check for all n such that $F_n \le 4\cdot 9^3 = 2916$, for if F_n has n digits, $n \ge 5$, then $n\cdot 9^3 < 10^{n-1} \le F_n$.

$$n = 15 \quad 16 \quad 17 \quad 18$$

$$F_n = 610 \quad 987 \quad 1597 \quad 2584$$

Now,

$$2584 > 2^3 + 5^3 + 8^3 + 4^3 ,$$

and

$$1597 > 1^3 + 5^3 + 9^3 + 7^3 ,$$

but

$$987 < 9^3 + 8^3 + 7^3 .$$

Hence the largest n is 16, and $F_n = 987$.

I wrote a short computer program to determine the largest n for an arbitrary power, and obtained the following results:

Power	n	F_n
2	11	89
3	16	987
4	19	4,181
5	24	46,368
6	29	514,229
7	34	5,702,887
8	39	63,245,986
9	42	267,914,296

Also solved by Paul S. Bruckman, Paul Salomaa, Charles W. Trigg, and the Proposer.

TWO HEADS NOT BETTER THAN ONE

B-236 Proposed by Paul S. Bruckman, San Rafael, California.

Let P_n denote the probability that, in n throws of a coin, two consecutive heads will not appear. Prove that

$$P_n = 2^{-n} F_{n+2} .$$

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.

A sequence of H's and T's of length n is called admissible if two heads do not appear together anywhere in the sequence. Let α_n be the number of admissible sequences of length n. Then there are α_n admissible sequences of length $n+1$ which end with a T, while there are α_{n-1} admissible sequences of length $n+1$ ending in an H (since any such sequence must actually end in TH). Thus $\alpha_{n+1} = \alpha_n + \alpha_{n-1}$. Combined with the initial values $\alpha_1 = 2$ and $\alpha_2 = 3$, we find $\alpha_n = F_{n+2}$ for $n \geq 1$ and the required probability becomes $F_{n+2}/2^n$ as stated.

NOTE: Essentially the same problem occurs as Problem 62-6 in SIAM Review (solution in Vol. 6, No. 3, July 1964, p. 313) and as Problem E2022 in American Mathematical Monthly (solution in Vol. 74, No. 10, December, 1968, p. 1117). See also Problem 1, p. 14 in An Introduction to Combinatorial Analysis by John Riordan (J. Wiley and Sons, Inc., 1958) and Problem B-5 in the Fibonacci Quarterly (solution in Vol. 1, No. 3, October, 1963, p. 79).

Also solved by Ashok K. Chandra, Ralph Garfield, Peter A. Lindstrom, Graham Lord, Bob Prielipp, Paul Salomaa, Richard W. Sielaff, and the Proposer.

G. C. D. PROBLEM

B-237 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let (m,n) denote the greatest common divisor of the integers m and n.

(i) Given $(a,b) = 1$, prove that $(a^2 + b^2, a^2 + 2ab)$ is 1 or 5.

(ii) Prove the converse of Part (i).

Solution by Paul Salomaa, Junior, M. I. T., Cambridge, Massachusetts.

$$(a^2 + b^2, a^2 + 2ab) = (a^2 + b^2 - [a^2 + 2ab], a^2 + 2ab) = (b^2 - 2ab, a^2 + 2ab) .$$

Let p be a prime (or 1) such that

$$p \mid (a^2 + b^2, a^2 + 2ab) .$$

Then

$$p \mid (b^2 - 2ab) \quad \text{and} \quad p \mid (a^2 + 2ab) .$$

But $(p,a) = (p,b) = 1$, for if not, $p|(a^2 + b^2)$ would imply $p|(a,b)$ forcing $p = 1$. Hence $p|(b - 2a)$ and $p|(a + 2b)$. So

$$p \mid [b - 2a + 2(a + 2b)] ,$$

i.e., $p|5b$. Since $(p,b) = 1$, we have that $p|5$, so $p = 5$ or $p = 1$. If

$$p^2 \mid (a^2 + b^2, a^2 + 2ab) ,$$

then

$$p^2 \mid (b - 2a) \quad \text{and} \quad p^2 \mid (a + 2b) ,$$

hence $p^2|5b$, so in this case $p = 1$. In particular, $5^2 \nmid (a^2 + b^2, a^2 + 2ab)$. For the converse, let $(a,b) = d$. Then $d^2|(a^2 + b^2, a^2 + 2ab)$. Hence $d^2|5$ or $d^2|1$, and in either case, $d = 1$.

Also solved by Ashok K. Chandra, Herta T. Freitag, Graham Lord, and the Proposer.