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ON THE USE OF FIBONACCI RECURRENCE RELATIONS IN THE DESIGN OF LONG WAVELENGTH FILTERS AND INTERFEROMETERS

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(Submitted May 1979)

1. INTRODUCTION

The frequent occurrence of Fibonacci related numbers in art and nature has been of interest to readers of this journal since the very first issue when Basin [1] wrote such an article.

The purpose of this paper is to explain briefly the use of Horadam's generalized Fibonacci recurrence relation [2]

$$u_n = pu_{n-1} - qu_{n-2} \qquad (n \geq 2) \tag{1.1}$$

in a problem in optics. If we consider the sequence $\{u_n\}$ whose elements satisfy (1.1) with initial conditions $u_0 = 1$ and $u_1 = S_0$, then, from Horadam, we have that

$$u_n = V\alpha^n + W\beta^n$$

where

$$V = \frac{\beta - S_0}{\beta - \alpha}, \quad W = \frac{S_0 - \alpha}{\beta - \alpha},$$

in which α, β are the roots of the characteristic equation

$$x^2 - px + q = 0.$$

2. NONLINEAR RECURRENCE RELATION

To show the relationship with the optics problem, we need some more preliminary results.

Put $p = B - A$, $q = C - AB$, and the recurrence relation can be rewritten as

$$\frac{u_{n+1}}{u_{n-1}} - (B - A)\frac{u_n}{u_{n-1}} + (C - AB) = 0.$$

We then add and subtract the term

$$A\left(\frac{u_{n+1}}{u_n} + A\right)$$

to get

$$\frac{u_{n+1}}{u_{n-1}} + A\frac{u_n}{u_{n-1}} + A\frac{u_{n+1}}{u_n} + A^2 - A\frac{u_{n+1}}{u_n} - A^2 - B\frac{u_n}{u_{n-1}} - AB + C = 0,$$

which can be rewritten as

$$\left(\frac{u_{n+1}}{u_n} + A\right)\left(\frac{u_n}{u_{n-1}} + A\right) - A\left(\frac{u_{n+1}}{u_n} + A\right) - B\left(\frac{u_n}{u_{n-1}} + A\right) + C = 0.$$

From this we obtain the non-linear recurrence relation

$$R_n R_{n-1} - AR_n - BR_{n-1} + C = 0, \tag{2.1}$$

in which

(2.2) $$R_n = \frac{u_{n+1}}{u_n} + A.$$

Thus,

(2.3) $$R_n = \frac{(A + \alpha)V\alpha^n + (A + \beta)W\beta^n}{V\alpha^n + W\beta^n}$$

where the term S_0 in the definitions of V and W is given by

$$S_0 = R_0 - A.$$

3. THE OPTICS PROBLEM

3.1 General Remarks

In this section, we will show that Eq. (2.1) occurs in the theory of multi-element optical filters and interferometers. However, before launching into its derivation, it is necessary to acquaint the reader with the background to the problem.

Devices such as beam splitters, filters, and interferometers are common tools in all forms of experimental optics. In the fields of infrared physics and microwave engineering, the construction of such apparatus relies upon the use of wire meshes (or grids) [3]. It is possible to classify these structures into two distinct classes according to their spectral properties. These are:

a. inductive grids, made by perforating (in doubly-periodic fashion) a thin metal plate with aperatures, and
b. capacitive grids, the natural complement of inductive grids, which are composed of a periodic array of metal inclusions immersed in an insulating material.

The transmittance of inductive structures approaches zero at long wavelengths, whereas that of capacitive grids approaches unity at those wavelengths. In the far infrared and microwave regions of the electromagnetic spectrum, absorption within the metal is negligible and so we need only concern ourselves with the reflectance and transmittance of these structures.

With these prefatory remarks, let us now concern ourselves with the design of interferometers and low-frequency pass filters. These consist of a stack of many such grids separated from one another by a distance of s. In the case of the interferometer, the stack is composed of purely inductive elements, while the low-pass filter is composed of a stack of captive structures.

Each of the grids in the stack acts as a diffraction grating and gives rise to an infinite set of diffracted plane waves (orders) excited by the incident plane wave field. Interferometers and filters are operated with wavelengths in excess of the grid period (d), and so it may be deduced that only the single undispersed wave is propagating (i.e., capable of carrying energy away from the grids). All of the other orders are said to be evanescent and decay exponentially as they propagate away from a grid. Provided that the ratio s/d exceeds 0.5, the evanescent orders provide no significant mechanism for communication between the grids [4] and so we need only consider the zeroth (or undispersed) order in our derivation.

3.2 Derivation of a Non-Linear Difference Equation

Let R_0 and T_0 be the amplitude reflection and transmission coefficients of the zeroth order of one of these grids. Now consider a wave incident upon the $(n+1)$-grid structure of Figure 1. Let R_n and T_n be the amplitude reflection and transmission coefficients of this device relative to a phase origin at the center of the uppermost grid. We now regard this $(n + 1)$-grid structure as a single grid displaced by a distance s from an n-grid structure. By adopting a multiple scat-

tering approach, we can now trace the path of a wave through this system. This is a sophistication of the ray trace mentioned by Huntley [5].

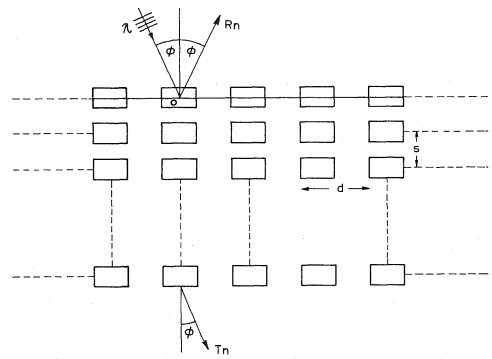

Fig. 1. Side view of the $(n + 1)$ grid stack with a plane wave of wavelength λ indicent at an angle of ϕ. R_n and T_n are reflection and transmission coefficients measured relative to a phase origin at 0.

A wave of amplitude 1 incident upon the top surface is reflected with amplitude R_0 and transmitted with amplitude T_0. The transmitted component then traverses an optical path length of $s \cos \phi$ (where ϕ is the angle of incidence). Thus, the wave incident upon the n grid structure has amplitude $T_0\rho$ where

$$\rho = \exp\ (i\delta)$$

and

$$\delta = \frac{2\pi}{\lambda} s \cos\ \phi$$

for a field of wavelength λ. This wave is then transmitted and reflected by the n grid structure with the reflected amplitude being given by $T_0 R_{n-1}\rho$. The reflected component then propagates toward the top surface, advancing in phase by δ, where it is partially transmitted out into free space with amplitude $T_0^2 R_{n-1}\rho^2$ and partially reflected back into the cavity with amplitude $T_0 R_{n-1} R_0 \rho^2$; we continue this process ad infinitum and arrive at the series:

$$R_n = R_0 + R_{n-1} T_0^2 \rho^2 \sum_{k=0}^{\infty} (R_0 R_{n-1} \rho^2)^k. \tag{3.1}$$

Since all of the reflection coefficients have magnitude less than unity, we write

$$R_n = R_0 + \frac{R_{n-1} T_0^2 \rho^2}{1 - R_0 R_{n-1} \rho^2}.$$

This may be reduced to the simpler form

$$R_n = \frac{R_0 - R_{n-1}\rho^2 \xi^2}{1 - R_0 R_{n-1} \rho^2} \tag{3.2}$$

where

$$\xi = \exp\ (i\psi_r)$$

with

$$\psi_r = \arg\ (R_0).$$

This simplification is a consequence of

a. conservation of energy,

$$|R_0|^2 + |T_0|^2 = 1$$

and

b. the phase constraint [6, 7]

$$\arg(T_0) = \arg(R_0) + \pi/2$$

appropriate to all lossless up-down symmetric structures.

Equation (3.2) is of the same form as (2.1) with

$$A = \frac{1}{R_0\rho^2}, \quad B = \frac{\xi^2}{R_0}, \quad C = \frac{1}{\rho^2};$$

constants which are only dependent upon the geometry and the reflection coefficient R_0, which may be found using a rigorous electromagnetic scattering theory. Having derived R_n, the transmittance is then

$$|T_n|^2 = 1 - |R_n|^2.$$

3.3 Interferometers

The basic component of any interferometer is an inductive element. In Figure 2 are shown transmission spectra for a typical inductive grid and its associated two-grid interferometer. The transmittance of this interferometer is given by

$$|T_1|^2 = 1/[1 + F\sin^2(\chi)]$$

where

$$F = \frac{4|R_0|^2}{(1 - |R_0|^2)^2}$$

and

$$\chi = \delta + \psi_r.$$

Clearly, it can be seen that this is a wavelength selection device with interference maxima for normally incident radiation, at

$$\lambda_{max} = \frac{2s}{(\ell - \psi_r/\pi)} \qquad (\ell = 0, 1, 2, \ldots).$$

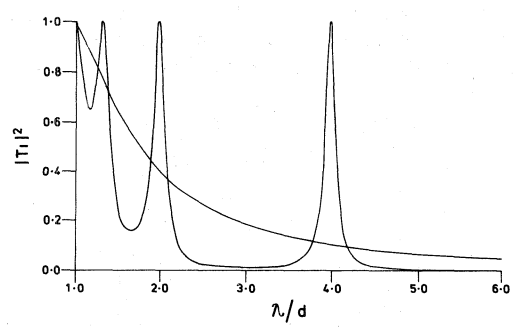

Fig. 2. Normal incidence wavelength spectra for a typical inductive grid [the curve whose decay is of the approximate form $|T_0|^2 \propto (d/\lambda)^2$] and its associated two-grid interferometer (the curve exhibiting resonant behavior at $\lambda = 2s/\ell$). For this structure, $s/d = 2.0$.

The resonance width $\Delta\lambda$ is governed by F, the finesse of the instrument, and decreases as the transmittance of a single grid decreases. This feature is illustrated in Figure 3, where a grid of substantially lower transmittance is used. Also shown on Figure 3 are spectra for three-, four-, and five-grid interferometers. For an $(n + 1)$-grid interferometer, the single transmission resonance for the two-grid device splits into n peaks, each having a significantly higher resolution factor Q,

$$Q = \frac{\lambda}{\Delta\lambda}.$$

The locations of these peaks are given by the n solutions of the equation

$$\beta^{n+1} = \alpha^{n+1} \qquad (\beta \neq \alpha). \tag{3.3}$$

This reduces to the simpler and more explicit form,

$$\delta + \arg(R_0) = k\pi \pm \frac{1}{2}\arccos\left[|R_0|^2 - (1 - |R_0|^2)\cos\left(\frac{2\Pi\ell}{n+1}\right)\right] \tag{3.4}$$

where $\ell = 1, 2, \ldots, n$ and k is a nonnegative integer.

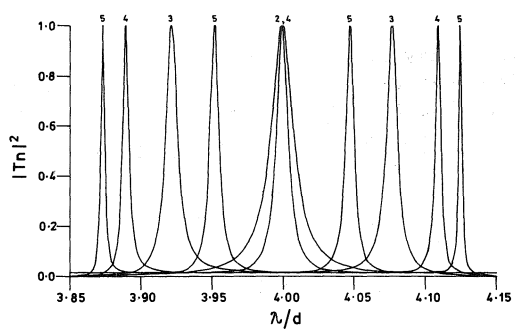

Fig. 3. Normal incidence wavelength spectra for an inductive grid stack composed of 2, 3, 4, and 5 elements (indicated by an integer above the resonance peaks). The separation of the individual elements is $s/d = 2.0$.

By considerably reducing the long wavelength filtering action of the inductive grids that compose the interferometer, we can obtain a broadband-pass filter. The spectrum of such a structure is illustrated in Figure 4.

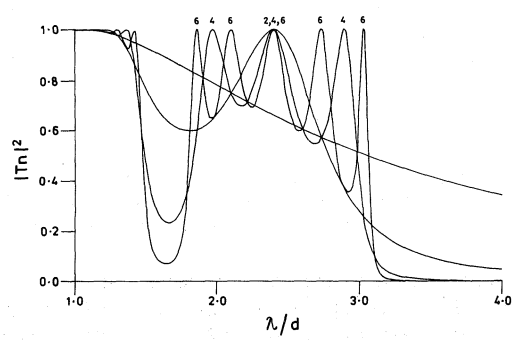

Fig. 4. Normal incidence wavelength spectra for 2-, 4-, and 6-grid band-pass filters. Here $s/d = 1.2$.

3.5 Low Pass Filters

The aim of such filters is to exclude any high-frequency components from the transmission spectrum. To achieve this objective, it is necessary to select a capacitive grid as the basic component of the stack. In Figure 5 we present typical spectra for four multi-element filters. Note that as the number of grids in the stack is increased, the cut-off between the transmission and rejection regions is sharpened.

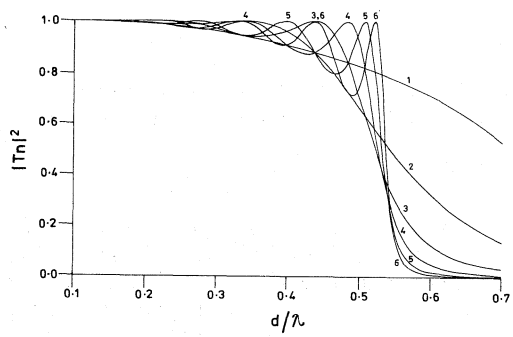

Fig. 5. Normal incidence frequency spectra of typical low-pass filters composed of up to 6 capacitive elements separated by $s/d = 1.0$.

4. CONCLUDING REMARKS

The solution of the nonlinear difference equations relying upon the use of Horadam's generalized Fibonacci recurrence relation discussed here totally circumvents the explicit and inelegant treatments of earlier, less general attempts [8]. It also facilitates the calculation of the positions of the transmission maxima [see Eq. (3.3)] for a grid stack containing an arbitrary number of elements.

ACKNOWLEDGMENT

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ON THE CONVERGENCE OF ITERATED EXPONENTIATION—III

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(Submitted May 1979)

The present paper can be considered as an extension of two previous papers in which the properties of the following function were discussed (see [1] and [2]):

$$F(x, y) = x^{y^{x^{y^{\cdot^{\cdot^{\cdot^{x^y}}}}}}}, \tag{1}$$

where an infinite number of exponentiations is understood. Equation (1) is the function specifically studied in [2], whereas in [1] we considered the simpler function

$$f(x) \equiv F(x, x), \tag{2}$$

i.e., the case of Eq. (1) where $x = y$. For both Eqs. (1) and (2), the ordering of the exponentiations is important, and for Eq. (1) and throughout this paper, we mean a bracketing order "from the top down," i.e., x raised to the power y, followed by y raised to the power x^y, and then x raised to the power $y^{(x^y)}$, and so on, all the way down to the x which is at the lowest position of the "ladder."

In the present paper, we study the properties of a function which is obtained by forming an infinite sequence of roots. We have restricted ourselves to a single (positive) variable x, i.e., the analogue of Eq. (2). We will call this function $\phi(x)$, and it is defined as follows:

$$\phi(x) = \sqrt[x^{\sqrt[x^{\sqrt{x}}]{x}}]{x}, \tag{3}$$

where an infinite number of roots is understood. The bracketing is again from "the top down," i.e., we mean $\sqrt[x]{x}$, followed by the $\sqrt[x]{x}$-th root of x, which can be written as $\xi(x)$, followed by the root $\sqrt[\xi]{x}$, and so on, down to the lowest x in the "ladder."

From Eq. (3), it can be see that we have:

$$\phi(x) = x^{\frac{1}{\phi(x)}} = \sqrt[\phi(x)]{x}, \tag{4}$$

provided that the sequence (3) has a nontrivial limit. From Eq. (4), we obtain the equation:

$$\phi(x)^{\phi(x)} = x. \tag{5}$$

Values of $\phi(x)$ were calculated by means of a simple program embodying the sequential operations of Eq. (3) on a Hewlett-Packard calculator. In this manner, we have obtained the graph of Figure 1, in which $\phi(x)$ is shown as a function of x. We note that for $x < e^{-1/e}$, i.e., $x < 0.692200\ldots$, $\phi(x) = 0$, and at $x = e^{-1/e}$, $\phi(x)$ has the value $1/e = 0.36788$. Indeed, for $\phi = e^{-1}$, Eq. (5) gives

$$\phi^{\phi} = e^{-1(1/e)} = e^{-1/e} = x.$$

The reason for the abrupt decrease of $\phi(x)$ to zero below $x = e^{-1/e}$ is illustrated

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in Figure 2, in which we have plotted ϕ^ϕ as a function of ϕ. It can be seen that ϕ^ϕ has a minimum value of $e^{-1/e}$ which is attained at $\phi = 1/e$. Indeed, the derivative $d\phi^\phi/d\phi$ is zero at this point, as can be seen from the following equation:

$$\frac{d\phi^\phi}{d\phi} = \frac{d\exp(\phi\log\phi)}{d\phi} = \phi^\phi(\log\phi + 1). \tag{6}$$

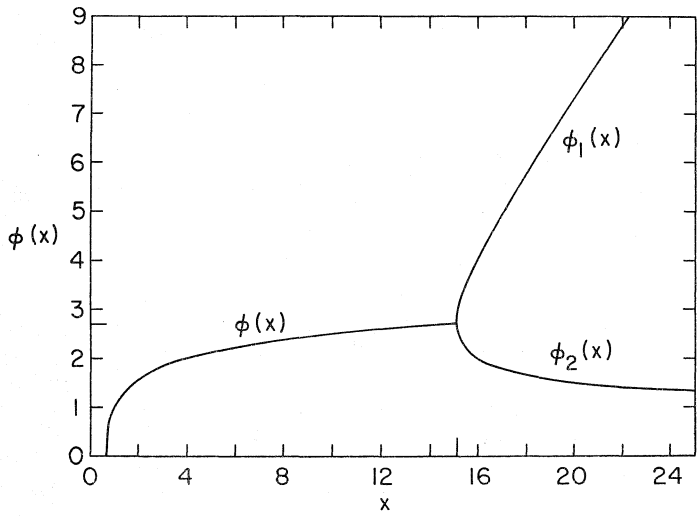

Fig. 1. The curve of the function $\phi(x)$ as a function of x. For $x < e^{-1/e} = 0.6922$, $\phi(x) = 0$. At $x = e^{-1/e}$, $\phi(x) = 1/e \cong 0.36788$, so that $\phi(x)$ has an abrupt discontinuity at $x = e^{-1/e}$. For $x > e^e = 15.1542$, the sequence $\phi(x)$ defined by Eq. (3) converges to two different values $\phi_1(x)$ and $\phi_2(x)$, depending on whether the number n of x's is odd or even, respectively. This property can be called "dual convergence" and has been described previously in [1-3].

Thus, for $x < e^{-1/e}$, Eq. (5) has no solution with $\phi(x) > 0$. At $\phi = 0$, the derivative $d\phi^\phi/d\phi \to -\infty$, since $\log\phi \to -\infty$. We also note from Figure 2 that for $e^{-1/e} < \phi^\phi < 1$, there are two values of ϕ for a given value of ϕ^ϕ. Thus, we can divide the curve of Figure 2 into two branches, the one to the left of $\phi = 1/e$, and the other to the right of $\phi = 1/e$. The branch to the right of $\phi = 1/e$, i.e., the branch with $\phi > 1/e$, gives the value of ϕ for a given x, as obtained from Eq. (3). The meaning of the other (left) branch will be discussed below. We note that for $\phi > 1$, there is a unique value of ϕ for a given $\phi^\phi = x$, as shown in Figure 2.

Returning now to Figure 1, we note that for $x > e^e = 15.1542\ldots$, we have a dual convergence of Eq. (3), namely a convergence to two values $\phi_1(x)$ and $\phi_2(x)$ depending upon whether the number n of x's in Eq. (3) is odd or even. This property of dual convergence has been discussed previously in connection with the function $f(x) = F(x, x)$ of [1] for $x < e^{-e} = 0.06599$. The concept of dual convergence was actually introduced in an earlier paper by the authors [3] which was circulated as a Brookhaven Informal Report [4].

At the point $x = e^e$, $\phi(x)$ has the unique value $\phi(x) = e$, which is marked on the ordinate axis of Figure 1. For very large x, it is easy to show that $\phi_1(x)$ approaches x, whereas $\phi_2(x)$ approaches 1. In order to illustrate this property, we consider the choice $x = 10{,}000$. Now $\sqrt[x]{x} = 10{,}000^{0.0001} = 1.000922$, and the next

step calls for the calculation of

$$10{,}000^{1/1.000922} = 9915.53,$$

followed by

$$10{,}000^{1/9915.53} = 1.000929.$$

The actual values to which the infinite sequence of Eq. (3) converges for $x = 10{,}000$ are:

$$\phi_1(x) = 9914.85 \quad \text{and} \quad \phi_2(x) = 1.0009294.$$

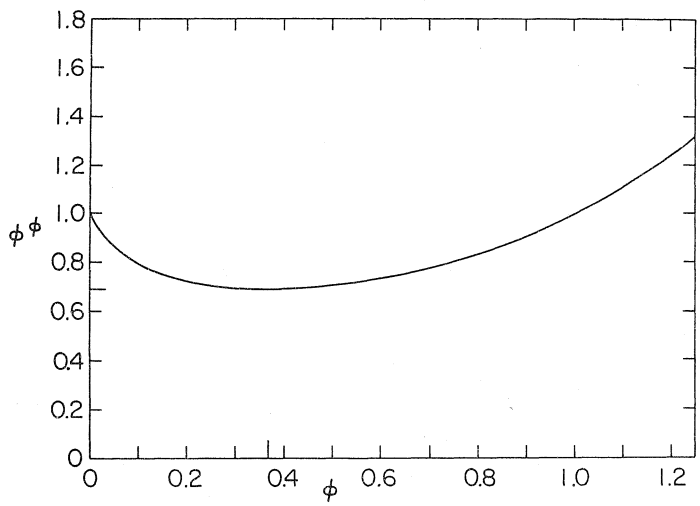

Fig. 2. The function ϕ^ϕ as a function of ϕ for ϕ in the region $0 < \phi < 1.25$. This function is of interest in connection with Eq. (5), according to which $\phi^\phi = x$. We note that the minimum value of ϕ^ϕ is $e^{-1/e} = 0.6922$ and is attained at $\phi = 1/e$. Thus, for $x < 1$, the function ϕ^ϕ can be considered as having two branches, the one to the left of $\phi = 1/e$ and the one to the right of $\phi = 1/e$. The right-hand branch gives the value of ϕ as a function of $x = \phi^\phi$, e.g., for $x = 0.8$, we have $\phi(x) = 0.7395$. The left-hand branch gives the value of $N_{\min}$, as explained in the text [see Eqs. (12)-(18)]. Thus, for values of x between $e^{-1/e}$ and 1, $\phi_N(x) = \phi(x)$, provided $N \geq N_{\min}$. For $N < N_{\min}$, $\phi_N(x) = 0$. As an example, $N_{\min}(x = 0.8) = 0.09465$.

Obviously, from the definition of $\phi_1(x)$ and $\phi_2(x)$, we have the relations:

$$\phi_1(x)^{\phi_2(x)} = \phi_2(x)^{\phi_1(x)} = x \tag{7}$$

for $x > e^e$. Incidentally, the equation $\phi(x)^{\phi(x)} = x$ continues to have a solution for $x > e^e$, but this solution does not give the values of $\phi(x)$ to which the sequence (3) approaches by dual convergence. As examples of values of $\phi_1(x)$ and $\phi_2(x)$ for $x > e^e$, we may cite:

$$\text{for } x = 20: \quad \phi_1(20) = 7.2802, \quad \phi_2(20) = 1.50907;$$
$$\text{for } x = 100: \quad \phi_1(100) = 76.379, \quad \phi_2(100) = 1.06215.$$

The occurrence of $x = e^{-1/e}$ and $x = e^e$ as limiting values for $\phi(x)$ and the similar occurrence of $x = e^{1/e}$ and $x = e^{-e}$ as limiting values for $f(x)$ suggests a recriprocal relationship between the functions $\phi(x)$ and $f(x)$. This conjecture is strengthened by the fact that the values of $f(x)$ and $\phi(x)$ at corresponding points

are the reciprocals of one another. Thus, we have:

$$\phi(x = e^{-1/e}) = 1/e, \qquad f(x = e^{1/e}) = e, \tag{8}$$

$$\phi(x = e^{e}) = e, \qquad f(x = e^{-e}) = 1/e. \tag{9}$$

We now prove the following relation between $\phi(x)$ and $f(x)$:

$$\phi(x) = \frac{1}{f(1/x)}. \tag{10}$$

Thus, the region of dual convergence of $\phi(x)$ for $x > e^e$ corresponds point-for-point to the region of dual convergence of $f(x)$ for $x < e^{-e}$, in which $f(x)$ has two branches $f_1(x)$ and $f_2(x)$, which approach the limiting values $f_1(x) \to x$ as $x \to 0$ for an odd number of x's in Eqs. (1) and (2), and $f_2(x) \to 1$ as $x \to 0$ for an even number of x's.

In order to prove the relation of Eq. (10), we simply note that:

$$\phi(x) = x^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{x}}}}}} = 1 \Big/ \frac{1}{x}^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{x}}}}}} = \left[f\left(\frac{1}{x}\right)\right]^{-1} \tag{11}$$

where the bracketing is "from the top down" the ladder, as in all of the present work. Thus, all of the arguments given for the single or dual convergence of $f(x)$ in [1] apply to the present case, provided that $x > 0$.

We now wish to consider a generalization of $\phi(x)$ to be denoted by $\phi_N(x)$, analogously to the generalization of $f(x)$ to the function $f_N(x)$ of [2]. Thus, we define $\phi_N(x)$ as follows:

$$\phi_N(x) = \sqrt[N]{x}^{\sqrt{x}^{\cdot^{\cdot^{\cdot^{\sqrt{x}}}}}}, \tag{12}$$

where N is an arbitrary positive quantity. By the same procedure as in Eq. (11), we can rewrite Eq. (12) as follows:

$$\phi_N(x) = x^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{x}}}}}} = 1 \Big/ \frac{1}{x}^{\frac{1}{x}^{\cdot^{\cdot^{\cdot^{\frac{1}{x}}}}}} = \left[f_{1/N}\left(\frac{1}{x}\right)\right]^{-1} \tag{13}$$

[see Eq. (26) of 2]. For values of $x > 1$, we have $1/x < 1$, and as shown in [2, discussion following Eq. (29)], we have

$$f_{1/N}\left(\frac{1}{x}\right) = f\left(\frac{1}{x}\right) \tag{14}$$

for all values of N, and correspondingly: $\phi_N(x) = \phi(x)$. This statement applies both to the region $1 < x \le e^e$, where $\phi(x)$ is single-valued, and to the region $x > e^e$, where we have dual convergence. In this case:

$$\phi_{1,N}(x) = \phi_1(x) \qquad \text{and} \qquad \phi_{2,N}(x) = \phi_2(x).$$

The situation is different when $x < 1$. As shown above, $\phi(x)$ is nonzero only in the limited region extending from $x = e^{-1/e} = 0.6922$ to $x = 1$. The corresponding values of $1/x$ are larger than 1, and hence $f_{1/N}(1/x)$ may diverge, depending on the value of N, giving $\phi_N(x) = 0$.

It has been shown in [2] that, for the function $f_{\overline{N}}(\overline{x})$, the upper limit on $\overline{N}$ is given by the root of the equation

$$\overline{x}^{f} = f, \tag{15}$$

where we must choose the upper branch of the curve of f vs. $\overline{x}$, i.e., the branch for which $f > e$, which we have denoted by $f^{(2)}$ [2, see the discussion following Eq. (28)]. We can therefore write $f^{(2)} = \overline{N}$. Now in view of Eq. (13), the lower limit on N for $\phi_N(x)$ is given by $N = 1/\overline{N}$, and the value of x is given by $x = 1/\overline{x}$. Upon inserting these substitutions into Eq. (15), we obtain:

$$\left(\frac{1}{x}\right)^{1/N} = \frac{1}{N}. \tag{16}$$

Upon taking the reciprocal on both sides of this equation, we find

$$x^{1/N} = N, \tag{17}$$

whence

$$x = N^N. \tag{18}$$

This equation for N is identical to the equation for $\phi(x)$ given in Eq. (5). Since $\overline{N} > e$ by the previous argument, we find $N < 1/e$, and therefore the relation of Eq. (18) for N, i.e., $N_{\min}$ (minimum value of N) corresponds to the part of the curve of $\phi^\phi = x$ which lies to the left of the point $\phi = 1/e$. Thus, the values of Figure 2 for $\phi < 1$ give both the value of $\phi(x)$ (right part of the curve) and the value of $N_{\min}(x)$ (left part of the curve), such that for $N < N_{\min}$, the function $\phi_N(x)$ of Eq. (13) is zero, even though the simple function $\phi(x)$ (with an x on top of the ladder) is convergent and nonzero, and in fact $\phi(x) \geq 1/e$.

In connection with the iterated root-taking which is implied by Eq. (3) for the function $\phi(x)$, we have considered another possible function obtained by iteration, namely:

$$R(n, a, x) \equiv \sqrt[n]{a + x\sqrt[n]{a + x\sqrt[n]{\ldots}}}. \tag{19}$$

Assuming the convergence of Eq. (19), we find:

$$R^n = a + xR. \tag{20}$$

For the case $n = 2$ (repeated square roots), Eq. (20) can be solved directly, with the result:

$$R(2, a, x) = \frac{x}{2} + \left(\frac{x^2}{4} + a\right)^{\frac{1}{2}}. \tag{21}$$

Also, for the special case that $a = 0$ in Eq. (19), we obtain, for arbitrary (positive) n:

$$R^n = xR, \tag{22}$$

which gives

$$R(n, 0, x) = x^{1/(n-1)}. \tag{23}$$

If, furthermore, we take $n = x$, we obtain:

$$R(x, 0, x) = x^{1/(x-1)}. \tag{24}$$

It can be easily shown that the function $R(x, 0, x)$ decreases monotonically from $\sim 1/x$ near $x = 0$ to $R = e$ at $x = 1$ and, further, to $R = 1$ as $x \to \infty$.

In Eq. (23), we note that $R(2, 0, x) = x$, i.e.,

$$x = \sqrt[2]{x\sqrt[2]{x\sqrt[2]{\ldots}}}. \tag{25}$$

Finally, we wish to show the connection of $R(2, a, x)$ to the continued fraction $F_c(a, x)$ defined as follows:

$$F_c(a, x) = x + \cfrac{a}{x + \cfrac{a}{x + \ldots}}. \tag{26}$$

From Eq. (26), we obtain the following equation determining the value of $F_c(a, x)$:

$$F_c - x = \frac{a}{F_c}, \tag{27}$$

whence:

$$F_c^2 - xF_c - a = 0. \tag{28}$$

This equation is identical to the one which determines the continued square root $R(2, a, x)$, and correspondingly

$$F_c(a, x) = R(2, a, x). \tag{29}$$

An interesting result of Eq. (28) is that in the limit that $x \to 0$, we find

$$\lim_{x \to 0} F_c(a, x) = a^{\frac{1}{2}}, \tag{30}$$

which does not seem obvious from the definition of $F_c(a, x)$ by Eq. (26).

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GENERALIZED FERMAT AND MERSENNE NUMBERS

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1. INTRODUCTION

The numbers $F_n = 1 + 2^{2^n}$ and $M_p = 2^p - 1$, where n is a nonnegative integer and p is a prime, are called Fermat and Mersenne numbers, respectively. Properties of these numbers have been studied for centuries and most of them are well known. At present, the number of known Fermat and Mersenne primes are five and twenty-seven, respectively. It is well known that if $2^n - 1 = p$, a prime, then n is a prime. It is quite easy to show that if $2^n - 1 = pq$, p and q are primes, then either n is a prime or $n = v^2$, where v is a prime. Thus

$$2^{v^2} - 1 = pq = (2^v - 1)(2^{v(v-1)} + \cdots + 2^v + 1),$$

where $2^v - 1 = p$ is a Mersenne prime. This leads to the following definition. Let k and n be positive integers. The number $L(k, n)$ is defined as follows:

$$L(k, n) = 1 + 2^n + (2^n)^2 + \cdots + (2^n)^{k-1}.$$

The purpose of this paper is to study the numbers $L(k, n)$, which contain both the Fermat, $L(2, 2^n)$, and Mersenne, $L(k, 1)$, numbers. We will show that while $L(k, n)$ possesses many interesting properties, there remain unanswered some very elementary questions about this class of numbers.

2. PRIME NUMBERS OF THE FORM $L(k, n)$

In this section, we shall show that if $L(k, n)$ is a prime, then either $L(k, n)$ is a Mersenne prime, or $n = p^i$ and $k = p$. But first we need a lemma which follows from Theorem 10 in [4, p. 17].

LEMMA 1: If $(a, b) = d$, then $(2^a - 1, 2^b - 1) = 2^d - 1$.

THEOREM 1: If $L(k, n)$ is a prime, then either $L(k, n)$ is a Mersenne prime, or $n = p^i$ and $k = p$, where p is a prime and i is a positive integer.

PROOF: If $n = 1$, then $L(k, 1)$ is a Mersenne prime. So suppose $n > 1$. If k is not a prime, then $k = ab$, where $a > 1$ and $b > 1$. Then

$$\begin{aligned} L(k, n)(2^n - 1) &= 2^{nk} - 1 = 2^{n(ab)} - 1 = (2^{na})^b - 1 \\ &= (2^{na} - 1)(2^{na(b-1)} + \cdots + 2^{na} + 1) \\ &= (2^n - 1)(2^{n(a-1)} + \cdots + 2^a + 1) \\ &\quad \cdot (2^{na(b-1)} + \cdots + 2^{na} + 1). \end{aligned}$$

Thus, cancelling $(2^n - 1)$ from both sides, $L(k, n)$ is not a prime; a contradiction. Thus $k = p$ for some prime p.

Next we wish to show that $n = p^i$. Suppose $n = p_1^{d_1} \cdots p_j^{d_j}$ and $p \neq p_k$ for any k. Then

$$L(k, n)(2^n - 1) = (2^n)^p - 1 = (2^p - 1)(2^{p(n-1)} + \cdots + 1).$$

Since $(p, n) = 1$, by Lemma 1, $(2^p - 1, 2^n - 1) = 1$. It follows that

$$(2^p - 1) \mid L(k, n).$$

If $(2^p - 1)$ is a proper divisor of $L(k, n)$, then $L(k, n)$ is not a prime; a contradiction. If $2^p - 1 = L(k, n)$, then

$$1 + 2 + \cdots + 2^{p-1} = 1 + 2^n + \cdots + (2^n)^{p-1};$$

impossible, since $n > 1$. Thus $p = p_i$ for some i.

Finally, suppose n has more than one prime factor, say $n = p^a x$, $x > 1$. Hence

$$\begin{aligned} L(k, n)(2^n - 1) &= (2^n)^p - 1 = (2^{p^{a+1}})^x - 1 \\ &= (2^{p^{a+1}} - 1)(\cdots) = (2^{p^a} - 1)(\cdots)(\cdots). \end{aligned}$$

Since $(n, p^{a+1}) = p^a$, it follows from Lemma 1 that

$$(2^n - 1, 2^{p^{a+1}} - 1) = 2^{p^a} - 1.$$

Thus

$$(2^{p^a(p-1)} + \cdots + 1) \mid L(k, n).$$

If $(2^{p^a(p-1)} + \cdots + 1)$ is a proper divisor of $L(k, n)$, then $L(K, n)$ is not a prime; a contradiction. On the other hand,

$$(2^{p^a(p-1)} + \cdots + 1) \neq L(k, n) = 1 + 2 + \cdots + (2^n)^{p-1}$$

because $n > p^a$. Thus $n = p^a$ and this completes the proof.

For the remainder of this paper, we shall employ the following notation:

$$L(p^i) = 1 + 2^{p^i} + (2^{p^i})^2 + \cdots + (2^{p^i})^{p-1}.$$

REMARK: By looking at the known factors of $2^n - 1$, the following are prime numbers: $L(3), L(3^2)$, and $L(7)$. The numbers $L(3^3), L(5), L(5^2), L(7^2), L(11), L(13)$, and $L(19)$ are *not* primes.

Motivated by the properties of Fermat and Mersenne numbers, we shall investigate the numbers $L(p^i)$ and present a list of unanswered problems concerning $L(p^i)$.

Problem 1. Determine for which primes p there exists a positive integer i for which $L(p^i)$ is a prime.

Problem 2. For each prime p, determine all the primes of the form $L(p^i)$.

3. RELATIVELY PRIME

It is well known [4, pp. 13 and 18] that each pair of Fermat numbers (also the Mersenne numbers) are relatively prime. We show below that for $i \neq j$, $L(p^i)$ and $L(p^j)$ are relatively prime for any prime p.

The proof of the following lemma can be deduced from Theorem 48 [4, p. 105].

LEMMA 2: For each prime p and each positive integer i, $p \nmid L(p^i)$.

THEOREM 2: The numbers $L(p^i)$ and $L(p^{i+k})$ are relatively prime, if $k > 0$.

PROOF: First we show that for any positive integer j, the numbers $L(p^j)$ and $(2^{p^j} - 1)$ are relatively prime. Suppose $m = \big(L(p^j), (2^{p^j} - 1)\big)$. Since

$$m \mid 2^{p^j} - 1, \; m \mid 2^{p^j n} - 1$$

for any positive integer n. Thus

$$m \mid (2^{p^j(p-1)} - 1) + (2^{p^j(p-2)} - 1) + \cdots + (2^{p^j} - 1) + (1 - 1)$$

implies that $m \mid L(p^j) - p$. Hence $m \mid p$ implies $m = 1$ or p. By Lemma 2, $p \nmid L(p^j)$ and thus $m = 1$. Now

$$\begin{aligned} 2^{p^{i+k}} - 1 &= 2^{(p^{i+k-1})p} - 1 = (2^{p^{i+k-1}} - 1)(2^{p^{i+k-1}(p-1)} + \cdots + 1) \\ &= (2^{p^{i+k-1}} - 1)L(p^{i+k-1}) = (2^p - 1)L(p)L(p^2) \cdots L(p^{i+k-1}). \end{aligned}$$

Suppose the g.c.d. of $L(p^i)$ and $L(p^{i+k})$ is d. Since $L(p^i) \mid 2^{p^{i+k}} - 1$, it follows that $d \mid (2^{p^{i+k}} - 1)$. But $L(p^{i+k})$ and $(2^{p^{i+k}} - 1)$ are relatively prime, thus $d = 1$. The proof is complete.

4. PSEUDOPRIMES

Recall that a number n is called a pseudoprime if $n \mid 2^n - 2$. It is well known [4, p. 115] that each of the Fermat and Mersenne numbers is a pseudoprime. We now show that $L(p^i)$ is a pseudoprime for each i. But first a lemma is needed. It is a consequence of Theorem 48 [4, p. 105].

LEMMA 3: For each prime p and each positive integer i, each prime factor of $L(p^i)$ is of the form $1 + kp^{i+1}$ for some positive integer k.

THEOREM 3: For each prime p and each positive integer i, $L(p^i)$ is a pseudoprime.

PROOF: By Lemma 3, each prime factor of $L(p^i)$ is of the form $1 + kp^{i+1}$ for some positive integer k. Thus, there exists a positive integer x such that

$$L(p^i) = 1 + xp^{i+1}$$

and hence

$$L(p^i) - 1 = xp^{i+1}.$$

Now

$$2^{L(p^i)-1} - 1 = 2^{xp^{i+1}} - 1 = (2^{p^{i+1}} - 1)(2^{p^{i+1}(x-1)} + \cdots + 1).$$

Since $L(p^i)(2^{p^i} - 1) = (2^{p^{i+1}} - 1)$, it follows that

$$L(p^i) \mid (2^{p^{i+1}} - 1)$$

and hence, from above,

$$L(p^i) \mid 2^{L(p^i)-1} - 1.$$

Thus $L(p^i)$ is a pseudoprime.

5. POWERS OF $L(p^i)$

The Fibonacci sequence is defined recursively:

$$F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n.$$

It was shown in [1] and [6] that the only Fibonacci squares are F_1, F_2, and F_{12}. In [3] necessary conditions are given for Fibonacci numbers that are prime powers of an integer. It is well known that each Fermat or Mersenne number [2] cannot be written as a power (greater than one) of an integer. We shall show that $L(3^i)$ also shares this property. However, whether $L(p^i)$, for arbitrary p, has this property or not is an open question.

THEOREM 4: Let q and j be positive integers. If $L(3^i) = q^j$, then $j = 1$.

PROOF: In fact, we prove a bit more. Let n be a positive integer. Suppose $1 + 2^n + (2^n)^2 = q^j$ and $j > 1$. Note that q is odd.

Case 1. $j > 1$ is odd. Let $x = 2^n$. Then

$$x(1 + x) = q^j - 1 = (q - 1)(q^{j-1} + \cdots + q + 1).$$

Since $(q - 1)$ is even, $L = (q^{j-1} + \cdots + 1)$ is odd. It follows that

$$x \mid (q - 1) \quad \text{and} \quad x \le q - 1.$$

Hence $x + 1 \le q < L$; a contradiction.

Case 2. $j > 1$ is even. It suffices to take $j = 2$. Thus

$$1 + 2^n + (2^n)^2 = q^2 \quad \text{or} \quad 2^n(1 + 2^n) = q^2 - 1 = (q - 1)(q + 1).$$

Since both $(q - 1)$ and $(q + 1)$ are even, and $1 + 2^n$ is odd, it follows that

$$q - 1 = 2^a Q \quad \text{and} \quad q + 1 = 2^b V,$$

where both Q and V are odd and $a + b = n$. Now

$$2 = (q + 1) - (q - 1) = 2^b V - 2^a Q = 2(2^{b-1}V - 2^{a-1}Q).$$

Hence $1 = 2^{b-1}V - 2^{a-1}Q$. It is clear that either $a = 1$ or $b = 1$. Suppose $a = 1$. Then $1 + Q = 2^{b-1}V$. If $Q = 1$, then $2 = 2^{b-1}V$ implies that $V = 1$, and this cannot happen. Thus $Q > 1$. Now

$$2^n(1 + 2^n) = 2Q2^{n-1}V = 2^n QV,\ 1 + 2^n = QV = V(2^{n-2}V - 1),$$

and

$$V + 1 = 2^{n-2}V^2 - 2^n = 2^{n-2}(V^2 - 2^2) = 2^{n-2}(V - 2)(V + 2).$$

Clearly, this is a contradiction. The case $b = 1$ is similar. This completes the proof.

We can also show that, for $p = 5, 7, 11$, $L(p^i)$ is not the power (greater than one) of any positive integer. The general case has, so far, eluded our investigation. It is so intriguing that we shall state it as a conjecture.

CONJECTURE 1: Let p be an arbitrary prime and q and j be positive integers. If $L(p^i) = q^j$, then $j = 1$.

6. COMMENTS

Even though each Fermat or Mersenne number is not the power (greater than one) of an integer, it is not known whether they are square-free. Naturally, we make a similar conjecture.

CONJECTURE 2: For each prime p and positive integer i, the number $L(p^i)$ is square-free.

REMARK: It has been shown in [5] that the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ is closely related to the square-freeness of the Fermat and Mersenne numbers. We have shown, by a similar method, that this is also the case for the numbers $L(p^i)$.

It is well known that $(p, 2^p - 1) = 1$ and $(n, 1 + 2^{2^n}) = 1$. Since the prime divisors of $L(p^i)$ are of the form $1 + kp^{i+1}$ [4, p. 106], it follows that

$$(i, L(p^i)) = 1.$$

Finally, we see that while $L(p^i)$ possesses many interesting properties, there remain unanswered some very elementary questions about this class of numbers.

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FIBONACCI NUMBERS OF GRAPHS

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1. INTRODUCTION

According to [1, p. 45], the total number of subsets of $\{1, \ldots, n\}$ such that no two elements are adjacent is F_{n+1}, where F_n is the nth Fibonacci number, which is defined by

$$F_0 = F_1 = 1, \; F_n = F_{n-1} + F_{n-2}.$$

The sequence $\{1, \ldots, n\}$ can be regarded as the vertex set of the graph P_n in Figure 1. Thus, it is natural to define the Fibonacci number $f(X)$ of a (simple) graph X with vertex set V and edge set E to be the total number of subsets S of V such that any two vertices of S are not adjacent.

The Fibonacci number of a graph X is the same as the number of complete (induced) subgraphs of the complement graph of X. (Our terminology covers the empty graph also.)

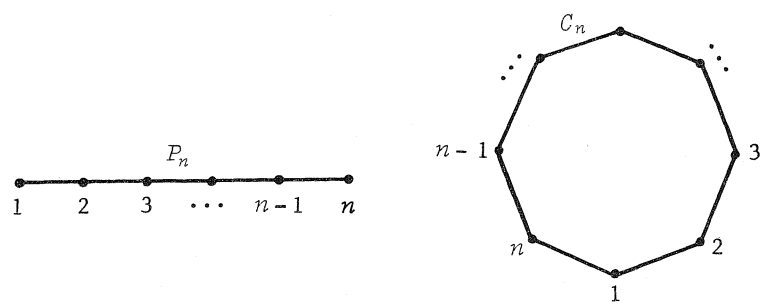

Fig. 1

In [1, p. 46] the case of a cycle C_n with n vertices is considered, as in Figure 1. The Fibonacci number $f(C_n)$ of such a cycle equals the nth Lucas number F_n^*, defined by

$$F_0^* = 2,\ F_1^* = 1,\ F_n^* = F_{n-1}^* + F_{n-2}^*.$$

Let $X_1 = (V, E_1)$ and $X_2 = (V, E_2)$ be two graphs with $E_1 \subseteq E_2$, then

$$f(X_1) \geq f(X_2).$$

So the following simple estimation results:

$$n + 1 = f(K_n) \leq f(X) \leq f(\overline{K_n}) = 2^n, \tag{1.1}$$

where X is a graph with n vertices, and K_n is the complete graph with n vertices and $\overline{K_n}$ its complement.

If X, Y are disjoint graphs, then we trivially obtain, for the Fibonacci number of the union $X \cup Y$,

$$f(X \cup Y) = f(X) \cdot f(Y).$$

2. THE FIBONACCI NUMBERS OF TREES

Trivially, the graph P_n is a tree with $f(P_n) = F_{n+1}$. Another simple example for a tree is the star S_n:

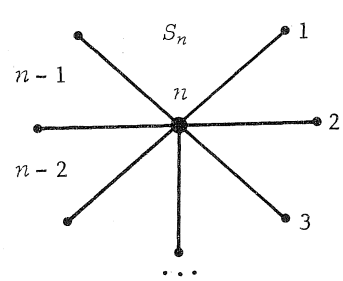

Fig. 2

The Fibonacci number $f(S_n)$ can be computed by counting the number of admissible vertex subsets (they do not contain two adjacent vertices) containing the vertex n or not containing n. Thus

$$f(S_n) = 1 + 2^{n-1}.$$

THEOREM 2.1: Let X be a tree with n vertices, then

$$F_{n+1} \leq f(X) \leq 2^{n-1} + 1.$$

PROOF: First, we prove the second inequality by induction. For $n = 1, 2$, it is trivial. Let X be a tree with $n + 1$ vertices and let v be an endpoint of X. The Fibonacci number $f(X)$ can be computed by counting the number of admissible vertex subsets containing v or not containing v. The number of admissible subsets containing v can trivially be estimated by 2^{n-1} and the number of admissible subsets not containing v can be estimated by $2^{n-1} + 1$ using the induction hypothesis. So we obtain

$$f(X) \leq 2^{n-1} + (2^{n-1} + 1) = 2^n + 1.$$

To prove the first inequality, it is necessary to prove a more general form; hence, we assume X to be a forest. We use induction and, for $n = 1, 2$, the estimation is trivial. Now we proceed by the same argument as above. Let $X = (V, E)$ be a forest with $n + 1$ vertices and v be an endpoint of X. Let X_1 be the induced subgraph of the set $V - \{v\}$ and let w be the adjacent vertex of v. Then X_2 denotes the induced subgraph of the set $V - \{v, w\}$. Trivially, X_1 and X_2 are forests with n and $n - 1$ vertices, respectively. By the induction hypothesis, we obtain

$$f(X) = f(X_1) + f(X_2) \geq F_{n+1} + F_n = F_{n+2},$$

and so the theorem is proved.

REMARK 2.2: There are natural numbers m such that no tree X exists with $f(X) = m$. This is evident because natural numbers m exist not contained in intervals of the form $[F_n, 2^{n-1} + 1]$. Further, there are numbers m contained in such intervals that are not Fibonacci numbers of trees.

EXAMPLE 2.3: Let R_n be the graph with $2n$ vertices as in Figure 3.

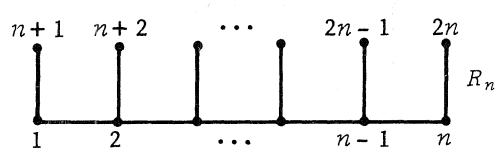

Fig. 3

For the Fibonacci numbers of R_n, we obtain the following recursion

$$f(R_{n+1}) - 2f(R_n) - 2f(R_{n-1}) = 0, \; f(R_1) = 3, \; f(R_2) = 8.$$

The solution of this recursion is

$$f(R_n) = \frac{3 + 2\sqrt{3}}{6}(1 + \sqrt{3})^n + \frac{3 - 2\sqrt{3}}{6}(1 - \sqrt{3})^n.$$

Some other examples are treated in more detail in Section 3.

3. EXAMPLES

Let $X = (V, E)$ be a graph and $y_1, \ldots, y_s$ vertices not contained in V. Then, $Y = (V_1, E_1)$ denotes the graph with

$$V_1 = V \cup \{y_1, \ldots, y_s\} \quad \text{and} \quad E_1 = E \cup \left\{\{y_1, v_j\} \middle| 1 \leq i \leq s, \; v \in V\right\}.$$

By the usual recursion argument, we obtain

$$f(Y) = f(X) + 2^s - 1. \tag{3.1}$$

For an example, we take the following graphs:

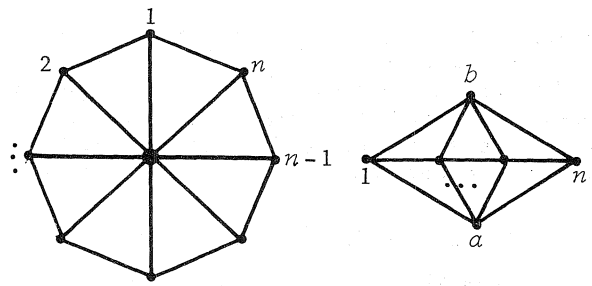

Fig. 4

EXAMPLE 3.2: We consider the graphs Q_n with $2n$ vertices and Q_n' with $2n - 1$ vertices as in Figure 5.

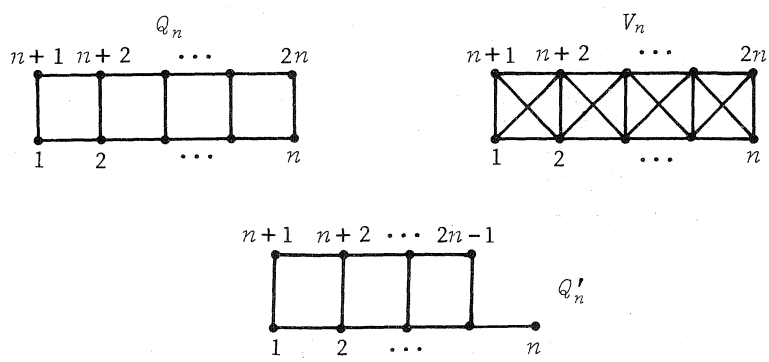

Fig. 5

Let a_n and b_n denote the Fibonacci numbers of Q_n and Q_n', respectively. By our usual recursion argument, we obtain

1. $$a_n = b_n + b_{n-1}, \text{ and}$$
2. $$b_n = a_{n-1} + b_{n-1}.$$

We now have

3. $$b_{n-1} = a_{n-2} + b_{n-2},$$

and by adding (2) and (3),

$$b_n + b_{n-1} = a_{n-1} + a_{n-2} + b_{n-1} + b_{n-2},$$

and so

$$a_n = 2a_{n-1} + a_{n-2};\ a_1 = 3,\ a_2 = 7.$$

This recursion has the solution

$$a_n = \frac{1}{2}(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1} = f(Q_n).$$

EXAMPLE 3.3: Now we consider the graph V_n with $2n$ vertices, as in Figure 5. By the usual recursion argument, we obtain

$$f(V_n) = f(V_{n-1}) + 2f(V_{n-2});\ f(V_1) = 3,\ f(V_2) = 5,$$

and so

$$f(V_n) = \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right).$$

4. PROBLEMS

PROBLEM 4.1: Compute the Fibonacci number $f(L_n)$ of the lattice graph L_n with n^2 vertices in Figure 6.

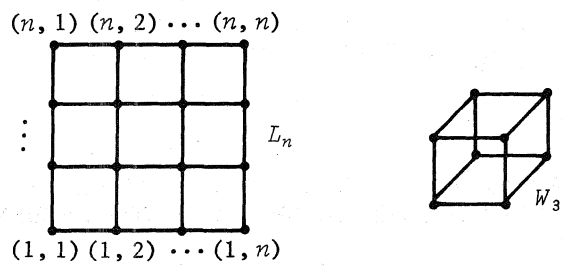

Fig. 6

PROBLEM 4.2: Compute the Fibonacci number of the n-dimensional cube W_n with 2^n vertices in Figure 6.

PROBLEM 4.3: Compute the Fibonacci number of the generalized Peterson graph Pet_n with $4n + 6$ vertices ($n \geq 1$).

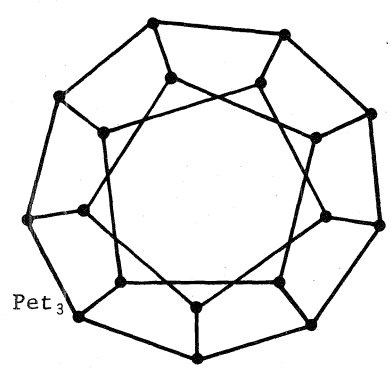

Fig. 7

PROBLEM 4.4: Give a lower bound for $f(X)$ in the case of a planar graph X with n vertices. Give estimations for $f(X)$ if X denotes a regular graph X of degree r or if X denotes an exactly k-connected graph.

PROBLEM 4.5: Let $\omega = (k_n)$ be an increasing sequence of natural numbers, then a sequence Ω of graphs $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ with $F(X_n) = k_n$ exists such that X_i is embedded as an induced subgraph in X_{i+1}. This is trivial if we take for X_n the complete graph $K_{k_n - 1}$.

We define

$$\delta(\omega) = \inf_{\substack{\Omega = (X_n) \\ f(X_n) = k_n}} \left\{\alpha : |E(X_n)| = O\left(|V(X_n)|^{\alpha}\right)\right\}$$

If γ is a class of increasing sequences of natural numbers (e.g., all increasing sequences or the arithmetic progressions), then we define

$$\Delta(\gamma) = \sup_{\omega \in \gamma} \delta(\omega).$$

Trivially, we obtain $\Delta(\gamma) \leq 2$.

The problem is to give better estimations for $\Delta(\gamma)$ in the general case or in the case where γ is the class of all arithmetic progressions.

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SOME IDENTITIES AND DIVISIBILITY PROPERTIES OF LINEAR SECOND-ORDER RECURSION SEQUENCES

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INTRODUCTION

Following Lucas [5], let P and Q be integers such that

$$\text{(i)} \qquad (P, Q) - 1 \quad \text{and} \quad D = P^2 + 4Q \neq 0.$$

Let the roots of

$$\text{(ii)} \qquad x^2 = Px + Q$$

be

$$\text{(iii)} \qquad a = (P + D^{\frac{1}{2}})/2, \; b = (P - D^{\frac{1}{2}})/2.$$

Consider the sequences

$$\text{(iv)} \qquad u^n = (a^n - b^n)/(a - b), \; v_n = a^n + b^n.$$

In this article, we examine sums of the form

$$\sum \binom{k}{j} x_n^j (Qx_{n-1})^{k-j} u_j,$$

where $x_n = u_n$ or v_n, and prove that

$$\text{g.c.d. } (u_n, u_{kn}/u_n) \text{ divides } k,$$

and that

$$\text{g.c.d. } (v_n, v_{kn}/v_n) \text{ divides } k \text{ if } k \text{ is odd.}$$

PRELIMINARIES

(1) $(u_n, Q) = (v_n, Q) = 1$
(2) $(u_n, u_{n-1}) = 1$
(3) $D = (a - b)^2$
(4) $P = a + b, \; Q = -ab$
(5) $v_n = u_{n+1} + Qu_{n-1}$
(6) $au_n + Qu_{n-1} = a^n, \; bu_n + Qu_{n-1} = b^n$
(7) $av_n + Qv_{n-1} = a^n(a - b), \; bv_n + Qv_{n-1} = -b^n(a - b)$
(8) $v_n = Pv_{n-1} + Qv_{n-2}$
(9) P even implies v_n even

(10) k odd implies $v_{kn}/v_n = \sum_{j=0}^{(k-3)/2} v_{(k-1-2j)n}Q^{jn} + Q^{(k-1)n/2}$

REMARKS: (1) is Carmichael [2, Th. I], and (2) follows from [2, Corollary to Th. VI]. (3) follows from (iii), (4) follows from (i) and (iii). (5) follows from (iv) and (4). (6) can be proved by induction, while (7) follows from (5) and (6). (8) follows from (iii) and (iv), (9) follows from Carmichael [2, Th. III], and (10) is Lucas [5, Eq. (44), p. 199].

THE MAIN THEOREMS

THEOREM 1: $u_{kn} = \sum_{j=1}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} u_j.$

PROOF: (iv) implies

$$(a - b)u_{kn} = a^{kn} - b^{kn} = (a^n)^k - (b^n)^k;$$

(6) implies

$$\begin{aligned}(a - b)u_{kn} &= (au_n + Qu_{n-1})^k - (bu_n + Qu_{n-1})^k \\ &= \sum_{j=0}^{k} \binom{k}{j} (au_n)^j (Qu_{n-1})^{k-j} - \sum_{j=0}^{k} \binom{k}{j} (bu_n)^j (Qu_{n-1})^{k-j} \\ &= \sum_{j=0}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} a^j - \sum_{j=0}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} b^j \\ &= \sum_{j=0}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} (a^j - b^j) = \sum_{j=1}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} (a^j - b^j).\end{aligned}$$

Therefore,

$$u_{kn} = \sum_{j=1}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} (a^j - b^j)/(a - b) = \sum_{j=1}^{k} \binom{k}{j} u_n^j (Qu_{n-1})^{k-j} u_j.$$

THEOREM 2: $(u_n, u_{kn}/u_n) | k.$

PROOF: Theorem 1 implies

$$u_{kn}/u_n = \sum_{j=1}^{k} \binom{k}{j} u_n^{j-1} (Qu_{n-1})^{k-j} u_j = k(Qu_{n-1})^{k-1} + \sum_{j=1}^{k} \binom{k}{j} u_n^{j-1} (Qu_{n-1})^{k-j} u_j.$$

Let $d = (u_n, u_{kn}/u_n)$, so that $d|u_n$, $d|u_{kn}/u_n$. Therefore, we have $d|k(Qu_{n-1})^{k-1}$. Now (1), (2) imply $(d, Q) = (d, u_{n-1}) = 1$. Therefore, $d|k$.

THEOREM 3: If k is odd, then

$$D^{(k-1)/2} v_{kn} = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} u_j.$$

PROOF: Together, (iv) and (3) imply

$$\begin{aligned}(a - b)D^{(k-1)/2} v_{kn} &= (a - b)^k (a^{kn} + b^{kn}) \\ &= (a - b)^k a^{kn} + (a - b)^k b^{kn} \\ &= \{(a - b)a^n\}^k - \{-(a - b)b^n\}^k.\end{aligned}$$

(7) implies

$$(a - b)D^{(k-1)/2} v_{kn} = (av_n + Qv_{n-1})^k - (bv_n + Qv_{n-1})^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} (av_n)^j (Qv_{n-1})^{k-j} - \sum_{j=0}^{k} \binom{k}{j} (bv_n)^j (Qv_{n-1})^{k-j}$$

$$= \sum_{j=0}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} a^j - \sum_{j=0}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} b^j$$

$$= \sum_{j=0}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} (a^j - b^j) = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} (a^j - b^j).$$

Therefore,

$$D^{(k-1)/2} v_{kn} = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} (a^j - b^j)/(a - b) = \sum_{j=1}^{k} \binom{k}{j} v_n^j (Qv_{n-1})^{k-j} u_j.$$

LEMMA 1: $(v_n, v_{n-1}) = \begin{cases} 1 \text{ if } P \text{ is odd} \\ 2 \text{ if } P \text{ is even.} \end{cases}$

PROOF: Let $d = (v_n, v_{n-1})$, $d^* = (v_{n-1}, v_{n-2})$. (8) and (1) imply $d|d^*$, while (8) implies $d^*|d$, so that $d = d^*$. Repeating this argument $n - 1$ times, one has $d = (v_1, v_0)$. But (iv) and (4) imply $v_1 = P$ and $v_0 = 2$, so that $d = (P, 2)$. Therefore, P odd implies $d = 1$, P even implies $d = 2$.

LEMMA 2: k odd, P even imply v_{kn}/v_n odd.

PROOF: The hypothesis and (10) imply

$$v_{kn}/v_n - Q^{(k-1)n/2} = \sum_{j=0}^{(k-3)/2} v_{(k-1-2j)n} Q^{jn}$$

The hypothesis and (9) imply $v_{kn}/v_n - Q^{(k-1)n/2}$ is even, whereas the hypothesis and (9) imply Q is odd. Therefore, v_{kn}/v_n is odd.

LEMMA 3: $(v_{n-1}, v_n, v_{kn}/v_n) = 1$ if k is odd.

PROOF: Let $d = (v_n, v_{kn}/v_n)$, so that $d|v_n$ and (d, v_{n-1}) (v_n, v_{n-1}). Now Lemma 1 implies $(v_n, v_{n-1})|2$. Therefore $(d, v_{n-1})|2$. If P is even, Lemma 2 implies d is odd, which implies (d, v_{n-1}) is odd. Therefore $(d, v_{n-1}) = 1$. If P is odd, then Lemma 1 implies $(v_n, v_{n-1}) = 1$. Therefore $(d, v_{n-1}) = 1$.

THEOREM 4: k odd implies $(v_n, v_{kn}/v_n)|k$.

PROOF: The hypothesis and Theorem 2 imply

$$D^{(k-1)/2} v_{kn}/v_n = \sum_{j=1}^{k} \binom{k}{j} v_n^{j-1} (Qv_{n-1})^{k-j} u_j$$

$$= k(Qv_{n-1})^{k-1} + \sum_{j=2}^{k} \binom{k}{j} v_n^{j-1} (Qv_{n-1})^{k-j} u_j.$$

If $d = (v_n, v_{kn}/v_n)$, we have $d|k(Qv_{n-1})^{k-1}$. Now (1) and Lemma 3 imply

$$(d, Q) = (d, v_{n-1}) = 1.$$

Therefore, $d|k$.

CONCLUDING REMARKS

Theorem 1 generalizes a result pertaining to Fibonacci numbers, i.e., the case $P = Q = 1$, by Carlitz and Ferns [1, Eq. (1.6), p. 62] with $k = 0$; by Vinson [6, p. 38] with $r = 0$; and by Halton [3, Eq. (35), p. 35]. Theorem 2 generalizes Halton [4, Lem. XVI] as well as Carmichael [2, Th. XVII]. Theorems 1 and 3 remain valid if u_{kn}, v_{kn} are replaced by u_{kn+r}, v_{kn+r}, while u_j is replaced by u_{j+r}.

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POLYGONAL PRODUCTS OF POLYGONAL NUMBERS AND THE PELL EQUATION

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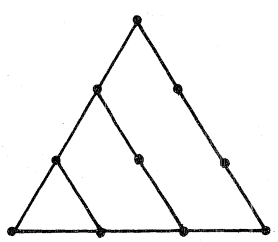
1. INTRODUCTION

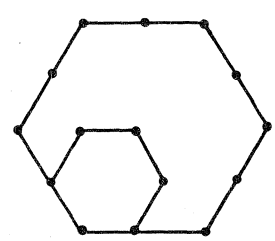
The kth polygonal number of order n (or the kth n-gonal number) P_k^n is given by the equation

$$P_k^n = P_k^n = k[(n-2)(k-1)+2]/2.$$

Diophantus (c. 250 A.D.) noted that if the arithmetic progression with first term 1 and common difference $n - 2$ is considered, then the sum of the first k terms is P_k^n. The usual geometric realization, from which the name derives, is obtained by considering regular polygons with n sides sharing a common angle and having points at equal distances along each side with the total number of points being P_k^n. Two pictorial illustrations follow.



$P_4^3 = 10$



$P_3^6 = 15$

The first forty pages of Dickson's *History of Number Theory*, Vol. II, is devoted to results on polygonal numbers.

In 1968, W. Sierpiński [6] showed that there are infinitely many triangular numbers which at the same time can be written as the sum, the difference, and the product of other triangular numbers. It is easy to show that $4(m^2 + 1)^2$ is the sum, difference, and product of squares. Since then, several authors have proved similar results for sums and differences of other polygonal numbers. R. T. Hansen [2] considered pentagonal numbers, W. J. O'Donnell [4, 5] considered hexagonal and septagonal numbers, and S. Ando [1] proved that for any n infinitely many n-gonal numbers can be written as the sum and difference of other n-gonal numbers. Although Hansen gives several examples of pentagonal numbers written as the product of two other pentagonal numbers, the existence of an infinite class was left in doubt.

In this paper we show that for every n there are infinitely many n-gonal numbers that can be written as the product of two other n-gonal numbers, and in fact show how to generate infinitely many such products. We suspect that our method does not generate all of the solutions for every n, but we have not tried to prove this. Perhaps some reader will be challenged to try to find a product which is not generated by our method. Moreover, except for $n = 3$ and 4, it is still not known whether there are infinitely many n-gonal numbers which at the same time can be written as the sum, difference, and product of n-gonal numbers.

Our proof uses the well-known theory of the Pell equation. We also use a result (not found by us in the literature) on the existence of infinitely many solutions of a Pell equation satisfying a congruence condition, given that one solution exists satisfying the congruence condition. In Section 2 we note some facts about the Pell equation and prove this latter result. In Section 3 we prove our theorem on products of polygonal numbers.

2. THE PELL EQUATION

Although it was first issued by Fermat as a challenge problem, and a complete theory was given by Lagrange, the equation

$$u^2 - Dv^2 = M, \tag{1}$$

where D is not a perfect square, is usually called the Pell equation. The special case

$$u^2 - Dv^2 = 1 \tag{2}$$

always has an infinite number of solutions when D is not a square. In fact, if (u_1, v_1) is the least solution of (2), then any solution (u_j, v_j) is given (see, e.g. [3, pp. 139-48]) by the equation

$$u_j + \sqrt{D}v_j = (u_1 + \sqrt{D}v_1)^j. \tag{3}$$

Also, it is easy to see that if (u^*, v^*) is any particular solution of (1), then (u_j^*, v_j^*), given by

$$u_j^* + \sqrt{D}v_j^* = (u^* + \sqrt{D}v^*)(u_j + \sqrt{D}v_j), \tag{4}$$

is also a solution. Thus, we can generate infinitely many solutions to (1) if we can find one solution.

In what follows, Z^+ denotes the positive integers and $(a, b) \equiv (c, d) \pmod{m}$ means that $a \equiv c$ and $b \equiv d \pmod{m}$. We first prove a result which is heavily dependent upon the representability given by (3) of the solutions to (2).

THEOREM 1: If $D \in Z^+$ is not a square, then for any $m \in Z^+$ there are infinitely many integral solutions to the Pell equation

$$u^2 - Dv^2 = 1 \text{ with } (u, v) \equiv (1, 0) \pmod{m}.$$

PROOF: Suppose (u_1, v_1) is the least solution to (2) and (u_j, v_j) is the solution given by (3). Since there are only m^2 distinct ordered pairs of integers modulo m, there must be $j, \ell \in Z$ such that $(u_j, v_j) \equiv (u_\ell, v_\ell) \pmod{m}$. Using (3) we notice that, for any $t \in Z$,

$$u_t + \sqrt{D}v_t = (u_1 + \sqrt{D}v_1)(u_{t-1} + \sqrt{D}v_{t-1})$$

so

$$u_t = u_1u_{t-1} + Dv_1v_{t-1} \quad \text{and} \quad v_t = v_1u_{t-1} + u_1v_{t-1}.$$

Applying these equations to the above congruence, we deduce

$$u_1u_{j-1} + Dv_1v_{j-1} \equiv u_1u_{\ell-1} + Dv_1v_{\ell-1} \pmod{m} \tag{5}$$

and

$$v_1u_{j-1} + u_1v_{j-1} \equiv v_1u_{\ell-1} + u_1v_{\ell-1} \pmod{m}. \tag{6}$$

Multiplying (6) by u_1 and subtracting v_1 times (5), we have

$$(u_1^2 - Dv_1^2)v_{j-1} \equiv (u_1^2 - Dv_1^2)v_{\ell-1} \pmod{m},$$

or since $u_1^2 - Dv_1^2 = 1$,

$$v_{j-1} \equiv v_{\ell-1} \pmod{m}.$$

Similarly, u_1 times (5) minus Dv_1 times (6) yields

$$u_{j-1} \equiv v_{\ell-1} \pmod{m},$$

so in fact

$$(u_{j-1}, v_{j-1}) \equiv (u_{\ell-1}, v_{\ell-1}) \pmod{m}.$$

We can conclude, therefore, that for $K = |j - \ell|$,

$$(u_0, v_0) \equiv (u_{sK}, v_{sK}) \pmod{m}$$

for any $s \in Z^+$. But $u_0 = 1$ and $v_0 = 0$, so the theorem is proved.

As a corollary we can prove the following theorem about the general Pell equation showing infinitely many solutions in prescribed congruence classes.

THEOREM 2: If $m, D \in Z^+$, D is not a square, and the Pell equation $u^2 - Dv^2 = M$ has a solution

$$(u^*, v^*) \equiv (a, b) \pmod{m},$$

then it has infinitely many solutions

$$(u_t^*, v_t^*) \equiv (a, b) \pmod{m}.$$

PROOF: Let (u^*, v^*) be the solution to (1) provided in the hypothesis, and, for $t \in Z^+$, let (u_t, v_t) be solutions of (2) guaranteed by Theorem 1, that is,

$$(u_t, v_t) \equiv (1, 0) \pmod{m}.$$

Then the solutions (u_t^*, v_t^*) of (1) obtained from these solutions by applying (4) are such that

$$u_t^* = u^*u_t + Dv^*v_t \equiv a \cdot 1 + D \cdot b \cdot 0 \equiv a \pmod{m}$$

and

$$v_t^* = v^*u_t + u^*v_t \equiv b \cdot 1 + a \cdot 0 \equiv b \pmod{m},$$

as desired.

The following corollary follows by taking m in the previous theorem to be the least common multiple of m_1 and m_2.

COROLLARY: If $m_1, m_2, D \in Z^+$, D is not a square, and $a^2 - Db^2 = M$, then there are infinitely many solutions to the Pell equation $u^2 - Dv^2 = M$ with $u \equiv a \pmod{m_1}$ and $v \equiv b \pmod{m_2}$.

3. POLYGONAL PRODUCTS

In this section we first show that any nonsquare n-gonal number is infinitely often the quotient of two n-gonal numbers. The theorem that n-gonal products are infinitely often n-gonal and a remark on the solvability of a related equation complete this section.

THEOREM 3: If the n-gonal number $P = P_s$ is not a square, then there exist infinitely many distinct pairs (P_x, P_y) of n-gonal numbers such that

$$P_x = P_s P_y. \tag{7}$$

PROOF: Recalling that $P_x = \frac{1}{2}x[(n-2)(x-1)+2]$ and setting $n - 2 = r$, Eq. (7) becomes

$$rx^2 - (r-2)x = P[ry^2 - (r-2)y].$$

Multiplying by $4r$ to complete the square gives

$$(2rx - (r-2))^2 - (r-2)^2 = P[(2ry - (r-2))^2 - (r-2)^2].$$

Setting

$$\begin{aligned} u &= 2rx - (r-2), \\ v &= 2ry - (r-2), \end{aligned} \tag{8}$$

we get the Pell equation

$$u^2 - Pv^2 = M, \tag{9}$$

with $M = (r-2)^2 - P(r-2)^2$.

Thus, in order to ensure infinitely many solutions (x, y) to (7), it suffices to have infinitely many solutions (u, v) to (9) for which the pair (x, y) obtained from (8) is integral. Put another way, it suffices to show the existence of infinitely many solutions (u^*, v^*) of (9) for which the congruence

$$(u^*, v^*) \equiv (-(r-2), -(r-2)) \equiv (r+2, r+2) \pmod{2r}$$

holds.

But notice that, since $P_1 = 1$, a particular solution of (7) is $x = s$, $y = 1$, and these values of x and y give

$$\begin{aligned} u &= (2s-1)r + 2, \\ v &= r + 2, \end{aligned}$$

as a particular solution of (9). Thus, we have a solution (u^*, v^*) of (9) with $(u^*, v^*) \equiv (r+2, r+2) \pmod{2r}$. Theorem 2 guarantees the infinitely many solutions we are seeking.

Our final theorem is now a straightforward corollary.

THEOREM 4: For any $n \geq 3$, there are infinitely many n-gonal numbers which can be written as a product of two other n-gonal numbers.

PROOF: The case $n = 4$ is trivial. By the previous theorem, we need only show that P_s is not a square for some s. But for $n \neq 4$, at least one of $P_2 = n$ and $P_9 = 9(4n-7)$ is not a square.

REMARK 1: We originally tried to prove that

$$P_k = k[(n-2)(k-1)+2]/2 = P_x \cdot P_y$$

infinitely often by setting $P_x = k$ and

$$P_y = ((n-2)(P_x - 1) + 2)/2,$$

and solving the Pell equation that results from this last equation. This method works if $n \neq 2t^2 + 2$, and thus, for these values of n, there are infinitely many solutions to the equation $P_{P_x} = P_x P_y$.

REMARK 2: There are 51 solutions of $P_x^3 = P_s^3 P_y^3$ with $P_x < 10^6$. There are 43 solutions of $P_x^n = P_s^n P_y^n$ with $5 \le n \le 36$ and $P_x^n < 10^6$. In just two of these, $x = P_s$:

$$P_{477}^5 = P_{18}^5 P_{22}^5 \quad \text{and} \quad P_{946}^6 = P_{22}^6 P_{31}^6.$$

For $36 \le n \le 720$, there are no solutions with $P_x^n < 10^6$.

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WAITING FOR THE KTH CONSECUTIVE SUCCESS AND THE FIBONACCI SEQUENCE OF ORDER K

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1. INTRODUCTION AND SUMMARY

In the sequel, k is a fixed integer greater than or equal to 2, and n is an integer as specified. Let N_k be a random variable denoting the number of trials until the occurrence of the kth consecutive success in independent trials with constant success probability p $(0 < p < 1)$. Shane [6] and Turner [7] considered the problem of obtaining the distribution of N_k. The first author found a formula for $P[N_k = n]$ $(n \ge k)$, as well as for $P[N_k \le x]$ $(x \ge k)$, in terms of the polynacci polynomials of order k in p. Turner derived a formula for $P[N_k = n + k - 1]$ $(n \ge 1)$ in terms of the entries of the Pascal-T triangle. Both Shane and Turner first treated the special cases $p = 1/2$, $k = 2$, and $p = 1/2$, general k. For these cases, their formulas coincide.

Presently, we reconsider the problem and derive a new and simpler formula for $P[N_k = n + k]$ $(n \ge 0)$, in terms of the multinomial coefficients (see Theorem 3.1). The method of proof is also new. Interestingly enough, our formula includes as corollaries the special formulas of Shane and Turner. We present these results in Section 3. In Section 2, we obtain an expansion of the Fibonacci sequence of order k in terms of the multinomial coefficients (see Theorem 2.1), which is of interest in its own right and instrumental in deriving one of the corollaries.

2. THE FIBONACCI SEQUENCE OF ORDER K

In this section, we consider the Fibonacci sequence of order k and derive an expansion of it, in terms of the multinomial coefficients.

DEFINITION 1: The sequence $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ is said to be the Fibonacci sequence of order k if $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_1^{(k)} & \text{if } 2 \le n \le k \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \ge k+1. \end{cases} \tag{2.1}$$

Turner [7] calls $\left\{f_n^{(k)}\right\}_{n=1}^{\infty}$ the kth-order Fibonacci-T sequence. With $f_n^{(k)} = 0$ $(n \le -1)$, $\left\{f_n^{(k)}\right\}_{n=-\infty}^{\infty}$ is called by Gabai [4] the Fibonacci k-sequence. The shift version of the last sequence, obtained by setting $F_{n,k} = f_{n+1}^{(k)}$, is called by Shane [6] the polynacci sequence of order k. See, also, Fisher and Kohlbecker [3], and Hoggatt [5].

Denoting by F_n and T_n, as usual, the Fibonacci and Tribonacci numbers, respectively, it follows from (2.1) that

$$f_n^{(2)} = F_n \quad \text{and} \quad f_n^{(3)} = T_n,\ n \ge 0. \tag{2.2}$$

The Tribonacci numbers seem to have been introduced by Agronomoff [1]. Their name, however, is due to Feinberg [2], who rediscovered them.

We now proceed to establish the following lemma.

LEMMA 2.1: Let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, and assume that

$$c_n^{(k)} = \begin{cases} 1 & \text{if } n = 0, 1 \\ 2c_{n-1}^{(k)} & \text{if } 2 \le n \le k \\ 2c_{n-1}^{(k)} - c_{n-1-k}^{(k)} & \text{if } n \ge k+1. \end{cases} \tag{2.3}$$

Then

$$c_n^{(k)} = f_{n+1}^{(k)},\ n \ge 0.$$

PROOF: From (2.1) and (2.3), it follows that

$$c_n^{(k)} = f_{n+1}^{(k)},\ 0 \le n \le k. \tag{2.4}$$

Suppose next that

$$c_n^{(k)} = f_{n+1}^{(k)},\ k+1 \le n \le m, \tag{2.5}$$

for some integer $m \ge k+1$. Then

$$\begin{aligned} c_{m+1}^{(k)} &= 2c_m^{(k)} - c_{m-k}^{(k)}, \text{ by (2.3)}, \\ &= 2f_{m+1}^{(k)} - f_{m+1-k}^{(k)}, \text{ by (2.4) and (2.5)}, \\ &= f_{m+2}^{(k)}, \text{ by (2.1)}. \end{aligned} \tag{2.6}$$

Relations (2.4)-(2.6) show the lemma.

We will employ Lemma 2.1 to prove the following lemma.

LEMMA 2.2: Let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, and denote by $A_n^{(k)}$ the number of arrangements of $n+k$ elements (f or s), such that no k adjacent ones are all s, but the last k. Then

$$A_n^{(k)} = f_{n+1}^{(k)},\ n \ge 0.$$

PROOF: For each $n \ge 0$, define $a_n^{(k)}(f)$, $a_n^{(k)}(s)$, and $a_n^{(k)}$ as follows:

(2.7) $a_n^{(k)}(f)$ = the number of arrangements of n elements (f or s), such that the last is f and no k adjacent ones are all s $(n \ge 1)$; $a_0^{(k)}(f) = 1$.

(2.8) $a_n^{(k)}(s)$ = the number of arrangements of n elements (f or s), such that the last is s and no k adjacent ones are all s $(n \ge 1)$; $a_0^{(k)}(s) = 0$.

(2.9) $a_n^{(k)}$ = the number of arrangements of n elements (f or s), such that no k adjacent ones are all s ($n \geq 1$); $a_0^{(k)} = 1$.

Relations (2.7)-(2.9) imply

$$a_n^{(k)} = a_n^{(k)}(f) + a_n^{(k)}(s),\ n \geq 0, \tag{2.10}$$

$$a_{n+1}^{(k)}(f) = a_n^{(k)},\ n \geq 0, \tag{2.11}$$

and

$$a_{n+1}^{(k)}(s) = \begin{cases} a_n^{(k)}, & 0 \leq n \leq k-2 \\ a_n^{(k)} - a_{n+1-k}^{(k)}(f), & n \geq k-1. \end{cases} \tag{2.12}$$

The second part of (2.12) is due to the fact that, for $n \geq k - 1$, $a_{n+1}^{(k)}(s)$ equals $a_n^{(k)}$ minus the number of arrangements among the $a_n^{(k)}(s)$ whose last $k - 1$ elements are all s. Adding (2.11) and (2.12) and utilizing (2.10) and (2.11), we obtain

$$a_{n+2}^{(k)}(f) = \begin{cases} 2a_{n+1}^{(k)}(f), & 0 \leq n \leq k-2 \\ 2a_{n+1}^{(k)}(f) - a_{n+1-k}^{(k)}(f), & n \geq k-1. \end{cases} \tag{2.13}$$

We also have, by the definition of $A_n^{(k)}$ and (2.7), that

$$a_n^{(k)}(f) = A_n^{(k)},\ n \geq 0, \tag{2.14}$$

with $A_0^{(k)} = A_1^{(k)} = 1$. The last two relations give

$$A_n^{(k)} = \begin{cases} 1, & n = 0, 1 \\ 2A_{n-1}^{(k)}, & 2 \leq n \leq k \\ 2A_{n-1}^{(k)} - A_{n-1-k}^{(k)}, & n \geq k+1, \end{cases}$$

which establishes Lemma 2.2, by means of Lemma 2.1

The following theorem gives a formula for the Fibonacci numbers of order k in terms of the multinomial coefficients.

THEOREM 2.1: Let $\{f_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k. Then

$$f_{n+1}^{(k)} = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k},\ n \geq 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ satisfying the relation $n_1 + 2n_2 + \cdots + kn_k = n$.

PROOF: By Lemma 2.2 and (2.14),

$$f_{n+1}^{(k)} = a_n^{(k)}(f),\ n \geq 0. \tag{2.15}$$

Next, observe that an arrangement of n elements (f or s) is one of the $a_n^{(k)}(f)$ if and only if n_1 of its elements are $e_1 = f$; n_2 of its elements are $e_2 = sf, \ldots$; n_k of its elements are $e_k = \underbrace{ss\ldots s}_{k-1}f$ ($n_1 + 2n_2 + \cdots + kn_k = n$). Now, for fixed nonnegative integers $n_1, \ldots, n_k$, the number of arrangements of the $n_1 + \cdots + n_k$ e's is $\binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}$, However, $n_1, \ldots, n_k$ are allowed to vary, subject to the condition $n_1 + 2n_2 + \cdots + kn_k = n$. Therefore,

$$a_n^{(k)}(f) = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k},\ n \geq 0, \tag{2.16}$$

where the summation is taken over all nonnegative integers $n_1, \ldots, n_k$, such that $n_1 + 2n_2 + \cdots + kn_k = n$. Relations (2.15) and (2.16) establish the theorem.

Setting $k = 2$ and $k = 3$ in Theorem 2.1, and defining $[x]$ to be the greatest integer in x, as usual, we obtain the following corollaries, respectively.

COROLLARY 2.1: Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence. Then

$$F_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i}, \quad n \geq 0.$$

COROLLARY 2.2: Let $\{T_n\}_{n=0}^{\infty}$ be the Tribonacci sequence. Then

$$T_{n+1} = \sum_{i=0}^{[n/2]} \sum_{j=0}^{[(n-2i)/3]} \binom{i+j}{i}\binom{n-i-2j}{i+j}, \quad n \geq 0.$$

The first result is well known. The second, however, does not appear to have been noticed.

3. WAITING FOR THE KTH CONSECUTIVE SUCCESS

In this section we state and prove Theorem 3.1, which expresses $P[N_k = n + k]$ $(n \geq 0)$ in terms of the multinomial coefficients. We also give two corollaries of the theorem, which re-establish all the special formulas of Shane [6] and Turner [7] for the probability density function of N_k.

THEOREM 3.1: Let N_k be a random variable denoting the number of trials until the occurrence of the kth consecutive success in independent trials with success probability p $(0 < p < 1)$. Then

$$P[N_k = n + k] = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} (1-p)^{n_1 + \cdots + n_k} p^{n+k-(n_1 + \cdots + n_k)}, \quad n \geq 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$, such that $n_1 + 2n_2 + \cdots + kn_k = n$.

PROOF: A typical element of the event $[N_k = n + k]$ is an arrangement

$$x_1 x_2 \ldots x_{n_1 + \cdots + n_k} \underbrace{ss \ldots s}_{k},$$

such that n_1 of the x's are $e_1 = f$; n_2 of the x's are $e_2 = sf$, ...; n_k of the x's are $e_k = \underbrace{ss \ldots s}_{k-1}f$, and $n_1 + 2n_2 + \cdots + kn_k = n$. Fix $n_1, \ldots, n_k$. Then the number of the above arrangements is $\binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}$, and each one of them has probability

$$P\left[x_1 x_2 \ldots x_{n_1 + \cdots + n_k} \underbrace{ss \ldots s}_{k}\right] = \left[P\{e_1\}\right]^{n_1} \left[P\{e_2\}\right]^{n_2} \ldots \left[P\{e_k\}\right]^{n_k} P\{\underbrace{ss \ldots s}_{k}\}$$
$$= (1-p)^{n_1 + \cdots + n_k} p^{n+k-(n_1 + \cdots + n_k)}, \quad n \geq 0,$$

by the independence of the trials, the definition of e_j $(1 \leq j \leq k)$, and $P\{s\} = p$. Therefore,

$$P\left[\text{all } x_1 x_2 \ldots x_{n_1 + \cdots + n_k} \underbrace{ss \ldots s}_{k};\ n_j \geq 0 \text{ fixed } (1 \leq j \leq k),\ \sum_{j=1}^{k} jn_j = n\right]$$
$$= \binom{n_1 + \cdots + n_k}{n_1, \ldots n_k} (1-p)^{n_1 + \cdots + n_k} p^{n+k-(n_1 + \cdots + n_k)}, \quad n \geq 0.$$

But the nonnegative integers $n_1, \ldots, n_k$ may vary, subject to the condition $n_1 + 2n_2 + \cdots + kn_k = n$. Consequently,

$$\begin{aligned} P[N_k &= n + k] \\ &= P\left[\text{all } x_1x_2 \ldots x_{n_1+\cdots+n_k}\underbrace{ss \ldots s}_{k};\ n_j \geq 0\ (1 \leq j \leq n),\ \sum_{j=1}^{k} jn_j = n\right] \\ &= \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}(1 - p)^{n_1+\cdots+n_k}p^{n+k-(n_1+\cdots+n_k)},\ n \geq 0, \end{aligned}$$

where the summation is over all $n_1, \ldots, n_k$ as above, and this establishes the theorem.

We now have the following obvious corollary to the theorem.

COROLLARY 3.1: Let N_k be as in Theorem 3.1, and assume $k = 2$. Then

$$P[N_2 = n + 2] = \sum_{i=0}^{[n/2]} \binom{n - i}{i} p^{i+2}(1 - p)^{n-i},\ n \geq 0.$$

This result is a simpler version of Turner's [7] formula for general p and $k = 2$ (our notation).

We also have the following corollary, by means of Theorem 2.1.

COROLLARY 3.2: Let N_k be as in Theorem 3.1, and assume $p = 1/2$. Then

$$P[N_k = n + k] = f_{n+1}^{(k)}/2^{n+k},\ n \geq 0.$$

This result is a version of formula (12) of Shane [6] and of Turner's [7] formula for general k and $p = 1/2$.

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THE LENGTH OF THE FOUR-NUMBER GAME

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INTRODUCTION

Let D be the operator defined on 4-tuples of nonnegative integers by

$$D(w, x, y, z) = (|w - z|, |w - x|, |x - y|, |y - z|).$$

Given any initial 4-tuple $S = S_0 = (w_0, x_0, y_0, z_0)$, we obtain a sequence $\{S_n\}$, where $S_{n+1} = DS_n$. This sequence is sometimes called the four-number game. The following curious fact seems to have been discovered and rediscovered several times—[3], [4], [5]—$S_n = (0, 0, 0, 0)$ for all sufficiently large n. We can thus make the following definition.

DEFINITION: The length of the sequence $\{S_n\}$, denoted $L(S)$, is the smallest n such that $S_n = (0, 0, 0, 0)$.

A natural question to ask is: "How long can a game continue before all zeros are reached?" Again, it is well known that the length can be arbitrarily long if the numbers in S_n are sufficiently large [4]. One of the easiest ways to see this makes use of the so-called Tribonacci numbers:

$$t_0 = 0,\ t_1 = 1,\ t_2 = 1 \quad \text{and} \quad t_n = t_{n-1} + t_{n-2} + t_{n-3} \quad \text{for } n \geq 3.$$

If we let $T_n = (t_n, t_{n-1}, t_{n-2}, t_{n-3})$, then a simple calculation shows that

$$D^3T_n = 2T_{n-2},$$

and so

$$L(T_n) = 3\left[\frac{n}{2}\right].$$

It has also been noticed that the sequence beginning with some T_n seems to have the longest length of any sequence whose original elements do not exceed t_n. We will prove that this is almost true.

It should be pointed out that if we allow the elements of S_0 to be real, then we can obtain a game of infinite length by taking $S_0 = (r^3, r^2, r, 1)$, where $r = 1.839\ldots$ is the real root of the equation $x^3 - x^2 - x - 1 = 0$ (see [2], [6], [7]). Moreover, this is essentially the only way to obtain a game of infinite length [7]. To obtain a long game with integer entries, we should pick the initial terms to have ratios approximating r [1]. The Tribonacci numbers do this very nicely.

MAIN RESULT

Before proving our main theorem, we need a few easy observations. If

$$|S| = \max(w, x, y, z),$$

then

$$|S_0| \geq |S_1| \geq |S_2| \ \ldots .$$

The games having initial elements

$$(w, x, y, z),\ (x, y, z, w),\ (y, z, w, x),\ (z, w, x, y);$$

$$(z, y, x, w);\ (w + k, x + k, y + k, z + k);$$

and $$(kw, kx, ky, kz),\ k > 0;$$

all have the same length. We now state our main theorem which will be an immediate consequence of Theorem 2.

THEOREM 1: If $|S| \le |T_n|$, then $L(S) \le L(T_n) + 1 = 3\left[\frac{n}{2}\right] + 1$.

One of the first things to notice is that $L(S) \le 6$, unless the elements of S are monotonically decreasing, $w > x > y > z$. [Remember, cyclic permutations and reversals yield equivalent games, so $(5, 7, 12, 2) \sim (2, 5, 7, 12) \sim (12, 7, 5, 2)$, which is monotonically decreasing.) This can be checked by simply calculating the first six S_n if S_0 is not monotonic [1]. Also, if S_n is monotonic decreasing, then S_{n+1} cannot be monotonic increasing. Therefore, in a long game, all of the S_n at the beginning must be monotonic decreasing.

Let $S_n = (w_n, x_n, y_n, z_n)$. We say that S_n is additive if $w_n = x_n + y_n + z_n$. If S_{n-1} is monotonic (decreasing), then a trivial calculation shows that S_n is additive. Thus, although $S = S_0$ may not be additive, $S_1, S_2, \ldots, S_n$ will be additive as long as $S_0, S_1, \ldots, S_{n-1}$ are monotonic.

LEMMA: If $S_1, S_2, \ldots, S_{10}$ are all monotonic (decreasing), S_1 is additive, and $|S_1| \le t_n$, then either $|S_4| \le 2t_{n-2}$ or $|S_7| \le 4t_{n-4}$ or $|S_{10}| \le 8t_{n-6}$.

PROOF: Write $S_1 = (a + b + c, a, b, c)$ and assume $a + b + c \le t_n$,

$$|S_4| > 2t_{n-2},\ |S_7| > 4t_{n-4},\ \text{and}\ |S_{10}| > 8t_{n-6}.$$

Since we know that $S_1 \ldots S_{10}$ are all monotonic, they can be explicitly calculated, and we find that

$$|S_4| = 2b,\ |S_7| = 4a - 4b - 4c,\ \text{and}\ |S_{10}| = 16c - 8b.$$

$|S_4| > 2t_{n-2}$ implies $2b \ge 2t_{n-2} + 2$ or $3b \ge 3t_{n-2} + 3$; $|S_7| > 4t_{n-4}$ implies $a - b - c \ge t_{n-4} + 1$; $|S_{10}| > 8t_{n-6}$ implies $2c - b \ge t_{n-6} + 1$. Adding these three inequalities, we obtain

$$a + b + c \ge 3t_{n-2} + t_{n-4} + t_{n-6} + 5.$$

But since $a + b + c \le t_n$, we have

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} \ge 3t_{n-2} + t_{n-4} + t_{n-6} + 5.$$

Using the defining relation of the Tribonacci numbers repeatedly, we get

$$2t_{n-3} \ge 2t_{n-3} + 5,$$

which is an obvious contradiction. This proves the lemma.

THEOREM 2: If S_1 is additive and $|S_1| \le t_n$, then $L(S_1) \le L(T_n) = 3\left[\frac{n}{2}\right]$, $n \ge 2$.

PROOF: Since S_1 is additive, we may write $S_1 = (a + b + c, a, b, c)$, where $t_{n-1} < a + b + c \le t_n$. We use induction on n. We can check the first 'few' cases (by computer) and see that the theorem is true for $n = 2, 3, \ldots, 9$. (That is, $|S_1| \le 81$.) Now, assume the result is true for all S_1 such that $|S_1| \le t_k$, where $k < n$, $n \ge 10$.

If $S_1, \ldots, S_{10}$ are all monotonic, then, by the induction hypothesis and the lemma, either

$$L(S_1) = L(S_4) + 3 \le 3\left[\frac{n-2}{2}\right] + 3 = 3\left[\frac{n}{2}\right]$$

or

$$L(S_1) = L(S_7) + 6 \le 3\left[\frac{n-4}{2}\right] + 6 = 3\left[\frac{n}{2}\right]$$

or

$$L(S_1) = L(S_{10}) + 9 \le 3\left[\frac{n-6}{2}\right] + 9 = 3\left[\frac{n}{2}\right].$$

Here we have used the fact that 2^t divides every element of S_{3t+1}, $t \ge 1$. Thus, for example, $S_4 = 2S_4^*$ and $L(S_4) = L(S_4^*)$. If $|S_4| \le 2t_{n-2}$, then $|S_4^*| \le t_{n-2}$, and so

$$L(S_4^*) \le L(T_{n-2}) = 3\left[\frac{n-2}{2}\right],$$

by the induction hypothesis, taking S_4^* as our 'new' S_1. Thus, in any case,

$$L(S_1) \leq 3\left[\frac{n}{2}\right].$$

If $S_1, \ldots, S_{10}$ are not all monotonic, let S_j be the first which is nonmonotonic. Then

$$L(S_1) = L(S_j) + (j - 1) \leq 6 + j - 1 = j + 5 \leq 15,$$

since $L(S_j) \leq 6$ whenever S_j is not monotonic. But since $n \geq 10$,

$$L(T_n) = 3\left[\frac{n}{2}\right] \geq 15,$$

so $L(S_1) \leq L(T_n)$.

This completes the proof of Theorem 2.

Theorem 1 is now an easy corollary, since: if S_0 is monotonic decreasing and $|S_0| \leq |T_n|$, then $|S_1| \leq |T_n|$ and S_1 is additive. If S_0 is not monotonic decreasing, then $L(S_0) \leq 6$.

There actually are examples where $L(S) = L(T_n) + 1$:

$$L(T_6) = L(13, 7, 4, 2) = 9 \quad \text{and} \quad L(13, 6, 2, 0) = 10.$$

ly,

$$L(a + b + c, b + c, c, 0) = L(a + b + c, a, b, c) + 1;$$

$$L(t_n, t_{n-2} + t_{n-3}, t_{n-3}, 0) = L(T_n) + 1 = 3\left[\frac{n}{2}\right] + 1.$$

If we begin with a k-tuple of nonnegative integers, then it is known that $S_n = (0, 0, \ldots, 0)$ for sufficiently large n, provided $k = 2^t$. (If $k \neq 2^t$, the sequence $\{S_n\}$ may cycle [3], [4], [9].) Thus, a natural question to ask is: "What is the maximum length of the eight number game, or, more generally, the 2^t-number game?"

It was already mentioned that if S_1 is additive and leads to a long four-number game, then the ratios of the elements of S_1 should be close to the number $r = 1.839\ldots$. How accurately can the length of the game be predicted if one knows these ratios?

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SUMS OF CONSECUTIVE INTEGERS

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The purpose of this note is to simplify and extend the results in [1]. Given a positive integer n, let $C_e(n)$, $C_o(n)$ denote the number of representations of as a sum of an even, odd number of consecutive positive integers.

THEOREM 1: $C_o(n)$ is the number of odd divisors d of n such that $\frac{d(d+1)}{2} \le n$ and $C_e(n)$ is the number of odd divisors d of n such that $\frac{d(d+1)}{2} > n$.

PROOF: If n is a sum of an odd number b of consecutive integers, then there exists an integer $a \ge 1$ such that

$$n = \sum_{i=0}^{b-1} (a+i) = b\left(a + \frac{b-1}{2}\right).$$

Hence b is an odd divisor of n with $\frac{b(b+1)}{2} \le n$, since

$$\frac{b+1}{2} \le a + \frac{b-1}{2} = \frac{n}{b}.$$

If b is an odd divisor of n such that $\frac{b(b+1)}{2} \le n$, let $a = \frac{n}{b} - \frac{b-1}{2}$. Then $a \ge 1$ and

$$n = b\,a + \frac{b-1}{2} = \sum_{i=0}^{b-1} (a+i),$$

so that n is the sum of an odd number of consecutive positive integers.

If n is a sum of an even number b of consecutive positive integers, then there exists an integer $a \ge 1$ such that

$$n = \sum_{i=0}^{b-1} (a+i) = \frac{b}{2}(2a+b-1).$$

Let $d = 2a + b - 1$, then d is odd, d divides n, and $\frac{d(d+1)}{2} > n$, since

$$d + 1 = 2a + b > b = \frac{2n}{d}.$$

If d is an odd divisor of n such that $\frac{d(d+1)}{2} > n$, let $b = \frac{2n}{d}$ and $a = \frac{(d+1-b)}{2}$. Then $a \ge 1$, b is even, and

$$n = \frac{bd}{2} = \frac{b}{2}(2a+b-1) = \sum_{i=0}^{b-1} (a+i),$$

so that n is a sum of an even number of consecutive positive integers. □

An immediate consequence of Theorem 1 is the following corollary.

COROLLARY 1: Let $n = 2\,m$, $r \ge 0$, m odd. The number of representations of n as a sum of consecutive positive numbers is $\tau(m)$ (the number of divisors of m). □

This result is also in [2], which of course gives the results in [1].

We also find a characterization of primes.

COROLLARY 2: Let n be an odd positive integer. Then n is composite if and only if there is a pair of positive numbers u, v such that

$$8n = u^2 - v^2;\ u - v \geq 6. \tag{1}$$

PROOF: If n is odd composite, then n is the sum of at least three consecutive integers by Theorem 1. That is

$$n = a + (a + 1) + \cdots + (a + k),\ k \geq 2.$$

Hence $2n = (k + 1)(2a + k)$. Let $v = 2a - 1$ and $u = 2k + 2a + 1$. Then

$$k + 1 = \frac{u - v}{2} \quad \text{and} \quad 2a + k = \frac{u + v}{2},$$

so that $8n = u^2 - v^2$ and $u - v \geq 6$. Note that u, v are odd. Conversely, given an odd integer n satisfying (1), we find

$$8n = (u + v)(u - v).$$

If n is prime and $u - v$ is even, then $u - v = 8$, $2n$, or $4n$. When $u - v = 8$, we have $2u = n + 8$ so that $n = 2$, while $u - v = 2n$ implies that $u = 2 + n$, and hence $v = 2 - n \leq 0$. If $u - v = 4n$, then $u + v = 2$ and $u = v = 1$, which says that $n = 0$. Thus, if n is a prime, we must have $u + v = 8$ and $u - v = n$, which implies once again that $n = 2$.

We conclude that n must be composite. It is also simple to solve the above system for a and k. □

It is not easy to find $C_o(n)$ explicitly. For instance, let $\tau_o(n, x)$ denote the number of odd positive divisors of n which are $\leq x$. One finds

$$\tau_o(n, x) = \sum_{\substack{d \leq x \\ d \text{ odd}}} \frac{c_d(n)}{d} \sum_{\substack{k \leq x/d \\ k \text{ odd}}} \frac{1}{k},$$

where $c_d(n)$ is the Ramanujan function. This is not altogether satisfactory, but it will yield an estimate. One direct but very elaborate way to find $\tau_o(n, x)$ explicitly is by counting lattice points as follows. Write $n = 2^{a_0} p_1^{a_1} \ldots p_k^{a_k}$ as a product of primes. An odd divisor d of n is of the form $d = p_1^{b_1} \ldots p_k^{b_k}$, where $0 \leq b_i \leq a_i$. The inequality $d \leq x$ means

$$b_1 \log p_1 + \cdots + b_k \log p_k \leq \log x.$$

Let $e_1, \ldots, e_k$ be the standard basis of $\mathbb{R}^k$. Consider the parallel-piped P determined by $a_1 e_1, \ldots, a_k e_k$ and the hyperplane H with equation

$$x_1 \log p_1 + \cdots + x_k \log p_k = \log x.$$

Then $\tau_o(n, x)$ is the number of lattice points in the region "below" H which are also contained in P. There are of course k^2 possible intersections of H with P to consider, a formidable task! However, we have, perhaps a little surprisingly,

COROLLARY 3: Write $n = 2^k m$, where m is odd. Then

$$C_o(n) \quad \frac{1}{2}\tau(m);\ C_1(n) \leq \left(k + \frac{1}{2}\right)\tau(m).$$

In particular, when n is odd, we have

$$C_e(n) \leq \frac{\tau(n)}{2} \leq C_o(n).$$

PROOF: It is very easy to show that

$$\sqrt{n} \leq \frac{-1 + \sqrt{1 + 8n}}{2}, \tag{1}$$

and if $d > 0$, then

$$\frac{d(d+1)}{2} \leq n \Longleftrightarrow d \leq \frac{-1+\sqrt{1+8n}}{2} \tag{2}$$

Thus $C_0(n)$ is at least the number of odd divisors d of n that are $\leq \sqrt{n}$, so *a fortiori* we have

$$C_0(n) \geq \tau_0(m, \sqrt{m}).$$

If $d|m$ and $d \leq \sqrt{m}$, then $m/d|m$ and $m|d \geq \sqrt{m}$. Thus

$$\tau(m, \sqrt{m}) = \begin{cases} \dfrac{\tau(m)}{2} & \text{if } m \text{ is not a square} \\ \dfrac{\tau(m)+1}{2} & \text{if } m \text{ is a square.} \end{cases}$$

Hence $C_0(n) \geq \tau(m)/2$. We have $C_1(n) = \tau(n) - C_0(n)$, and thus

$$C_1(n) \leq (k+1)\tau(m) - \frac{\tau(m)}{2} = \left(k + \frac{1}{2}\right)\tau(m).$$

This completes the proof. □

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CONCERNING A PAPER BY L. G. WILSON

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1. INTRODUCTION

Wilson [3] uses the expression (2.1) below, which approximates the Fibonacci and Lucas sequences $\{F_r\}$ and $\{L_r\}$, respectively, for r sufficiently large. The object of this paper is to make known this and another expression (3.1) by applying techniques different from those used in [3]. In particular, we need

$$\beta_i = 4\cos^2\frac{i\pi}{2n}. \tag{1.1}$$

Special attention is directed to the sequence (2.4).

2. A GENERATING EXPRESSION

Consider

$$F_r(x, y) \equiv T_r = \left(\frac{x+\sqrt{x^2+4x}}{2}\right)^{r-1} y^{-1/2}, \tag{2.1}$$

in which x and y are real numbers and $r \to \infty$. Some applications of this expression are given in Examples 1-3.

EXAMPLE 1: Let $x = 1$ and $y = 5$, then

$$F_{r+1}(1,\ 5) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^r = \frac{\alpha^r}{\sqrt{5}}, \text{ where } \alpha = \frac{1+\sqrt{5}}{2}. \tag{2.2}$$

Using Binet's formula, we see that

$$\lim_{r\to\infty} F_{r+1}(1,\ 5) = F_r.$$

EXAMPLE 2: Let $x = 1$ and $y = \dfrac{3-\sqrt{5}}{2} = 4\cos^2\dfrac{2\pi}{5}$, then $y^{-1} = \alpha^2$. Hence,

$$\begin{cases} F_{r+1}(1,\ 1) = \left(\dfrac{1+\sqrt{5}}{2}\right)^r = \alpha^r \\ \lim\limits_{r\to\infty} F_{r+1}(1,\ 1) = L_r. \end{cases} \tag{2.3}$$

EXAMPLE 3: Let x be the real root of $t^3 + t^2 - 1 = 0$ and $y = x$. It can be verified that

$$x = \sqrt[3]{\frac{25}{54} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{25}{54} - \sqrt{\frac{23}{108}}} - \frac{1}{3},$$

and it is shown in [2] that the reciprocal of x is the real root of the characteristic equation for (2.5) below. For r sufficiently large, (2.1) approximates the Neumann sequence discussed in [2] and [3] and given by

$$\begin{array}{cccccccccccccccccccc} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} & T_{17} & \cdots \\ 3 & 0 & 2 & 3 & 2 & 5 & 5 & 7 & 10 & 12 & 17 & 22 & 29 & 39 & 51 & 68 & 90 & 119 & \cdots \end{array} \tag{2.4}$$

where

$$T_n = T_{n-2} + T_{n-3} \qquad (n \geq 3). \tag{2.5}$$

This is possibly the slowest growing integer sequence for which $p|T_p$ for all prime (see [2]).

3. COMPLEX SEQUENCES

Write

$$F_{mn} = \left(\frac{x + \sqrt{x^2 + 4x}}{2}\right)^n, \tag{3.1}$$

where

$$x = z_m = -\beta_m = -4\cos^2\frac{m\pi}{2n} \text{ [by (1.1)]}, \ m = 1,\ 2,\ \ldots,\ n-1. \tag{3.2}$$

Then

$$\begin{cases} F_{mn} = (-1)^n 2^n \cos^{2n}\left(\dfrac{m\pi}{2n}\right)\left\{1 + i\tan\dfrac{m\pi}{2n}\right\} \\ \qquad = (-1)^n 2^n \cos^n\left(\dfrac{m\pi}{2n}\right)e^{im\pi/2} \qquad \text{by Euler's Theorem} \\ \qquad = (-1)^n \beta_m^{n/2}\, e^{im\pi/2} \end{cases} \tag{3.3}$$

so

$$\begin{cases} \sum_{m=1}^{n-1} F_{mn}^2 = \sum_{m=1}^{n-1} (-1)^{2n} \beta_m^n e^{im\pi} \\ \qquad = \pm \sum_{m=1}^{n-1} (-1)^n \beta_m^n \qquad \text{according as } n \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases} \\ \qquad = \pm \sum_{m=1}^{n-1} z_m^n \text{ [by (3.2)], according as } n \text{ is } \begin{cases}\text{odd}\\ \text{even.}\end{cases} \end{cases}$$

When $n = 6$, (3.3) gives

$$F_{16} = (26 + 15\sqrt{3})i,\ F_{26} = -27,\ F_{36} = -8i,\ F_{46} = 1,\ F_{56} = (26 - 15\sqrt{3})i.$$

From (3.2),

$$z_1 = -(2 + \sqrt{3}),\ z_2 = -3,\ z_3 = -2,\ z_4 = -1,\ z_5 = -(2 - \sqrt{3}).$$

Hence, by (3.4),

$$\sum_{m=1}^{5} F_{m6}^2 = -\sum_{m=1}^{5} z_m^6 = -6^6.$$

From (3.3), it is clear that the F_{mn} are, alternately, purely real and purely imaginary.

Together, (2.1) and (3.1) yield

$$T_{r+1} = F_{mn}^{r/n} y^{-1/2}. \tag{3.5}$$

Wilson [3] also gives the cases $y = -1$ and $y = -3$ with $n = 6$, $m = 1$, so

$$x = -4 \cos^2 \frac{\pi}{12} = -(2 + \sqrt{3}) \quad \text{[by (3.2)]}.$$

This produces what he calls "regular complex Fibonacci sequences," by which he means that terms at regular intervals are either purely real or purely imaginary (while in all other cases the terms are of the form $a + ib$, where a, b are real). The period of these "cycles" is, in both cases, 6, beginning with T_1. Details of the computation involved are omitted here in the interest of brevity, and are left to the reader's curiosity.

4. CONCLUDING COMMENTS

Return now to Examples 1-3 in Section 2.

In (3.5), take $n = 5$, $y = 5$, $m = 2$, i.e., $\beta_2 = 4 \cos^2 \frac{\pi}{5}$ by (1.1). Then Example 1, (2.2), results.

Next, with $n = 5$, $m = 2$ again, but with $y = \beta_4 = 4 \cos^2 \frac{2\pi}{5}$ by (1.1), Example 2, (2.3), results.

Furthermore, observe that, when $x = 1$ and $y = 5$, the recurrence relation for $T_r = F_r(1, 5)$ in (2.2) is given by

$$T_{r+1} = \sum_{n=0}^{r} (-1)^n \binom{r}{n} T_{2r-2n+1} \left[= \frac{\alpha^r}{\sqrt{5}} = \frac{(-1 + \alpha^2)^r}{\sqrt{5}}\right]. \tag{4.1}$$

This is related to the more general

$$T_n = \sum_{r=2}^{d} a_r T_{n-r}, \tag{4.2}$$

given in Neumann and Wilson [2], where $T_0 = d$, $T_1 = 0$, $T_2 = 2a_2$, $T_3 = 3a_3$, ...

When $d = 3$, $a_2 = 1$, $a_3 = 1$, we obtain from (4.2) the Neumann sequence (2.4), which, as we have noted, can also be generated by Wilson's function (2.1).

Finally, we observe that

$$(4.3)\qquad \begin{cases} f_r = F_r(1,\ 5) = (-1)^r/5 \cdot F_{r+1}(1,\ 5) \\ \ell_r = F_r(1,\ 1) + (-1)^r/F_r(1,\ 1). \end{cases}$$

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A GENERALIZATION OF THE DIRICHLET PRODUCT

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1. INTRODUCTION

If f is the Dirichlet product of arithmetical functions g and h, then by definition

$$f(n) = \sum_{d|n} g(d)h(n/d).$$

In this paper we define a convolution of two arithmetical functions that generalizes the Dirichlet product. With this new convolution, which we shall refer to as the the "k-prime product," it is possible to define arithmetical functions which are analogs of certain well-known functions such as Euler's function $\phi(n)$, defined implicitly by the relation

$$(1.1)\qquad \sum_{d\delta=n} \phi(d) = n.$$

Other well-known functions to be considered in this paper include $\tau(n)$ and $\sigma(n)$ given by $\tau(n) = \Sigma 1$ and $\sigma(n) = \Sigma d$, where the summations are over the positive divisors of n. The familiar Moebius function $\mu(n)$ is defined as the multiplicative function with the evaluation $\mu(p) = -1$ and $\mu(p^e) = 0$ if $e > 1$, and satisfies the relation

$$(1.2)\qquad \sum_{d\delta=n} \mu(d) = \varepsilon(n) \equiv \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu(1) = 1$, since μ is a nonzero multiplicative function. Upon applying the Moebius inversion formula to (1.1), one obtains the simple Dirichlet product representation for ϕ,

$$(1.3)\qquad \phi(n) = \sum_{d\delta=n} \mu(d)\delta.$$

Another function which may be defined by means of the Dirichlet product is $q(n)$, the characteristic function of the set Q of square-free integers,

$$q(n) = \sum_{d\delta = n} \nu_2(d) = \sum_{d^2\delta = n} \mu(d), \tag{1.4}$$

where $\nu_2(n) = \mu(m)$ if $n = m^2$ and $\nu_2(n) = 0$ otherwise. The representations (1.3) and (1.4) are extremely useful in the development of the theory of the Euler function and the set of square-free integers.

In Section 2, we define appropriate generalizations of the concepts mentioned above and prove a generalized Meobius inversion formula (Theorem 2.4).

Included in Section 3 is a short discussion of the usefulness of the results obtained in Section 2 and an indication of the direction in which further study should be directed.

2. THE GENERALIZED PRODUCT

For each integer $k \geq 1$, let L_k represent the set of positive integers n with the property that if a prime p divides n, then p^k also divides n. A number in L_k is said to be "k-full." Let Q_k be the set of positive integers n such that each prime divisor of n has multiplicity less than k. A number in Q_k is said to be "k-free." Any positive integer n can be written uniquely in the form $n = n_1n_2$, where $n_1 \in L_k$, $n_2 \in Q_k$, and $(n_1, n_2) = 1$. If m and n are positive integers with unique decompositions $m = m_1m_2$ and $n = n_1n_2$, then m and n are said to be "relatively k-prime" [notation: $(m, n)_k = 1$] provided that $(m_2, n_2) = 1$. Given arithmetical functions $f(n)$ and $g(n)$, we define the "k-prime product" of f and g (notation: $f \circ g$) as follows:

$$(f \circ g)(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = n}} f(d)g(\delta).$$

For $k = 1$, the k-prime product reduces to the Dirichlet product. The next two theorems are proved by arguments similar to those used in the case $k = 1$.

THEOREM 2.1: The k-prime product is an associative operation.

More can be said about the algebraic structure of our system. As is the case in the Dirichlet product, the arithmetic functions form a cummulation ring with unity under addition and the k-prime product.

THEOREM 2.2: If each of g and h is a multiplicative function, then $g \circ h$ is multiplicative.

We now define the generalization of the Moebius function which was mentioned earlier.

DEFINITION 2.1: Let $\mu_k(n)$ denote the multiplicative function for which $\mu_k(p^n)$ is -1, 1, 0 whenever $0 < n < k$, $k < n < 2k$, and $n \geq 2k$, respectively. Clearly, this is a valid generalization of Moebius' function, and we shall see later on that $\mu_k(n)$ plays much the same role in the development of the theory for the k-prime product as $\mu(n)$ does in the case of the Dirichlet product. In particular, we have the following two theorems.

THEOREM 2.3: $\displaystyle\sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \mu_k(d) = \varepsilon(n).$

PROOF: For $n = 1$ the theorem is obvious. By Theorem 2.2, we need only prove the theorem for prime powers, $n = p^e$, $e > 0$. Now, if $e < k$, we have

$$\sum_{\substack{d\delta = p^e \\ (d,\delta)_k = 1}} \mu_k(d) = \mu_k(1) + \mu_k(p^e) = 1 - 1 = 0,$$

by the definition of relatively k-prime and μ_k. In the case $e \geq k$, we have

$$\sum_{\substack{d\delta = p^e \\ (d,\delta)_k = 1}} \mu_k(d) = \sum_{\substack{a=0 \\ \max(a, e-a) \geq k}}^{e} \mu_k(p^a) = \sum_{\substack{a=0 \\ \max(a, e-a) \geq k \\ a < 2k}}^{e} \mu_k(p^a)$$

by definition of μ_k. And this expression is $k - k$ or $(e - k + 1) - (e - k + 1)$, according as $e \geq 2k$ or $k \leq e < 2k$. In either case, we have the desired result.

Let $\ell(n)$ denote the arithmetical function which is identically 1.

THEOREM 2.4: If both f_1 and f_2 are arithmetical functions, then $f_1 = f_2 \circ \ell$ if and only if $f_2 = \mu_k \circ f_1$.

PROOF: If

$$f_2(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \mu_k(d) f_1(\delta),$$

then

$$\sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} f_2(n) = \sum_{\substack{DE\delta = n \\ (DE,\delta)_k = 1 \\ (D,E)_k = 1}} \mu_k(D) f_1(E) = \sum_{\substack{DE\delta = n \\ (D,\delta)_k = 1 \\ (E,D\delta)_k = 1}} \mu_k(D) f_1(E)$$

$$= \sum_{\substack{E|n \\ (E,D\delta)_k = 1}} f_1(E) \sum_{\substack{D\delta = n/E \\ (D,\delta)_k = 1}} \mu_k(D).$$

The inner sum here is 1 is $n/E = 1$ and 0 otherwise, by Theorem 2.3, so the expression reduces to $f_1(n)$. The proof of the other half is similar.

It is interesting to note that a shorter proof of this theorem can be obtained by using only the algebraic structure that was mentioned following Theorem 2.1.

The last theorem corresponds to the Meobius inversion formula in the theory of the Dirichlet product.

From the familiar representation of Euler's function as a Dirichlet product, we are led to the following generalized ϕ function.

DEFINITION 2.2: $\phi_k^*(n) = \sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \mu_k(d)\delta.$

By Theorem 2.4 and the definition of $\phi_k^*(n)$, we have immediately

THEOREM 2.5: $\sum_{\substack{d\delta = n \\ (d,\delta)_k = 1}} \phi_k^*(n) = n.$

Also, by Theorem 2.2, we have

REMARK 2.1: $\phi_k^*(n)$ is multiplicative.

We now define the k-prime analog of the square-free numbers. An integer n is said to be "k-square-free" provided that if a prime p divides n, then the multiplicity of p is in the range $\{1, 2, \ldots, k - 1, k + 1, k + 2, \ldots, 2k - 1\}$. So if $q_k^*(n)$ denotes the characteristic function of the set Q_k^* of k-square-free numbers, then $q_k^*(n)$ is multiplicative and, for prime powers p^e, has the evaluation

$$q_k^*(p^e) = \begin{cases} 1 & \text{if } e \in \{0, 1, \ldots, k-1, k+1, k+2, \ldots, 2k-1\} \\ 0 & \text{otherwise.} \end{cases}$$

3. FURTHER RESULTS

The algebraic results above coincide with classical results in the study of arithmetical functions. Another area of interest is in the area of analytic number theory. An important technique for obtaining estimates on the asymptotic

average of an arithmetical function f is to express f as a Dirichlet product of functions g and h. Therefore, it is natural to investigate the possibility of expressing a function f as a product of two functions under our new convolution, and whenever such a representation exists, to use it to obtain asymptotic results for f. This would allow us to investigate certain functions which do not arise naturally as a Dirichlet product. Some results have been obtained by this method but more refinements are required.

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COMBINATORIAL ASPECTS OF AN INFINITE PATTERN OF INTEGERS

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1. INTRODUCTION

In two previous papers, [3] and [4], certain basic properties of the sequence $\{A_n(x)\}$ defined by

$$A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1, \text{ and}$$
$$A_n(x) = xA_{n-2}(x) - A_{n-4}(x) \tag{1.1}$$

were obtained by the authors.

Here, we wish to investigate further properties of this sequence using as our guide some of the numerical information given by L. G. Wilson [5]. Terminology and notation of [3] and [4] will be assumed to be available to the reader. In particular, let

(1.2) $$\beta_i = 4\cos^2\frac{i\pi}{2n},$$
then
(1.3) $$\beta_{n+i} = \beta_{n-i},$$
(1.4) $$\beta_i - 2 = 2\cos\frac{i\pi}{n},$$
and
(1.5) $$(\beta_i - 2)^2 = \beta_{2i}.$$

The main result in this paper is Theorem 6. Besides the proof given, another proof is available.

2. PROPERTIES OF $A_k(\beta_i - 2)$

The following theorems generalize computational details in [5]. In Theorems 1 and 2, we use results in [3] and [4] with the Chebyshev polynomial of the second kind, $U_n(x)$.

THEOREM 1: $$A_{2n-1}(\beta_i - 2) = \begin{cases} +1 & (i \text{ odd}) \\ -1 & (i \text{ even}) \end{cases} \qquad (i = 1, 2, 3, \ldots, n-1).$$

PROOF:
$$\begin{aligned} A_{2n-1}(\beta_i - 2) &= A_{2n}(\beta_i - 2) + A_{2n-2}(\beta_i - 2) \\ &= U_{n-1}\left(\cos\frac{i\pi}{n}\right) + U_{n-2}\left(\cos\frac{i\pi}{n}\right) \text{ by (1.4) and [4]} \\ &= \frac{\sin\left(n\cdot\frac{i\pi}{n}\right) + \sin(n-1)\frac{i\pi}{n}}{\sin\frac{i\pi}{n}} \\ &= \pm 1 \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases}. \end{aligned}$$

E.g., $$A_9\ 2\cos\frac{\pi}{5} = 1.$$

THEOREM 2: $$A_r(\beta_i - 2) = \pm A_{2n-r}(\beta_i - 2) \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases}$$
$$(r \text{ odd};\ i = 1, 2, 3, \ldots, n-1).$$

PROOF:
$$\begin{aligned} A_r(\beta_i - 2) &= A_{r+1}(\beta_i - 2) + A_{r-1}(\beta_i - 2) \\ &= \frac{U_{r-1}}{2}\left(\cos\frac{i\pi}{n}\right) + \frac{U_{r-3}}{2}\left(\cos\frac{i\pi}{n}\right) \text{ by (1.4) and [4]} \\ &= \frac{\sin\left(\frac{r+1}{2}\right)\frac{i\pi}{n} + \sin\left(\frac{r-1}{2}\right)\frac{i\pi}{n}}{\sin\frac{i\pi}{n}} = \frac{\sin\frac{ri\pi}{2n}}{\sin\frac{i\pi}{2n}} \\ &= \pm\frac{\sin\left(n - \frac{r}{2}\right)\frac{i\pi}{n}}{\sin\frac{i\pi}{2n}} \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases} \\ &= \pm\left[U_{n-\frac{r+1}{2}}\left(\cos\frac{i\pi}{n}\right) + U_{n-\frac{r+3}{2}}\left(\cos\frac{i\pi}{n}\right)\right] \\ &= \pm[A_{2n-r+1}(\beta_i - 2) + A_{2n-r-1}(\beta_i - 2)] \text{ by (1.4) and [4]} \\ &= \pm A_{2n-r}(\beta_i - 2) \text{ according as } i \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases}. \end{aligned}$$

COROLLARY 1: When $i = 1$, $A_r\, 2\left(\cos \frac{\pi}{n}\right) = A_{2n-r}\left(2 \cos \frac{\pi}{n}\right)$.

E.g., $A_3\left(2 \cos \frac{\pi}{5}\right) = A_7\left(2 \cos \frac{\pi}{5}\right) = \dfrac{\sin \frac{3\pi}{10}}{\sin \frac{\pi}{10}} = 2 \cos \frac{\pi}{5} + 1 = \dfrac{3 + \sqrt{5}}{2} = \left(\dfrac{1 + \sqrt{5}}{2}\right)^2$.

COROLLARY 2: $A_1(\beta_i - 2)$, $A_3(\beta_i - 2)$, ..., $A_{2n-1}(\beta_i - 2)$ for a cycle of period n.

E.g., for $n = 6$, $i = 1$, $A_1 = A_{11} = 1$, $A_3 = A_9 = 1 + \sqrt{3}$, $A_5 = A_7 = 2 + \sqrt{3}$.

Our next theorem involves $\phi(n)$, Euler's ϕ-function.

THEOREM 3: Let n be odd and $m = \frac{1}{2}\phi(n)$, then $\beta_{2^m} - 2 = -(\beta_1 - 2)$.

PROOF: By the Fermat-Euler Theorem, since $(2, n) = 1$, it follows that

$$2^m \equiv \pm 1 \pmod{n}.$$

Hence, there exists an odd integer t such that $2^m = nt \pm 1$. Therefore,

$$\begin{aligned}\beta_{2^m} - 2 = 2 \cos 2^m\left(\frac{\pi}{n}\right) &= 2 \cos(nt \pm 1)\frac{\pi}{n} \\ &= 2 \cos \pi t \cos\left(\pm \frac{\pi}{n}\right) \\ &= -(\beta_1 - 2).\end{aligned}$$

COROLLARY 3: When n is even, just one operator ("square and subtract 2") produces the $\beta_1 - 2$ for $n/2$.

This is obvious, because $(\beta_1 - 2)^2 - 2 = 2 \cos \frac{2\pi}{n} = 2 \cos \frac{\pi}{n/2}$.

3. SEMI-INFINITE NUMBER PATTERNS

Consider the pattern of numbers and their mode of generation given in Table 1 for a fixed number $k = 5$ of columns (Wilson [5]).

Row n \ Column m	1	2	3	4	5
0	1	1	1	1	1
	2	2	2	2	2
1	2	4	4	4	2
	6	8	8	6	
2	6	14	16	14	6
	20	30	30	20	
3	20	50	60	50	20
	70	110	110	70	
4	70	180	220	180	70
	250	400	400	250	
5	250	650	800	650	250
	900	1450	1450	900	
6	900	2350	2900	2350	900
	. . .	. . .	. . .	. . .	. . .

Table 1. Pattern of Integers for $k = 5$

Designate the row number by n and the column number by m ($n = 0, 1, 2, \ldots$; $m = 1, 2, \ldots, k$). The element in row n and column m is denoted by U_{nm}.

From Table 1, the following information may be gleaned:

$$\begin{bmatrix} U_{n1} \\ U_{n2} \\ U_{n3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} U_{n-1,1} \\ U_{n-1,2} \\ U_{n-1,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.1}$$

$$U_{nm} = 5(U_{n-1,m} - U_{n-2,m}),\ n > 2 \text{ and} \tag{3.2}$$

$$\begin{cases} U_{n1} = U_{n5} = \dfrac{2}{\sqrt{5}}(\alpha a^{n-1} - \beta b^{n-1}) \\ U_{n2} = U_{n4} = \dfrac{2}{\sqrt{5}}(Aa^{n-1} - Bb^{n-1}), \quad n \geq 1, \\ U_{n3} = \dfrac{2}{\sqrt{5}}(Ca^{n-1} - Db^{n-1}) \end{cases} \tag{3.3}$$

where

$$\begin{cases} a = \frac{1}{2}(5 + \sqrt{5}),\ b = \frac{1}{2}(5 - \sqrt{5}) \\ \alpha = \frac{1}{2}(1 + \sqrt{5}),\ \beta = \frac{1}{2}(1 - \sqrt{5}) \\ A = 2 + \sqrt{5}, \quad B = 2 - \sqrt{5} \\ C = 3 + \sqrt{5}, \quad D = 3 - \sqrt{5} \end{cases} \tag{3.4}$$

so that $A = 2\alpha + 1$, $B = 2\beta + 1$, $C = 2(\alpha + 1) = A + 1$, $D = 2(\beta + 1) = B + 1$.

It follows from (3.3) and (3.4) that

$$\lim_{n\to\infty}\left(\frac{U_{n2}}{U_{n1}}\right) = \frac{A}{\alpha} = \alpha + 1 = A_3\left(2\cos\frac{\pi}{5}\right), \tag{3.5}$$

and

$$\lim_{n\to\infty}\left(\frac{U_{n3}}{U_{n1}}\right) = \frac{C}{\alpha} = 2\alpha = A_5\left(2\cos\frac{\pi}{5}\right). \tag{3.6}$$

Extending Table 1 to the case $k = 6$, so that now, for example, $U_{51} = 252$ and $U_{43} = 236$, we eventually derive $U_{nm} = 6U_{n-1,m} - 9U_{n-2,m} + 2U_{n-3,m}$; thus

$$\begin{cases} U_{n1} = \frac{1}{3}\{2^n + (2 + \sqrt{3})^n + (2 - \sqrt{3})^n\} = U_{n6} \\ U_{n2} = \frac{1}{3}\{2^n + (1 + \sqrt{3})(2 + \sqrt{3})^n + (1 - \sqrt{3})(2 - \sqrt{3})^n\} = U_{n5} \\ U_{n3} = \frac{1}{3}\{-1.2^n + (2 + \sqrt{3})(2 + \sqrt{3})^n + (2 - \sqrt{3})(2 - \sqrt{3})^n\} = U_{n4}, \end{cases} \tag{3.7}$$

whence

$$\lim_{n\to\infty}\left(\frac{U_{n2}}{U_{n1}}\right) = \sqrt{3} + 1 = A_3\left(2\cos\frac{\pi}{6}\right), \tag{3.8}$$

and

$$\lim_{n\to\infty}\left(\frac{U_{n3}}{U_{n1}}\right) = 2 + \sqrt{3} = A_5\left(2\cos\frac{\pi}{6}\right). \tag{3.9}$$

Results (3.5), (3.6), (3.8), and (3.9) suggest a connection between various limits of ratios (as $n \to \infty$) and corresponding $A_r\left(2\cos\frac{\pi}{k}\right)$. This link is developed

in the next section. [In passing, we remark that for $k = 9$, $n = 13$, we calculate to two decimal places that

$$\frac{U_{13,2}}{U_{13,1}} = 2.85, \quad A_3\left(2 \cos \frac{\pi}{9}\right) = 2.88,$$

the common value to which they aspire as $n \to \infty$, $k \to \infty$ being 3 (cf. Theorem 6).]

4. AN INFINITE NUMBER PATTERN

For $1 \le m \le k$, we find [cf. (3.1)]

$$\text{(4.1)} \qquad \begin{cases} U_{nm} = U_{n-1,m-1} + 2U_{n-1,m} + U_{n-1,m+1} & 1 < m < k \\ U_{n1} = U_{n-1,1} + U_{n-1,2} & m = 1 \\ U_{nk} = U_{n-1,k} + U_{n-1,k-1} & m = k \end{cases}$$

with $U_n = U_{n,k+1-m}$. Also

$$\text{(4.2)} \qquad \begin{cases} U_{nm} = \displaystyle\sum_{r=1}^{[k/2]} (-1)^{r-1} v_{kr} U_{n-r,m} & n > [k/2] \\ U_{n1} = \dbinom{2n}{n} & n \le k - 1 \end{cases}$$

in which v_{nm} is an element of an array in row n and column m defined by

$$\text{(4.3)} \qquad \begin{aligned} v_{nm} &= v_{n-1,m} + v_{n-2,m-1} & n &\ge 2m \\ v_{nm} &= 0 & n &< 2m \\ v_{n1} &= n, \ v_{n0} = 1, \ v_{0m} = 0, \ v_{2n,2n-1} = 2. \end{aligned}$$

For example, if

$$k = 6, \ U_{nm} = 6U_{n-1,m} - 9U_{n-2,m} + 2U_{n-3,m},$$

and if

$$k = 9, \ U_{nm} = 9U_{n-1,m} - 27U_{n-2,m} + 30U_{n-3,m} - 9U_{n-4,m}.$$

We look briefly at the $\{v_{nk}\}$ in Section 5.

Notice in (4.2) that for $n \to \infty$, i.e., $k \to \infty$, U_{n1} are the *central binomial coefficients*.

Now let $n \to \infty$ and $k \to \infty$. We wish to obtain the limit of $U_{nm}/U_{nm'}$. But first, by easy calculation using (4.1) we derive

$$\text{(4.4)} \qquad \lim_{n\to\infty}\left(\frac{U_{n-1,1}}{U_{n1}}\right) = \frac{1}{4}.$$

THEOREM 4: $\displaystyle\lim_{n\to\infty}\left(\frac{U_{nm}}{U_{n1}}\right) = 2m - 1.$

PROOF: The result is trivially true for $m = 1$. Assume the theorem is true for $m = p$. That is, assume

$$\lim_{n\to\infty}\left(\frac{U_{np}}{U_{n1}}\right) = 2p - 1.$$

We test this hypothesis for $m = p + 1$, using (4.1) several times. Now

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left(\frac{U_{n,\,p+1}}{U_{n1}}\right)\\
&= \lim_{n\to\infty}\left\{\frac{U_{n-1,\,p} + 2U_{n-1,\,p+1} + (U_{n,\,p+1} - U_{n-1,\,p} - 2U_{n-1,\,p+1})}{U_{n1}}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{2(U_{np} - U_{n-1,\,p-1} - 2U_{n-1,\,p}) + (U_{n,\,p+1} - 2U_{n-1,\,p+1})}{U_{n1}}\right\}\\
&= \lim_{n\to\infty}\left\{2\left(\frac{U_{np}}{U_{n1}} - \frac{U_{n-1,\,p-1}}{U_{n-1,\,1}}\cdot\frac{U_{n-1,\,1}}{U_{n1}} - 2\frac{U_{n-1,\,p}}{U_{n-1,\,1}}\cdot\frac{U_{n-1,\,1}}{U_{n1}}\right)\right.\\
&\qquad\left. + \left(\frac{U_{n,\,p+1}}{U_{n1}} - 2\frac{U_{n-1,\,p+1}}{U_{n-1,\,1}}\cdot\frac{U_{n-1,\,1}}{U_{n1}}\right)\right\}\\
&= 2\left(2p - 1 - \left(\frac{2p-3}{4}\right) - 2\left(\frac{2p-1}{4}\right)\right) + \left(R - 2\frac{R}{4}\right) \qquad \text{by (4.4) and the inductive hypothesis,}
\end{aligned}
$$

whence $R = 2p + 1$, which establishes the theorem.

COROLLARY 4: $\lim_{n\to\infty}\left(\frac{U_{nm}}{U_{nm'}}\right) = \frac{2m-1}{2m'-1}$

THEOREM 5: $A_{2m-1}(\beta_1 - 2) = 2m - 1 = A_{2k-(2m-1)}(\beta_1 - 2)$, $1 \le m \le k$, $k \to \infty$.

PROOF:
$$
\begin{aligned}
A_{2m-1}(\beta_1 - 2) &= \frac{\sin(2m-1)\frac{i\pi}{2k}}{\sin\frac{i\pi}{2k}} = A_{2k-(2m-1)}(\beta_1 - 2) \text{ by Theorem 2}\\
&= 2m - 1
\end{aligned}
$$

on using a trigonometrical expansion for the numerator, simplifying, and then letting $k \to \infty$.

Clearly there is a connection between Theorems 4 and 5. We therefore assert:

THEOREM 6: $\lim_{n\to\infty}\left(\frac{U_{nm}}{U_{n1}}\right) = A_{2m-1}(\beta_1 - 2) = 2m - 1 \qquad (k \to \infty)$.

Observe that, with the aid of (4.1) and the manipulative technique of Theorem 4, we may deduce

$$\lim_{n\to\infty}\left(\frac{U_{n-1,\,2}}{U_{n1}}\right) = \frac{3}{4},\ \lim_{n\to\infty}\left(\frac{U_{n-1,\,3}}{U_{n1}}\right) = \frac{5}{4},\ \lim_{n\to\infty}\left(\frac{U_{n-1,\,4}}{U_{n1}}\right) = \frac{7}{4},\ \ldots, \tag{4.5}$$

and

$$\lim_{n\to\infty}\left(\frac{U_{n-2,\,2}}{U_{n1}}\right) = \frac{3}{16},\ \lim_{n\to\infty}\left(\frac{U_{n-2,\,3}}{U_{n1}}\right) = \frac{5}{16},\ \lim_{n\to\infty}\left(\frac{U_{n-2,\,4}}{U_{n1}}\right) = \frac{7}{16},\ \ldots\ . \tag{4.6}$$

Ultimately,

$$\lim_{n\to\infty}\left(\frac{U_{n-r,\,m}}{U_{n1}}\right) = \frac{2m-1}{4^r}, \tag{4.7}$$

from which Theorem 4 follows if we put $r = 0$.

This concludes the theoretical basis, with extensions, for the detailed numerical information given by Wilson [5].

5. SOME PROPERTIES OF $\{v_{kr}\}$

Define

$$\Delta v_{kr} = v_{kr} - v_{k-1,r}. \tag{5.1}$$

THEOREM 7: $\Delta^r v_{kr} = 1$.

PROOF: Use induction. When $r = 1$,

$$\Delta v_{k1} = k - (k - 1) = 1 \quad \text{by (5.1) and (4.3).}$$

Assume the result is true for $r = 2, 3, \ldots, s - 1$. Then

$$\begin{aligned}\Delta^s v_{ks} &= \Delta^{s-1}(\Delta v_{ks}) \\ &= \Delta^{s-1} \cdot (v_{k-1,s} + v_{k-2,s-1} - v_{k-1,s}) \quad \text{by (5.1) and (4.3)} \\ &= \Delta^{s-1} v_{k-2,s-1} \\ &= 1 \quad \text{from the inductive hypothesis.}\end{aligned}$$

Hence, the theorem is proved.

It can also be shown that

$$v_{nm} = \frac{n}{n-m}\binom{n-m}{m} = \binom{n-m}{m} + \binom{n-m-1}{m-1}, \tag{5.2}$$

whence

$$L_n = \sum_{m=0}^{[n/2]} v_{nm}, \tag{5.3}$$

in which L is the nth Lucas number defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \qquad (n > 2)$$

with initial conditions $L_1 = 1$, $L_2 = 3$.

Another property is

$$\sum_{m=0}^{n} v_{n+m,m} = 3 \cdot 2^{n-1}. \tag{5.4}$$

Table 2 shows the first few values of v_{kr} (see Hoggatt & Bicknell [2], where the v_{kr} occur as coefficients in a list of Lucas polynomials).

$k \backslash r$	0	1	2	3	4	5
1	1					
2	1	2				
3	1	3				
4	1	4	2			
5	1	5	5			
6	1	6	9	2		
7	1	7	14	7		
8	1	8	20	16	2	
9	1	9	27	30	9	
10	1	10	35	50	25	2
11	1	11	44	77	55	11
	. . .	. . .	. . .	. . .	. . .	. . .

Table 2. Values of v_{kr} ($k = 1, 2, \ldots, 11$)

Coefficients in the generating difference equations (4.2), as k varies, appear in Table 2 if we alternate + and - signs. Corresponding characteristic polynomials occur in [4] as proper divisors, or as products of proper divisors. Refer to Hancock [1], also.

Further, it might be noted that, if we employ the recurrence relation in (4.1) repeated.y, we may expand U_{nm} binomially as

$$U_{nm} = U_{n-t,\,m-t} + \binom{2t}{1}U_{n-t,\,m-t+1} + \binom{2t}{2}U_{n-t,\,m-t+2} + \cdots$$
$$+ \binom{2t}{1}U_{n-t,\,m+t+1} + U_{n-t,\,m+t} \quad (1 \le t < n,\ 1 \le t < m).$$

This is because the original recurrence relation (4.1) for U_{nm} is "binomial" ($t = 1$), i.e., the coefficients are 1, 2, 1.

Finally, we remark that the row elements in the first column, U_{n1}, given in (4.2), are related to the *Catalan numbers* C_n by

$$U_{n1} = (n + 1)C_n. \tag{5.5}$$

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ONE-PILE TIME AND SIZE DEPENDENT TAKE-AWAY GAMES

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1. INTRODUCTION

In a one-pile take-away game, two players alternately remove chips from a single pile of chips. Depending on the particular formulation of play, a *constraint function* specifies the number of chips which may be taken from the pile in each position. The game ends when no move is possible. In *normal* (*misère*) play, the player who makes the final move wins (loses). Necessarily, one of the players has a strategy which can force a win.

In this *Quarterly*, Whinihan [7], Schwenk [5], and Epp & Ferguson [2] have analyzed certain one-pile take-away games which can be represented by an ordered

triple (n, w, f). Here $n \in Z^+ \cup \{0\}$, $w \in Z^+$, and $f: Z^+ \to Z^+$ is nondecreasing. On the initial move in the game (n, w, f), a player takes from 1 to w chips from a pile of n chips. Subsequently, if a player takes t chips from the pile, then the next player to move may take from 1 to $f(t)$ chips. In [3], the author provides an analysis of a generalization of this formulation of a one-pile take-away game so as to allow for play with two piles of chips.

The purpose of this paper is to present a formulation and an analysis of another type of one-pile take-away game. The formulation in this paper is quite dissimilar to that studied in [2], [5], and [7]. In the present formulation, the constraint function f is a function of two variables. The first variable is equal to one plus the number of moves made since the start of play. Think of this variable as representing *time*. The second variable represents the number of chips in the pile, that is, pile *size*. We shall call this formulation the *one-pile time and size dependent take-away game*. It is nicknamed *tastag*.

For example, suppose the constraint function is

$$f(t, n) = t + 1 + \left[\frac{n}{2}\right].$$

Here, $[x]$ denotes the largest integer less than or equal to x. At the start of play (time $t = 1$), suppose that the pile contains 211 chips. The first player to move may take from 1 to 107 chips. Suppose that he takes 51 chips, say, so as to leave 160 chips in the pile. Then his opponent may reply (at time $t = 2$) by taking from 1 to 83 chips. In Section 4, it will be shown that for play beginning with a pile of 211 chips, the second player to move can force a win. In Section 5, it will be shown that if the first player opens play by taking 51 chips, then the second player possesses fifteen winning replies. To force a win, the second player should take from 43 to 57 chips. If the first player opens by taking 107 chips, say, then the second player has a unique winning reply, namely, to take a single chip.

2. THE RULES OF THE GAME

Let $f: Z^+ \times Z^+ \to Z^+$. Suppose that the pile contains n chips after $t - 1$ moves have been made, $t \geq 1$. On the tth move, the player to move must take from 1 to $f(t, n)$ chips. (t, n, f) will denote the position consisting of a pile of n chips after $t - 1$ moves have been made, with play governed by the constraint function f.

In this paper we restrict ourselves to tastags for which the constraint function f satisfies the following growth condition.

CONDITION 2.1: $\forall t \geq 1$, $\forall n \geq 1$

$$f(t, n) \leq f(t, n + 1) \leq f(t, n) + 1.$$

Set $\mathcal{C} = \{f \mid f: Z^+ \times Z^+ \to Z^+ \text{ and } f \text{ satisfies Condition 2.1}\}$.

Define the normal outcome sets h_+ and p_+ by

$$h_+ = \{(t, n, f) \mid t \geq 1,\ n \geq 0,\ f \in \mathcal{C} \text{ and the } \textit{first} \text{ player to move in } (t, n, f) \text{ can force a win in } \textit{normal} \text{ play}\}$$

and

$$p_+ = \{(t, n, f) \mid t \geq 1,\ n \geq 0,\ f \in \mathcal{C} \text{ and the } \textit{second} \text{ player to move in } (t, n, f) \text{ can force a win in } \textit{normal} \text{ play}\}.$$

We define the misère outcome sets h_- and p_- just as we define h_+ and p_+, respectively, except that we replace "normal" by "misère" in the definitions.

For $f \in \mathcal{C}$, define $\tilde{f}: Z^+ \times Z^+ \to Z^+$ by

$$\tilde{f}(t, n) = f(t, n + 1) \quad \forall t \geq 1,\ \forall n \geq 1.$$

In a straightforward manner, it can be shown that $\tilde{f} \in \mathcal{C}$. It is also not difficult to verify the following:

PROPOSITION 2.1: If $t \geq 1$, $n \geq 1$, and $f \in \mathfrak{C}$, then $(t, n, f) \in h_-$ if and only if $(t, n-1, \tilde{f}) \in h_+$.

An immediate consequence of Proposition 2.1 is the following: If we can analyze (t, n, f) for normal play for each $t \geq 1$, $n \geq 0$, and $f \in \mathfrak{C}$, then we can analyze (t, n, f) for misère play for each $t \geq 1$, $n \geq 0$, and $f \in \mathfrak{C}$.

In this paper attention is restricted to normal play. Our aim is the following:

1. Determine the outcome sets h_+ and p_+.
2. For each $(t, n, f) \in h_+$, prescribe a winning move for the player who moves next.

3. THE GAME TABLEAU

For fixed $f \in \mathfrak{C}$, to analyze all one-pile tastags (t, n, f), $t \geq 1$, $n \geq 0$, we construct a *game tableau for* f. The game tableau is an infinite array

$$\langle E_{t,r}\rangle_{t,r=1}^{\infty}$$

whose entries belong to the set $Z^+ \cup \{0, \infty\}$. For each $t \geq 1$, let D_t denote the tth diagonal of the tableau. That is, $D_t = \langle E_{t+1-r,r}\rangle_{r=1}^{t}$. For example, in the tableau in Figure 3.1, $D_8 = \langle 2, 3, 5, 0, 0, 0, 0, 0\rangle$.

In the sequel, the following conventions are adopted:

1. $E_{t,-1} = -1$, $E_{t,0} = 0 \ \forall\ t \geq 1$.
2. $\max Z = \infty$.
3. $n + \infty = \infty \ \forall\ n \in Z^+ \cup \{0, \infty\}$.
4. The domain of f is extended from $Z^+ \times Z^+$ to $Z^+ \times (Z^+ \cup \{\infty\})$, and

$$f(t, \infty) = \infty \qquad \forall t \geq 1.$$

Construct the game tableau for f by double induction as follows:

A. The sole entry of D_1 is $E_{11} = \max\{n \mid f(1, n) \geq n\}$.

B. Suppose that the entries for diagonals $D_1, D_2, \ldots, D_{t-1}$ have been computed for some $t \geq 2$. Then compute the entries of diagonal D_t as follows:

1. $E_{t,1} = \max\{n \mid f(t, n) \geq n\}$.
2. Suppose the entries $E_{t+1-u,u}$, $u = 1, 2, \ldots, r-1$, have been computed for some r, $2 \leq r \leq t$.

   a. If $E_{t-r+2,r-1} = 0$, put $E_{t-r+1,r} = 0$.

   b. If $E_{t-r+2,r-1} > 0$ and r is even, put

$$E_{t-r+1,r} = \begin{cases} 0, \text{ if } E_{t-r+2,r-1} + 1 \leq E_{t-r+1,u} \text{ for some } u,\ 1 \leq u \leq r-1. \\ E_{t-r+2,r-1} + 1, \text{ otherwise.} \end{cases}$$

   c. If $E_{t-r+2,r-1} > 0$ and r is odd, put

$$E_{t-r+1,r} = \begin{cases} 0, \text{ if } E_{t-r+2,r-1} + \max\{n \geq 1 \mid f(t-r+1, E_{t-r+2,r-1} + n) \geq n\} \\ \qquad\qquad \leq E_{t-r+1,u} \text{ for some } u,\ 1 \leq u \leq r-1. \\ E_{t-r+2,r-1} + \max\{n \geq 1 \mid f(t-r+1, E_{t-r+2,r-1} + n) \geq n\}, \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Let us illustrate this construction with an example.

EXAMPLE 3.1: Let $f: Z^+ \times Z^+ \to Z^+$ be defined as follows:

$$f(1, n) = \begin{cases} 3 & \text{for } n \leq 20, \\ n - 17 & \text{for } n \geq 21. \end{cases}$$

For $t = 2$ or 3, $f(t, n) = 5 - t + \left[\frac{n}{3}\right] \forall n \geq 1$.

$$f(4, n) = \begin{cases} 1 & \text{for } 1 \leq n \leq 9, \\ n - 9 & \text{for } n \geq 10. \end{cases} \quad f(5, n) \equiv 4. \quad f(6, n) = 1 + \left[\frac{n}{4}\right] \forall n \geq 1.$$

For $7 \leq t \leq 13$, $f(t, n) \equiv 2$. For $t \geq 14$, $f(t, n) = n \forall n \geq 1$. Condition 2.1 is satisfied by f. The complete game tableau for f is given in Figure 3.1.

$t \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	...
1	3	5	0	0	14	0	19	0	∞	0	0	0	...
2	4	0	0	11	0	16	0	20	0	0	∞	0	...
3	3	0	10	0	15	0	19	0	0	∞	0	0	...
4	1	5	0	8	0	11	0	0	∞	0	0	0	...
5	4	0	7	0	10	0	0	14	0	∞	0	0	...
6	1	3	5	6	9	0	13	0	∞	0	0	0	...
7	2	3	5	6	8	9	11	∞	0	0	0	...	
8	2	3	5	6	8	9	∞	0	0	0	...		
9	2	3	5	6	8	∞	0	0	0	...			
10	2	3	5	6	∞	0	0	0	...				
11	2	3	5	∞	0	0	0	...					
12	2	3	∞	0	0	0	...						
13	2	∞	0	0	0	...							
14	∞	0	0	0	...								
15	∞	0	0	...									
16	∞	0		...									
⋮	⋮	⋮	⋮										

Fig. 3.1. The game tableau for Example 3.1

For a large class of constraint functions in $\mathcal{C}$, the corresponding game tableaux have no zero entries. For any such game tableau, the entries of each row (column) form a strictly increasing (nondecreasing) sequence of positive integers. The tastags generated by such constraint functions will be called *escalation tastags*. Set

$$\mathcal{E} = \{f \in \mathcal{C} \mid \text{the game tableau of } f \text{ has no zero entries}\}.$$

EXAMPLE 3.2: Consider the constraint function $f(t) = t + 1 + [n/2]$ mentioned in Section 1. For $t \geq 1$ and $r \geq 1$, it can be shown that

$$E_{t,r} = \begin{cases} [2(r + t) - 3]2^{(r+1)/2} - 2(t - 2) & \text{if } r \text{ is odd}, \\ [2(r + t) - 3]2^{r/2} - 2t + 3 & \text{if } r \text{ is even}. \end{cases}$$

$f \in \mathcal{E}$. A portion of the tableau of f is shown in Figure 3.2.

EXAMPLE 3.3: On page 124 of [6], Silverman introduces a game called *Triskidekaphilia Escalation*. It was the challenge of this game for an arbitrary pile size $n \geq 0$ that motivated the present study of one-pile tastags. This game is equivalent to the one-pile tastag $(1, n, f)$, where $f(t, n) = t + 1$. $f \in \mathcal{E}$. For $t \geq 1$ and $r \geq 1$, it can be shown that

$$E_{t,r} = \begin{cases} \left(\frac{r+1}{2}\right)^2 + (t+1)\left(\frac{r+1}{2}\right) - 1 & \text{if } r \text{ is odd,} \\ \left(\frac{r}{2}\right)^2 + (t+2)\left(\frac{r}{2}\right) & \text{if } r \text{ is even.} \end{cases}$$

A portion of the game tableau of f is shown in Figure 3.3.

t \ r	1	2	3	4	5	6	7	8
1	4	7	22	29	74	89	210	241
2	6	9	28	35	88	103	240	271
3	8	11	34	41	102	117	270	301
4	10	13	40	47	116	131	300	331
5	12	15	46	53	130	145	330	361
6	14	17	52	59	144	159	360	391
7	16	19	58	65	158	173	390	421
8	18	21	64	71	172	187	420	451

Fig. 3.2. A portion of the game tableau for Examle 3.2

t \ r	1	2	3	4	5	6	7	8	9	10
1	2	4	7	10	14	18	23	28	34	40
2	3	5	9	12	17	21	27	32	39	45
3	4	6	11	14	20	24	31	36	44	50
4	5	7	13	16	23	27	35	40	49	55
5	6	8	15	18	26	30	39	44	54	60
6	7	9	17	20	29	33	43	48	59	65
7	8	10	19	22	32	36	47	52	64	70
8	9	11	21	24	35	39	51	56	69	75
9	10	12	23	26	38	42	55	60	74	80
10	11	13	25	28	41	45	59	64	79	85

Fig. 3.3 A portion of the game tableau for Example 3.3

4. DETERMINING THE NORMAL OUTCOME SETS

From the game tableau of f, $f \in \mathcal{C}$, the following theorem reveals the outcome set to which any tastag (t, n, f) belongs.

THEOREM 4.1: If $t \geq 1$, $n \geq 1$, and $f \in \mathcal{C}$, then $(t, n, f) \in h_+$ if and only if $\min\{r | E_{t,r} \geq n\}$ is odd.

As an illustration, return to Example 3.1. Is $(1, 22, f)$ a first-player win? Here

$$\min\{r | E_{1,r} \geq 22\} = 9,$$

which is odd. Thus, the first player to move in $(1, 22, f)$ can force a win.

How about the position $(5, 11, f)$? Here

$$\min\{r | E_{5,r} \geq 11\} = 8,$$

which is even. Thus, the second player to move in $(5, 11, f)$ can force a win.

As a final example, return to the tastag $(1, 211, f)$ mentioned in Section 1: $f(t, n) = t + 1 + [n/2]$. A portion of the game tableau for f is shown in Figure 3.2. We observe that $\min\{r | E_{1,r} \geq 211\} = 8$, which is even. As asserted in Section 1, $(1, 211, f)$ is a second-player win.

In the author's doctoral dissertation [4], it is shown that if $f \in \mathcal{C}$, then $\min\{r|E_{1,r} \geq n\}$ is, in fact, the normal *remoteness* number of (t, n, f). Moveover, if $f \in \mathcal{E}$, then $\min\{r|E_{1,r} \geq n\}$ is also the normal *suspense* number of (t, n, f).*

5. AN OPTIMAL STRATEGY

The proof of Theorem 4.1 will be *constructive*. Suppose that $(t, n, f) \in h_+$. Set $\beta(t, n, f) = \min\{r|E_{t,r} \geq n\}$. We prescribe the following *winning move*:

1. Take $n - E_{t+1,\, \beta(t,n,f)-1}$ chips if $n > E_{t+1,\, \beta(t,n,f)-1}$.
2. Take a single chip if $n \leq E_{t+1,\, \beta(t,n,f)-1}$.

As an illustration, return again to Example 3.1.

First consider the position $(3, 19, f)$. $\beta(3, 19, f) = 7$, so $(3, 19, f) \in h_+$. $19 > 11 = E_{4,6}$. The player whose turn it is to move should take $19 - 11 = 8$ chips. Since $f(3, 19) = 8$, seven other moves are also possible. Observe that each of the seven other moves is "bad," since $\beta(4, 19 - u, f) = 9\ \forall u,\ 1 \leq u \leq 7$.

Next consider the position $(4, 13, f)$. $\beta(4, 13, f) = 9$, so $(4, 13, f) \in h_+$. $13 \leq 14 = E_{5,8}$. The first player to move can make a winning move by taking a single chip. $f(4, 13) = 4$. Note that taking 2 chips is also a winning move. However, taking either 3 or 4 chips is a losing move.

Let u denote the move in which u chips are taken from the pile. The *set of winning moves* from the position (t, n, f) is

$$\{u|1 \leq u \leq f(t, n) \wedge n, \text{ and } (t + 1, n - u, f) \in p_+\}$$
$$= \{u|1 \leq u \leq f(t, n) \wedge n, \text{ and } \beta(t + 1, n - u, f) \text{ is even}\}.$$

When this set is nonempty, Condition 2.1 and a short argument assures us that it is a set of *consecutive integers*.

Return to the tastag discussed in Section 1. From Figure 3.2 we observe that $\beta(2, 160, f) = 7$, so $(2, 160, f) \in h_+$. The set of winning moves from $(2, 160, f)$ is

$$\{u|1 \leq u \leq 83, \text{ and } \beta(3, 160 - u, f) = 6\} = \{43, 44, \ldots, 57\}.$$

Next note that $\beta(2, 104, f) = 7$. The set of winning moves from $(2, 104, f)$ is

$$\{u|1 \leq u \leq 55, \text{ and } \beta(3, 104 - u, f) = 6\} = \{1\}.$$

6. THE PROOF OF THEOREM 4.1

Our proof of Theorem 4.1 takes the usual approach. Pick any $f \in \mathcal{C}$. To show that a set A satisfies

$$A = \{(t, n, f)|t \geq 1,\ n \geq 0\} \cap h_+,$$

it suffices to show each of the following:

a. No terminal position is in A.
b. For each position in A, there exists a move to a position not in A.
c. For each position not in A, every move results in a position in A.

Before proving Theorem 4.1, we introduce some notation and prove two lemmas.

For each $t \geq 1$, $n \geq 0$, *define*

$$\alpha(t, n, f) = \max[\{0\} \cup \{r|0 < E_{t,r} < n\}],$$
$$\beta(t, n, f) = \min\{r|E_{t,r} \geq n\}, \text{ and}$$
$$\gamma(t, n, f) = \max[\{0\} \cup \{r|r \text{ is even}, E_{t+1,r} > 0,\ r < \beta(t, n, t)\}].$$

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*Chapter 14 of [1] is a good reference for the reader who is not familiar with the concepts of remoteness and suspense numbers.

Since $f(t, n) \geq 1\ \forall\ t \geq 1,\ \forall\ n \geq 1$, it can be shown that $\beta(t, n, f) < \infty$ [in fact, $\beta(t, n, f) \leq n$] $\forall\ t \geq 1,\ \forall\ n \geq 0$. *Define* the set of "followers" of position (t, n, f) to be

$$F(t, n, f) = \{(t + 1, n - u, f) \mid 1 \leq u \leq f(t, n) \wedge n\}.$$

For each $\ell \geq 0$, *define* the set

$$A_\ell = \{(t, n, f) \mid t \geq 1,\ n \geq 0,\ \beta(t, n, f) \text{ is odd}\}.$$

Theorem 4.1 asserts that

$$\{(t, n, f) \mid t \geq 1,\ n \geq 0\} \cap h_+ = \bigcup_{r=0}^{\infty} A_{2r+1}.$$

Demanding that f satisfies Condition 2.1 forces the game tableau of f to possess two nice properties. Lemma 6.1 reveals the two properties.

LEMMA 6.1: Suppose $f \in \mathcal{C}$, $t \geq 1$, and $r \geq 0$.

a. If $0 < E_{t,\ 2r+1} < n$, then $n - f(t, n) > E_{t+1,\ 2r}$.
b. If $0 < n \leq E_{t,\ 2r+1}$, then $n - f(t, n) \leq E_{t+1,\ 2r}$.

PROOF: a. By the manner in which the tableau is constructed,

$$E_{t,\ 2r+1} > 0 \Rightarrow E_{t,\ 2r+1} = E_{t+1,\ 2r} + \delta,$$

where $\delta = \max\{n' \mid f(t, E_{t+1,\ 2r} + n') \geq n'\}$. Observe that

$$f(t, E_{t+1,\ 2r} + \delta + 1) < \delta + 1. \tag{1}$$

Since $n > E_{t,\ 2r+1}$, we have $n - E_{t+1,\ 2r} - \delta - 1 \geq 0$. Thus,

$$\begin{aligned} f(t, n) &= f[t, (E_{t+1,\ 2r} + \delta + 1) + (n - E_{t+1,\ 2r} - \delta - 1)] \\ &\leq f(t, E_{t+1,\ 2r} + \delta + 1) + (n - E_{t+1,\ 2r} - \delta - 1) \end{aligned} \tag{2}$$

by Condition 2.1. (1) and (2) yield

$$f(t, n) < (\delta + 1) + (n - E_{t+1,\ 2r} - \delta - 1) = n - E_{t+1,\ 2r}.$$

Thus, $n - f(t, n) > E_{t+1,\ 2r}$.

b. Since $E_{t,\ 2r+1} > 0$, we have $E_{t,\ 2r+1} = E_{t+1,\ 2r} + \delta$, where δ is as in the proof of part (a) of the Lemma. If $n - 1 \leq E_{t+1,\ 2r}$, then the assertion in part (b) of the Lemma is trivial. So suppose $n > E_{t+1,\ 2r} + 1$. Then $1 < n - E_{t+1,\ 2r} \leq \delta$, and so

$$f(t, n) = f[t, E_{t+1,\ 2r} + (n - E_{t+1,\ 2r})] \geq n - E_{t+1,\ 2r}. \qquad \text{Q.E.D.}$$

The second lemma we shall need is the following.

LEMMA 6.2: Suppose $f \in \mathcal{C}$, $t \geq 1$, $r \geq 1$, and $E_{t,u} < \infty$ for each u, $1 \leq u \leq 2r$. If $E_{t+1,\ 2r} > 0$, then $E_{t,\ 2r+1} > 0$.

PROOF: Suppose $E_{t,u} < \infty$ for each u, $1 \leq u \leq 2r$, and suppose $E_{t+1,\ 2r} > 0$. Then $E_{t,\ 2r+1} = 0$ if and only if

$$\exists u,\ 1 \leq u \leq 2r,\ \ni E_{t,u} \geq E_{t+1,\ 2r} + \delta,$$

where $\delta = \max\{n \mid f(t, E_{t+1,\ 2r} + n) \geq n\}$. *Assume* that there exists such an integer u. We consider two cases.

Case 1. u is even. Here $\exists r',\ 1 \leq r' \leq r,\ \ni u = 2r'$. Since

$$E_{t+1,\ 2r} > 0,\ E_{t+1,\ 2r'-1} < E_{t+1,\ 2r}.$$

Thus

$$E_{t+1,\ 2r} + \delta \geq E_{t+1,\ 2r} + 1 > E_{t+1,\ 2r'-1} + 1 = E_{t,\ 2r'} = E_{t,u},$$

a contradiction.

Case 2. u is odd. Here $\exists r'$, $0 \le r' < r$, $\ni u = 2r' + 1$. Let

$$\delta' = \max\{n \mid f(t, E_{t+1,\, 2r'} + n) \ge n\},$$

so $E_{t,\, 2r'+1} = E_{t+1,\, 2r'} + \delta'$. Then

$$\begin{aligned} &f[t, E_{t+1,\, 2r} + (E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1)] \\ &\quad = f[t, E_{t+1,\, 2r} + (E_{t+1,\, 2r'} + \delta' - E_{t+1,\, 2r} + 1)] \\ &\quad = f(t, E_{t+1,\, 2r'} + \delta' + 1) \ge f(t, E_{t+1,\, 2r'} + \delta') \quad \text{by Condition 2.1} \\ &\quad \ge \delta' \quad \text{by the definition of } \delta' \\ &\quad = E_{t,\, 2r'+1} - E_{t+1,\, 2r'} \quad \text{since } E_{t,\, 2r'+1} = E_{t+1,\, 2r'} + \delta' \\ &\quad \ge E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1 \quad \text{since } E_{t+1,\, 2r} > 0 \Rightarrow E_{t+1,\, 2r} > E_{t+1,\, 2r'}. \end{aligned}$$

Thus, $\delta \ge E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1$. Consequently,

$$E_{t+1,\, 2r} + \delta \ge E_{t+1,\, 2r} + (E_{t,\, 2r'+1} - E_{t+1,\, 2r} + 1) > E_{t,\, 2r'+1} = E_{t,\, u},$$

a contradiction.

In both Case 1 and Case 2, a contradiction has been observed. Thus, it must be that $E_{t,\, 2r+1} > 0$. Q.E.D.

PROOF OF THEOREM 4.1: Consider the set

$$A = \bigcup_{r=0}^{\infty} A_{2r+1}.$$

To prove Theorem 4.1, it suffices to establish statements (a), (b), and (c) in the first paragraph of this section. Figure 6.1 is intended as a guide.

$$\begin{array}{ccccccc} \cdots E_{t,\gamma+1} \cdots & & E_{t,\alpha} & \cdots & E_{t,\beta} \cdots \\ \cdots E_{t+1,\gamma} \cdots & & E_{t+1,\alpha-1} & \cdots & E_{t+1,\beta-1} \cdots \end{array}$$

Fig. 6.1. A portion of the game tableau for f

a. The set of terminal positions is $\{(t, 0, f) \mid t \ge 1\}$.

$$\beta(t, 0, f) = 0 \ \forall\ t \ge 1, \text{ since } E_{t,0} = 0 \ \forall\ t \ge 1.$$

Thus, $\underline{\{\text{terminal positions}\} \cap A = \emptyset}$. Statement (a) holds.

b. Suppose $\underline{(t, n, f) \in A}$. Then $\beta(t, n, f)$ is odd. Let $\alpha = \alpha(t, n, f)$ and $\beta = \beta(t, n, f)$. There are two cases to consider.

Case b.1. $n > E_{t+1,\, \beta-1}$. Since $0 < n \le E_{t,\beta}$, part (b) of Lemma 6.1 indicates that $n - f(t, n) \le E_{t+1,\, \beta-1}$. Thus, in position (t, n, f), a player may <u>take</u>

$$\underline{n - E_{t+1,\, \beta-1}}$$

<u>chips</u> to leave the position $(t + 1, E_{t+1,\, \beta-1}, f)$. $\beta(t + 1, E_{t+1,\, \beta-1}, f) = \beta - 1$ is even, so

$$\underline{(t + 1, E_{t+1,\, \beta-1}, f) \notin A}.$$

Case b.2. $n \le E_{t+1,\, \beta-1}$. <u>Taking a single chip</u> leaves the position

$$(t + 1, n - 1, f).$$

Let $\beta' = \beta(t + 1, n - 1, f)$. Since $E_{t+1,\, \beta-1} > n - 1$, we have $\beta' \le \beta - 1$, and so $\beta' + 1 \le \beta$.

Assume that $(t + 1, n - 1, f) \in A$. Then β' is odd. Set $\tilde{E}_{t,\beta'+1} = E_{t+1,\beta'} + 1$. Since $E_{t+1,\beta'} \geq n - 1$, we have

$$\tilde{E}_{t,\beta'+1} \geq n. \tag{3}$$

Consequently,

$$\tilde{E}_{t,\beta'+1} > E_{t,\alpha}. \tag{4}$$

By the maximality of α, the minimality of β, and (4), we conclude that $E_{t,\beta'+1} > 0$ (and, of course, $E_{t,\beta'+1} = \tilde{E}_{t,\beta'+1}$). But $\beta'+1$ is even, β is odd, and $\beta' + 1 \leq \beta$. Hence, we also have $\beta'+1 < \beta$. $\beta'+1 < \beta$ and (3) contradict the minimality of β. We conclude that $(t + 1, n - 1, f) \notin A$.

We have shown that, in both Case b.1 and Case b.2, statement (b) holds.

c. Suppose $\underline{(t, n, f) \notin A}$. If $n = 0$, statement (c) is vacuous. So assume $n > 0$. Observe that β is even and that $\beta > 0$. Let $\gamma = \gamma(t, n, f)$. If $\gamma = 0$, then $E_{t,\gamma+1} > 0$. If $\gamma > 0$, then γ even and $E_{t+1,\gamma} > 0$ imply that $E_{t,\gamma+1} > 0$ by Lemma 6.2. Thus, in either case, $E_{t,\gamma+1} > 0$. So $\underline{\gamma + 1 \leq \alpha}$ by the maximality of α, the minimality of β, and the fact that $\alpha + 1 < \beta$.

Now $0 < E_{t,\gamma+1} \leq E_{t,\alpha} < n$ and γ even imply that

$$n - f(t, n) > E_{t+1,\gamma} \tag{5}$$

by (a) of Lemma 6.1. Since $n \leq E_{t,\beta} = E_{t+1,\beta-1} + 1$, $n - 1 \leq E_{t+1,\beta-1}$. Combine this with (5) to get

$$\underline{E_{t+1,\gamma} < n - u \leq E_{t+1,\beta-1} \ \forall u \ni 1 \leq u \leq f(t, n).}$$

Thus, $\beta(t + 1, n - u, f)$ is odd $\forall u \ni 1 \leq u \leq f(t, n)$. We have shown that

$$\underline{F(t, n, f) \subseteq A,}$$

which verifies statement (c). Q.E.D.

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FIBONACCI-CAYLEY NUMBERS

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Horadam [2] defined and studied in detail the generalized Fibonacci sequence defined by

(1) $$H_n = H_{n-1} + H_{n-2} \qquad (n > 2),$$

with $H_1 = p$, $H_2 = p + q$, p and q being arbitrary integers. In a later article [3] he defined Fibonacci and generalized Fibonacci quaternions as follows, and established a few relations for these quaternions:

(2) $$P_n = H_n + H_{n+1}i_1 + H_{n+2}i_2 + H_{n+3}i_3,$$

and

(3) $$Q_n = F_n + F_{n+1}i_1 + F_{n+2}i_2 + F_{n+3}i_3,$$

where

$$i_1^2 = i_2^2 = i_3^2 = -1,\ i_1i_2 = -i_2i_1 = i_3,$$

and

$$i_2i_3 = -i_3i_2 = i_1,\ i_3i_1 = -i_1i_3 = i_2,$$

and $\{F_n\}$ is the Fibonacci sequence defined by

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \qquad (n > 2).$$

He also defined the conjugate quaternion as

(4) $$\overline{P}_n = H_n - H_{n+1}i_1 - H_{n+2}i_2 - H_{n+3}i_3,$$

and $\overline{Q}_n$ in a similar way. M. N. S. Swamy [5] obtained some additional relations for these quaternions.

In Section 1 of this paper we define the Fibonacci and generalized Fibonacci-Cayley numbers. In Section 2, we obtain a further generalization of these numbers as well as of the complex Fibonacci numbers and Fibonacci quaternions discussed in [3] and [5].

SECTION 1

We call

(5) $$R_n = F_n + F_{n+1}i_1 + \cdots + F_{n+7}i_7$$

and

(6) $$S_n = H_n + H_{n+1}i_1 + \cdots + H_{n+7}i_7,$$

where

$$i_1^2 = \cdots = i_7^2 = -1,\ i_1i_2 = i_3 = -i_2i_1,$$

$$i_2i_3 = i_1 = -i_3i_2,\ i_3i_1 = i_2 = -i_1i_3,$$

and six similar sets of six relations with 1, 2, 3 replaced by 1, 4, 5; 6, 2, 4; 6, 5, 3; 7, 2, 5; 7, 3, 4; and 1, 7, 6, respectively (see [1]), nth Fibonacci and generalized Fibonacci-Cayley numbers, respectively. We define conjugate Cayley numbers as

(7) $$\overline{S}_n = H_n - H_{n+1}i_1 - \cdots - H_{n+7}i_7,$$

and $\overline{R}_n$ in a similar way, so that

$$\begin{aligned} S_n\overline{S}_n &= \sum_{i=0}^{7} H_{n+i}^2 = [H_n^2 + H_{n+1}^2 + H_{n+2}^2 + H_{n+3}^2] + [H_{n+4}^2 + H_{n+5}^2 + H_{n+6}^2 + H_{n+7}^2] \\ &= P_n\overline{P}_n + P_{n+4}\overline{P}_{n+4} = 3[(2p - q)H_{2n+3} - (p^2 - pq - q^2)F_{2n+3}] \\ &\qquad + 3[(2p - q)H_{2n+11} - (p^2 - pq - q^2)F_{2n+11}] \\ &\qquad\qquad \text{(using Eq. 15 of [5])} \end{aligned}$$

(8) $$= 3[(2p - q)(H_{2n+3} + H_{2n+11}) - (p^2 - pq - q^2)(F_{2n+3} + F_{2n+11})].$$

$S_n + \overline{S}_n = 2H_n$ implies

(9) $$S_n^2 = 2H_nS_n - S_n\overline{S}_n.$$

Since

(10) $$H_{m+n+1} = F_{m+1}H_{n+1} + F_mH_n = F_{n+1}H_{m+1} + F_nH_m$$

(see [2]), we have

$$\begin{aligned} F_{m+1}S_{n+1} + F_mS_n &= (F_{m+1}H_{n+1} + F_mH_n) + \cdots + (F_{m+1}H_{n+4} + F_mH_{n+3})i_3 \\ &\quad + (F_{m+1}H_{n+5} + F_mH_{n+4})i_4 + \cdots + (F_{m+1}H_{n+8} + F_mH_{n+7})i_7 \\ &= H_{m+n+1} + H_{m+n+2}i_1 + \cdots + H_{m+n+8}i_7 \\ &= S_{m+n+1}, \end{aligned}$$

so that

(11) $$S_{m+n+1} = F_{m+1}S_{n+1} + F_mS_n = F_{n+1}S_{m+1} + F_nS_m.$$

This implies

$$S_{2n+1} = F_{n+1}S_{n+1} + F_nS_n$$

and

$$S_{2n} = F_{n+1}S_n + F_nS_{n-1} = F_nS_{n+1} + F_{n-1}S_n.$$

Again, since

(12) $$H_{n+1} = qF_n + pF_{n+1}$$

(Eq. 7 of [2]), we have

$$\begin{aligned} H_{m+1}S_{n+1} + H_mS_n &= (qF_m + pF_{m+1})S_{n+1} + (qF_{m-1} + pF_m)S_n \\ &= p(F_{m+1}S_{n+1} + F_mS_n) + q(F_mS_{n-1} + F_{m-1}S_n) \\ &= pS_{m+n+1} + qS_{m+n} \text{ [by (11)]}. \end{aligned}$$

Using (8) and (12) above and Eq. 17 of [5], we get

(14) $$\begin{aligned} S_n\overline{S}_n &= 3[p^2F_{2n+3} + 2pqF_{2n+2} + q^2F_{2n+1} + p^2F_{2n+11} + 2pqF_{2n+10} + q^2F_{2n+9}] \\ &= 3[p^2(F_{2n+3} + F_{2n+11}) + 2pq(F_{2n+2} + F_{2n+10}) + q^2(F_{2n+1} + F_{2n+9})]. \end{aligned}$$

Hence

(15) $$\begin{aligned} S_n\overline{S}_n + S_{n-1}\overline{S}_{n-1} &= 3[p^2(F_{2n+3} + F_{2n+11} + F_{2n+1} + F_{2n+9}) \\ &\quad + 2pq(F_{2n+2} + F_{2n+10} + F_{2n} + F_{2n+8}) \\ &\quad + q^2(F_{2n+1} + F_{2n+9} + F_{2n+1} + F_{2n+7})] \\ &= 3[p^2(L_{2n+2} + L_{2n+10}) + 2pq(L_{2n+1} + L_{2n+9}) \\ &\quad + q^2(L_{2n} + L_{2n+8})], \end{aligned}$$

since $L_n = F_{n-1} + F_{n+1}$, where $\{L_n\}$ is the Lucas sequence defined by

$$L_1 = 1,\ L_2 = 3,\ L_n = L_{n-1} + L_{n-2} \qquad (n > 2).$$

From (9), (13), and (15), we have

$$\begin{aligned} S_n^2 + S_{n-1}^2 &= 2(H_nS_n + H_{n-1}S_{n-1}) - (S_n\overline{S}_n + S_{n-1}\overline{S}_{n-1}) \\ &= 2(pS_{2n-1} + qS_{2n-2}) - 3[p^2(L_{2n+2} + L_{2n+10}) + 2pq(L_{2n+1} + L_{2n+9}) \\ &\qquad + q^2(L_{2n} + L_{2n+8})]. \end{aligned}$$

Analogous to Eq. 16 of [2], we have

(17) $$\{2S_{n+1}S_{n+2}\}^2 + \{S_nS_{n+3}\}^2 = \{2S_{n+1}S_{n+2} + S_n\}^2.$$

Using (11), we can establish the identity analogous to Eq. 17 of [2]:

(18) $$\frac{S_{n+t} + (-1)\ S_{n-t}}{S_n} = F_{t-1} + F_{t+1}.$$

If $p = 1$, $q = 0$, then we have the Fibonacci sequence $\{F_n\}$ and the corresponding Cayley number R_n for which we may write the following results:

(19) $$R_n\overline{R}_n = \overline{R}_nR_n = 3(F_{2n+3} + F_{2n+11}).$$

(20) $$R_n\overline{R}_n + R_{n-1}\overline{R}_{n-1} = 3(L_{2n+2} + L_{2n+10}).$$

(21) $$R_n^2 + R_{n-1}^2 = 2R_{2n-1} - 3(L_{2n+2} + L_{2n+10}).$$

Similar results may be obtained for the Lucas numbers and the corresponding Cayley numbers by letting $p = 1$ and $q = 2$ in the various results derived above.

SECTION 2

A. The following facts about composition algebras over the field of real numbers (the details of which can be found in [4]) are needed to obtain further generalization of complex Fibonacci numbers, Fibonacci quaternions, and Fibonacci-Cayley numbers.

1. The 2-dimensional algebra over the field R of real numbers with basis $\{1, i_1\}$ and multiplication table

	1	i_1
1	1	i_1
i_1	i_1	$-\alpha$

(α being any nonzero real number).

We denote this algebra by $C(\alpha)$. The conjugate of $x = a_0 + a_1i_1$ is $\overline{x} = a_0 - a_1i_1$ and $x\overline{x} = \overline{x}x = a_0^2 + \alpha a_1^2$.

2. The 4-dimensional algebra (over R) with basis $\{1, i_1, i_2, i_3\}$ and multiplication table

	1	i_1	i_2	i_3
1	1	i_1	i_2	i_3
i_1	i_1	$-\alpha$	i_3	$-\alpha i_2$
i_2	i_2	$-i_3$	$-\beta$	βi_1
i_3	i_3	αi_2	$-\beta i_1$	$-\alpha\beta$

(α, β any nonzero real numbers).

We denote this algebra by $C(\alpha, \beta)$. The conjugate of $x = a_0 + a_1i_1 + a_2i_2 + a_3i_3$ is $\overline{x} = a_0 - a_1i_1 - a_2i_2 - a_3i_3$ and $x\overline{x} = \overline{x}x = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2$.

3. The 8-dimensional algebra (over R) with basis $\{1, i_1, \ldots, i_7\}$ and multiplication table

	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	$-\alpha$	i_3	$-\alpha i_2$	i_5	$-\alpha i_4$	$-i_7$	αi_6
i_2	$-i_3$	$-\beta$	βi_1	i_6	i_7	$-\beta i_4$	$-\beta i_5$
i_3	αi_2	$-\beta i_1$	$-\alpha\beta$	i_7	$-\alpha i_6$	βi_5	$-\alpha\beta i_4$
i_4	$-i_5$	$-i_6$	$-i_7$	$-\gamma$	γi_1	γi_2	γi_3
i_5	αi_4	$-i_7$	αi_6	$-\gamma i_1$	$-\alpha\gamma$	$-\gamma i_3$	$\gamma\alpha i_2$
i_6	i_7	βi_4	$-\beta i_5$	$-\gamma i_2$	γi_3	$-\gamma\beta$	$-\gamma\beta i_1$
i_7	$-\alpha i_6$	βi_5	$\alpha\beta i_4$	$-\gamma i_3$	$-\gamma\alpha i_2$	$\gamma\beta i_1$	$-\alpha\beta\gamma$

(α, β, γ any nonzero real numbers).

We denote this algebra by $C(\alpha, \beta, \gamma)$. The conjugate of $x = a_0 + a_1 i_1 + \cdots + a_7 i_7$ is $\overline{x} = a_0 - a_1 i_1 - \cdots - a_7 i_7$ and $x\overline{x} = \overline{x}x = (a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2) + \gamma(a_4^2 + \alpha a_5^2 + \beta a_6^2 + \alpha\beta a_7^2)$.

B. Next we shall consider the following generalizations of H_n, F_n, and L_n, respectively:

$$h_n\colon\ h_1 = p,\ h_2 = bp + cq,\ h_n = bh_{n-1} + ch_{n-2} \qquad (n > 2)$$
$$f_n\colon\ f_1 = 1,\ f_2 = b,\ f_n = bf_{n-1} + cf_{n-2} \qquad (n > 2)$$
$$\ell_n\colon\ \ell_1 = b,\ \ell_2 = b^2 + 2c,\ \ell_n = b\ell_{n-1} + c\ell_{n-2} \qquad (n > 2)$$

(b, c, p, q being integers).

Then we have the following various relations:

$$h_n = pf_n + qcf_{n-1}$$
$$\ell_n = f_{n+1} + cf_{n-1}$$
$$ph_{2n-2} + cqh_{2n-3} = h_{n-1}(ch_{n-2} + h_n)$$
$$ch_n^2 + h_{n+1}^2 = ph_{2n+1} + cqh_{2n} = (2p - bq)h_{2n+1} - ef_{2n+1},$$

where $e = p^2 - bpq - cq^2$.

$$h_nh_{n+1} - c^2h_{n-2}h_{n-1} = b(ph_{2n-1} + qch_{2n-2})$$
$$h_{n+1}^2 - c^2h_{n-1}^2 = b(ph_{2n} + cqh_{2n-1}) = b(2p - bq)h_{2n} - bef_{2n}$$
$$h_{n-1}h_{n+1} - h_n^2 = (-c)^n e$$
$$f_{n-1}f_{n+1} - f_n^2 = (-c)^n$$
$$h_{n+t} = ch_{n-1}f_t + h_nf_{t+1} = ch_{t-1}f_n + h_tf_{n+1}$$
$$\frac{h_{n+t} - (-c)^{t+1}h_{n-t}}{h_n} = cf_{t-1} + f_{t+1}.$$

We now define the nth generalized complex Fibonacci number d_n as the element $h_n + h_{n+1}i_1$ of the algebra $C(1/c)$; the nth generalized Fibonacci quaternion p_n as the element $h_n + h_{n+1}i_1 + h_{n+2}i_2 + h_{n+3}i_3$ of the algebra $C(1/c, 1)$; and the nth generalized Fibonacci-Cayley number s_n as the element $h_n + h_{n+1}i_1 + \cdots + h_{n+7}i_7$ of the algebra $C(1/c, 1, 1)$.

The following is a list of relations for these numbers:

$$d_{n-1}d_{n+1} - d_n^2 = (-c)^n e(2 + bi_1).$$
$$\begin{aligned} d_n\overline{d}_n = \overline{d}_nd_n &= h_n^2 + \frac{1}{c}h_{n+1}^2 \\ &= \frac{1}{c}(ph_{2n+1} + cqh_{2n}) = \frac{1}{c}[(2p - bq)h_{2n+1} - ef_{2n+1}] \\ &= \frac{1}{c}[(p^2 + cq^2)f_{2n+1} + cq(2p - bq)f_{2n}]. \end{aligned}$$
$$\begin{aligned} d_n\overline{d}_n + cd_{n-1}\overline{d}_{n-1} &= \frac{1}{c}[(p^2 + cq^2)(f_{2n+1} + cf_{2n-1}) + qc(2p - bq)(f_{2n} + cf_{2n-2})] \\ &= \frac{1}{c}[(p^2 + cq^2)\ell_{2n} + qc(2p - bq)\ell_{2n-1}]. \end{aligned}$$
$$d_{m+n+1} = f_{m+1}d_{n+1} + cf_md_n = f_{n+1}d_{m+1} + cf_nd_m.$$
$$h_{m+1}d_{n+1} + ch_md_n = pd_{m+n+1} + qcd_{m+n}.$$
$$d_n^2 + cd_{n-1}^2 = 2(pd_{2n-1} + qcd_{2n-2}) - \frac{1}{c}[(p^2 + cq^2)\ell_{2n} + qc(2p - bq)\ell_{2n-1}].$$

$$d_{n+1}^2 - c^2 d_{n-1}^2 = -\frac{b^2}{c}(2p - bq)h_{2n+1} + \frac{b^2 e}{c} f_{2n+1} + 2b(ph_{2n+1} + qch_{2n})i_1.$$

$$\frac{d_{n+t} - (-c)^{t+1} d_{n-t}}{d_n} = cf_{t-1} + f_{t+1}.$$

$$p_n \overline{p}_n = d_n \overline{d}_n + d_{n+2}\overline{d}_{n+2} = \frac{1}{c}[(2p - bq)(h_{2n+1} + h_{2n+5}) - e(f_{2n+1} + f_{2n+5})]$$

$$= \frac{1}{c}[(p^2 + cq^2)(f_{2n+1} + f_{2n+5}) + cq(2p - bq)(f_{2n} + f_{2n+4})].$$

$$p_n \overline{p}_n + cp_{n-1}\overline{p}_{n-1} = \frac{1}{c}[(p^2 + cq^2)(\ell_{2n} + \ell_{2n+4}) + cq(2p - bq)(\ell_{2n-1} + \ell_{2n+3})].$$

$$p_{m+n+1} = f_{m+1}p_{n+1} + cf_m p_n = f_{n+1}p_{m+1} + cf_n p_m.$$

$$h_{m+1}p_{n+1} + ch_m p_n = pp_{m+n+1} + qcp_{m+n}.$$

$$p_n^2 + cp_{n-1}^2 = pp_{2n-1} + qcp_{2n-2} - (p_n\overline{p}_n + cp_{n-1}\overline{p}_{n-1}).$$

$$\frac{p_{n+t} - (-c)^{t+1} p_{n-t}}{p_n} = cf_{t-1} + f_{t+1}.$$

$$s_n \overline{s}_n = p_n \overline{p}_n + p_{n+4}\overline{p}_{n+4}.$$

$$s_n \overline{s}_n + cs_{n-1}\overline{s}_{n-1} = p_n\overline{p}_n + cp_{n-1}\overline{p}_{n-1} + p_{n+4}\overline{p}_{n+4} + cp_{n+3}\overline{p}_{n+3}.$$

$$s_{m+n+1} = f_{m+1}s_{n+1} + cf_m s_n = f_{n+1}s_{m+1} + cf_n s_m.$$

$$h_{m+1}s_{n+1} + ch_m s_n = ps_{m+n+1} + qcs_{m+n}.$$

$$s_n^2 + cs_{n-1}^2 = ps_{2n-1} + qcs_{2n-2} - (s_n\overline{s}_n + cs_{n-1}\overline{s}_{n-1}).$$

$$\frac{s_{n+t} - (-c)^{t+1} s_{n-t}}{s_n} = cf_{t-1} + f_{t+1}.$$

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GENERALIZATIONS OF SOME PROBLEMS ON FIBONACCI NUMBERS

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1. INTRODUCTION

In this paper, we obtain generalizations of some problems which have appeared in recent years in *The Fibonacci Quarterly*.

Throughout $\{F_n\}$ denotes the Fibonacci sequence defined by

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad (n > 2)$$

and $\{L_n\}$ denotes the Lucas sequence defined by

$$L_1 = 1,\ L_2 = 3, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad (n > 2).$$

Sequences $\{h_n\}$, $\{f_n\}$, and $\{\ell_n\}$ are defined as follows, respectively:

$$h_1 = p,\ h_2 = bp + cq,\ h_n = bh_{n-1} + ch_{n-2} \qquad (n > 2)$$

$$f_1 = 1,\ f_2 = b,\ f_n = bf_{n-1} + cf_{n-2} \qquad (n > 2)$$

$$\ell_1 = b,\ \ell_2 = b^2 + 2c,\ \ell_n = b\ell_{n-1} + c\ell_{n-2} \qquad (n > 2)$$

$$(b,\ c,\ p,\ q \text{ being integers}).$$

Note that for $b = c = p = 1$, $q = 0$ we will have $h_n = f_n$ and for $b = c = 1$ we will have $f_n = F_n$ and $\ell_n = L_n$.

The following relations will be used throughout:

$$h_n = \frac{\ell r^n - ms^n}{r - s}, \qquad f_n = \frac{r^n - s^n}{r - s},$$

$$\ell_n = r^n + s^n, \qquad \ell_n = cf_{n-1} + f_{n+1},$$

$$f_{2n} = f_n \ell_n, \qquad f_{2n} = -c^{2n} f_{-2n},$$

where

$$r + s = b,\ rs = -c,\ \ell = p - sq, \text{ and } m = p - rq.$$

2. GENERALIZATIONS

No proofs of the following generalizations are given, since they follow those of the original statements very closely. The original statements are referenced in parentheses, giving the Problem number, Volume number, and Year in which they appeared in *The Fibonacci Quarterly*.

H-263 (15, 1977): $\ell_{2mn}^2 \equiv 4c^{2mn} \pmod{\ell_m^2}$.

H-279 (17, 1979):

(a) $f_{n+6r}^4 - c^{2r}(\ell_{4r} + c^{2r})(f_{n+4r}^4 - c^{4r} f_{n+2r}^4) - c^{12r} f_n^4 = f_{2r} f_{4r} f_{6r} f_{4n+12r}$.

(b) $f_{n+6r+3}^4 + c^{2r+1}(\ell_{4r+2} - c^{2r+1})(f_{n+4r+2}^4 - c^{4r+2} f_{n+2r+1}^4) - c^{12r+6} f_n^4$
$= f_{2r+1} f_{4r+2} f_{6r+3} f_{4n+12r+6}$.

LEMMA 1: $\ell_{3m} - (-c)^m \ell_m = (b^2 + 4c) f_m f_{2m}$.

LEMMA 2: $(b^2 + 4c)(f_u^4 - c^{2u-2v} f_v^4) = f_{u-v} f_{u+v}[\ell_{u-v}\ell_{u+v} - 4(-c)^u]$.

LEMMA 3: $(-c)^m \ell_{2m} + c^{2m} = (-c)^m f_{3m}/f_m$.

B-271(b) (12, 1974): If k is even, then $\ell_k - 2c^k$ divides

$$h_{(n+2)k} - 2h_{(n+1)k}c^k + h_{nk}c^k.$$

(This generalization was suggested by the referee.)

B-275 (13, 1975): $h_{mn} = \ell_m h_{m(n-1)} - (-c)^m h_{m(n-2)}$.

B-277 (13, 1975): $\ell_{2n(2k+1)} \equiv c^{2nk}\ell_{2n} \pmod{f_{2n}}$.

B-282 (13, 1975): If $c = d^2$ $(d > 0)$, then $2d\ell_n\ell_{n+1}$, $|\ell_{n+1}^2 - c\ell_n^2|$, and $c\ell_{2n} + \ell_{2n+2}$ are the lengths of a right-angled triangle.

B-294 (13, 1975): $h_n\ell_k + h_k\ell_n = 2h_{n+k} + q(-c)^k\ell_{n-k}$.

B-298 (14, 1976): $(b^2 + 4c)h_{2n+3}h_{2n-3} = p^2\ell_{4n} + 2cpq\ell_{4n-1} + q^2c^2\ell_{4n-2} + ec^{2n-3}\ell_6$, where $e = p^2 - bpq - cq^2 = \ell m$.

B-323 (15, 1977): $h_{n+t}^2 - (-c)^t h_n^2 = f_t(ph_{2n+t} + cqh_{2n+t-1})$.

B-342 (15, 1977): $2c^3\ell_{n-1}^3 + b^3\ell_n^3 + 6c\ell_{n+1}^2\ell_{n-1} = (\ell_{n+1} + c\ell_{n-1})^3$.

B-343 (15, 1977): $\sum_{k=1}^{n}[cf_{2k-1}f_{2(n-k)+1} - f_{2k}f_{2(n-k+1)}] = \frac{1}{b^2+4c}\left(\frac{4c^2}{b}f_{2n} - bn\ell_{2n+1}\right)$.

B-354 (16, 1978): $h_{n+k}^3 - \ell_k^3h_n^3 + (-c)^k h_{n-k}[c^{2k}h_{n-k}^2 + 3h_{n+k}h_n\ell_k] = 0$.

B-355 (16, 1978): $h_{n+k}^3 - \ell_{3k}h_n^3 + (-c)^{3k}h_{n-k}^3 = 3e(-c)^n h_n f_k f_{2k}$.

B-379 (17, 1979): $f_{2n} \equiv nb(-c)^{n-1} \pmod{(b^2 + 4c)}$ for $n = 1, 2, \ldots$.

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A VARIANT OF THE FIBONACCI POLYNOMIALS WHICH ARISES IN THE GAMBLER'S RUIN PROBLEM

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In examining the gambler's ruin problem (a simple case of random walk) with a finite number of possible states, we were led to consider a sequence of linear recurrence relations that describe the number of ways to reach a given state. These recurrence relations have a sequence of polynomials as their auxiliary equations. These polynomials were unknown to us, but proved exceptionally rich in identities. We gradually noticed that these identities were analogous to well-known identities satisfied by the Fibonacci numbers. A check of back issues of *The Fibonacci Quarterly* then revealed that our sequence of polynomials differed only in sign from the Fibonacci polynomials studied in [1], [5], and several other papers.

In this paper we show, using graph theory and linear algebra, how the gambler's ruin problem gives rise to our sequence of polynomials. We then compare our polynomials to the Fibonacci polynomials and explain why the two sequences satisfy analogous identities. Finally, we use the Pascal arrays introduced in our analysis of gambler's ruin to give a novel proof of the divisibility properties of our sequence.

The Fibonacci numbers are defined recursively by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2},\ n \geq 2.$$

Likewise, the Fibonacci polynomials are defined by

$$F_0(x) = 0,\ F_1(x) = 1, \text{ and } F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ n \geq 2$$

(see [1, p. 407]).

1. GAMBLER'S RUIN AND PASCAL ARRAYS

A gambler whose initial capital is j dollars enters a game consisting of a sequence of discrete rounds. Each round is either won or lost. If the gambler wins a round, he is awarded one dollar; if he loses the round, he must forfeit one dollar. The game continues until either:

1. His capital reaches 0 for the first time. (Ruin.)
2. His capital reaches $b > 1$ for the first time. (Victory.)

Zero is called the *lower barrier* and b the *upper barrier*. Since the game ends as soon as either barrier is reached, these barriers are *absorbing* [3, p. 342].

We are interested in the number of ways the gambler's capital can reach i dollars, $0 < i < b$, in n rounds. Since he gains or loses one dollar in each round, this number equals the sum of the number of ways his capital can reach $i - 1$ or $i + 1$ dollars in $n - 1$ rounds, provided that $i - 1$ and $i + 1$ do not lie on the barriers. These numbers thus satisfy a recursive relation similar to that of the binomial coefficients in Pascal's triangle, except for the interference of the barriers.

Following Feller [3, Ch. 3], we use a "left-to-right" format for our truncated Pascal triangle rather than a "top-to-bottom" format. Thus in Diagram 1, we plot the numbers we have been describing on integer lattice points (n, i) with $b = 5$. We make the initial capital three dollars.

Capital									
5									
4	0	1	0	2	0	5	0	13	
3	1	0	2	0	5	0	13	0	Barriers
2	0	1	0	3	0	8	0	21	
1	0	0	1	0	3	0	8	0	
0									
	0	1	2	3	4	5	6	7	

Number of Rounds

Diagram 1

The appearance of the Fibonacci numbers F_n and F_{n+1} in the nth column is an accidental consequence of the selection of $b = 5$. In speaking of the point (n, i), we are using the "column first, row second" convention that is standard for coordinate systems, not the "row first, column second" convention of matrix theory.

It will be useful later to employ this rectangular lattice with more general initial values (the values in the 0th column). Given an integer $b > 1$ and a vector $\vec{X}_0 \in \mathbb{C}^{b-1}$, we define the *Pascal Array* P.A. $(b, \vec{X}_0)$, of height b and initial vector X_0, to be the (complex) array whose (n, i)-entry, for $n \geq 0$ and $0 < i < b$, is

$$(1) \qquad F(n, b, \vec{X}_0, i) = \begin{cases} \vec{X}_0 \cdot \vec{e}_i \text{ if } n = 0 \\ F(n-1, b, \vec{X}_0, i-1) + F(n-1, b, \vec{X}_0, i+1) \\ \text{if } n > 0 \text{ and } 1 < i < b-1 \\ F(n-1, b, \vec{X}_0, 2) \text{ if } n > 0 \text{ and } i = 1 \\ F(n-1, b, \vec{X}_0, b-2) \text{ if } n > 0 \text{ and } i = b-1. \end{cases}$$

($\vec{e}_i$ is the ith standard unit vector in $\mathbb{C}^{b-1}$.) Thus, in gambler's ruin, we are dealing with P.A. $(b, \vec{e}_j)$, $1 \le j \le b - 1$.

If the initial values are all nonnegative integers, we could interpret the Pascal array as representing many gamblers with different amounts of initial capital gambling at the same time. The full generality of complex entries will be needed in the last section of this paper.

LEMMA 1: If

$$\vec{X}_0 = \sum_{k=1}^{b-1} \alpha_k \vec{e}_k,$$

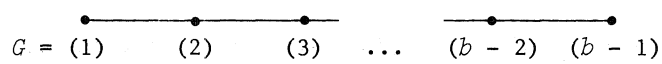
then

$$F(n, b, \vec{X}_0, i) = \sum_{k=1}^{b-1} \alpha_k F(n, b, \vec{e}_k, i).$$

PROOF: This is true for $n = 0$ since the 0th column of P.A. $(b, \vec{X}_0)$ consists of the coordinates $(\alpha_1, \ldots, \alpha_{b-1})$ of $\vec{X}_0$. The recursive definition (1) can then be used to establish the result for all n. □

2. GRAPH THEORY AND RECURRENCE RELATIONS

To learn more about Pascal arrays, it is useful to consider the labeled graph



A gambler could keep track of his gains and losses by moving a marker in a "random walk" along the vertices of this graph. (He would have to leave the graph when he achieved victory or ruin.)

The associated *adjacency matrix* A_b is the $(b - 1) \times (b - 1)$ matrix with

$$A_b(j, i) = \begin{cases} 1 \text{ if vertices } (j) \text{ and } (i) \text{ are connected by an edge,} \\ 0 \text{ otherwise.} \end{cases}$$

For example,

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

LEMMA 2: [2, Lemma 2.5, p. 11] For $n \ge 1$, the (j, i)-entry of the matrix power A_b^n equals the number of paths of G of length n starting at vertex (j) and ending at vertex (i). □

In Pascal array terminology, $A_b(j, i) = F(n, b, \vec{e}_j, i)$.

The *characteristic polynomial* of A_b is

$$P_b(\lambda) = \det(A_b - \lambda I_b),$$

where I_b is the $(b - 1) \times (b - 1)$ identity matrix. We have $P_2 = -\lambda$, $P_3 = \lambda^2 - 1$, and, in general, we expand the determinant by its first row to obtain the important recursive formula

$$P_k(\lambda) = -\lambda P_{k-1}(\lambda) - P_{k-2}(\lambda). \tag{2}$$

Note the similarity of this definition to that of the Fibonacci polynomials. Consistent with (2), we define

$$P_1(\lambda) = -P_3(\lambda) - \lambda P_2(\lambda) = 1 \quad \text{and} \quad P_0(\lambda) = -P_2(\lambda) - \lambda P_1(\lambda) = 0.$$

In Diagram 2, we give a chart of the $P_k(\lambda)$ for $0 \le k \le 10$.

k	$P_k(\lambda)$
0	0
1	1
2	$-\lambda$
3	$\lambda^2 - 1$
4	$-\lambda^3 + 2\lambda$
5	$\lambda^4 - 3\lambda^2 - 1$
6	$-\lambda^5 + 4\lambda^3 - 3\lambda$
7	$\lambda^6 - 5\lambda^4 + 6\lambda^2 - 1$
8	$-\lambda^7 + 6\lambda^5 - 10\lambda^3 + 4\lambda$
9	$\lambda^8 - 7\lambda^6 + 15\lambda^4 - 10\lambda^2 + 1$
10	$-\lambda^9 + 8\lambda^7 - 21\lambda^5 + 20\lambda^3 - 5\lambda$

Diagram 2

LEMMA 3: $P_k(\lambda) = i^{1-k}F_k(-i\lambda)$, where the F_k are the Fibonacci polynomials.

PROOF: By induction.

$$P_0(\lambda) = 0 = i^1F_0(-i\lambda) \quad \text{and} \quad P_1(\lambda) = 1 = i^0F_1(-i\lambda).$$

For $k \ge 2$,

$$\begin{aligned} P_k(\lambda) &= -\lambda P_{k-1}(\lambda) - P_{k-2}(\lambda) \quad \text{(by inductive assumption)} \\ &= (-\lambda)i^{1-(k-1)}F_{k-1}(-i\lambda) - i^{1-(k-2)}F_{k-2}(-i\lambda) \\ &= i^{1-k}[(-i\lambda)F_{k-1}(-i\lambda) - i^2F_{k-2}(-i\lambda)] \\ &= i^{1-k}[(-i\lambda)F_{k-1}(-i\lambda) + F_{k-2}(-i\lambda)] \\ &= i^{1-k}F_k(-i\lambda). \quad \square \end{aligned}$$

LEMMA 4: $P_k(\lambda)$ is a polynomial with integer coefficients having degree $k - 1$ and leading coefficient $(-1)^{k-1}$.

PROOF: These statements follow from Eq. (2) by induction. $\square$

The Cayley-Hamilton Theorem [4, Cor. 2, p. 244] states that $P_b(A_b)$ equals the zero matrix. Then for any $m \ge 0$, $A_b^m \cdot P_b(A_b)$ equals the zero matrix. Let

$$P_b(\lambda) = \sum_{k=0}^{b-1} \beta_k \lambda^k.$$

Thus

$$\sum_{k=0}^{b-1} \beta_k A_b^{m+k} \quad \text{equals the zero matrix for all } m \ge 0.$$

Looking at individual entries,

$$\sum_{k=0}^{b-1} \beta_k A_b^{m+k}(j, i) = 0 \text{ for all } m \ge 0,\ 0 < i, j < b.$$

By Lemma 2, this is equivalent to

$$\sum_{k=0}^{b-1} \beta_k F(m + k, b, \vec{e}_j, i) = 0 \text{ for all } m \ge 0,\ 0 < i, j < b. \tag{3}$$

When a sequence satisfies a linear recurrence relation such as (3), we say that

$$P_b(\lambda) = \sum_{k=0}^{b-1} \beta_k \lambda^k = 0$$

is its *auxiliary equation*.

We have proved the following.

THEOREM 5: Every row $\{F(n, b, \vec{e}_j, i)\}_{n=0}^{\infty}$ of the Pascal array P.A. $(b, \vec{e}_j)$ is a sequence which satisfies a linear recurrence relation with constant coefficients and auxiliary equation $P_b(\lambda) = 0$. □

COROLLARY 6: Every row of *any* Pascal array P.A. $(b, \vec{X}_0)$ satisfies the linear recurrence relation with auxiliary equation $P_b(\lambda) = 0$.

PROOF: This follows from Theorem 5, from Lemma 1, and from the superposition principle for solutions to linear recurrence relations. □

As a consequence of Corollary 6 and Lemma 4, if we know a row of a Pascal array P.A. $(b, \vec{X}_0)$ as far as the $(b - 2)$nd column, we can reconstruct the whole row uniquely.

We have not yet derived a closed-form expression for $P_k(\lambda)$. Following [4, pp. 267-70], we write

$$P_{k+1}(\lambda) = -\lambda P_k(\lambda) - P_{k-1}(\lambda)$$
$$P_k(\lambda) = P_k(\lambda)$$

In matrix terms,

$$\begin{bmatrix} P_{k+1}(\lambda) \\ P_k(\lambda) \end{bmatrix} = \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_k(\lambda) \\ P_{k-1}(\lambda) \end{bmatrix}.$$

A long calculation then produces the closed-form expression

$$P_k(\lambda) = \sum_{j=1}^{\left[\frac{k+1}{2}\right]} (-1)^{k-j} \binom{k-j}{j-1} \lambda^{k-(2j-1)}.$$

This parallels [1, Eq. (1.3), p. 409]. And, as in [1, Sec. 3], the matrix

$$M = \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix}$$

can be made to yield a great number of identities based on the iterative property

$$M^k = \begin{bmatrix} P_{k+1}(\lambda) & -P_k(\lambda) \\ P_k(\lambda) & -P_{k-1}(\lambda) \end{bmatrix}.$$

Unlike the Fibonacci polynomials [5, Theorem 1], the $P_k(\lambda)$ are reducible for $k \geq 3$. Their factors are interesting, and should be a subject of further study.

3. DIVISIBILITY PROPERTIES

In this section we will show how divisibility properties of $\{P_b(\lambda)\}$ similar to those of the Fibonacci polynomials [1, p. 415] follow from the consideration of Pascal arrays. Some of our theorems could also be derived using the above matrix M, but we wish to give proofs in the spirit of the gambler's ruin problem.

LEMMA 7: Let λ_0 be a root of $P_b(\lambda) = 0$. Then the sequence $\{1, \lambda_0, \lambda_0^2, \ldots$ satisfies the linear recurrence relation with auxiliary equation $P_b(\lambda) = 0$.

PROOF: Let

$$P_b(\lambda) = \sum_{k=0}^{b-1} \beta_k \lambda^k.$$

Then for any $m \geq 0$,

$$\sum_{k=0}^{b-1} \beta_k \lambda_0^{k+m} = \lambda_0^m \sum_{k=0}^{b-1} \beta_k \lambda_0^k = 0, \text{ since } \lambda_0 \text{ is a root of } P_b(\lambda) = 0. \ \square$$

LEMMA 8: There exists a vector $\vec{X}_1$ such that P.A. $(b, \vec{X}_1)$ has bottom row $\{1, \lambda_0, \lambda_0^2, \ldots\}$, where λ_0 is a root of $P_b(\lambda) = 0$.

PROOF: Suppose we are given

$$F(0, b, \vec{X}_1, 1) = 1,\ F(1, b, \vec{X}_1, 1) = \lambda_0,\ \ldots,\ F(b-2, b, \vec{X}_1, 1) = \lambda_0^{b-2},$$

and we wish to determine $\vec{X}_1$. Using Eq. (1) (third clause), we can determine $F(0, b, \vec{X}_1, 2)$ through $F(b-3, b, \vec{X}_1, 2)$. Then using Eq. (1) (second clause), we can determine $F(0, b, \vec{X}_1, 3), \ldots, F(b-4, b, \vec{X}_1, 3), \ldots, F(0, b, \vec{X}_1, b-2)$, $F(1, b, \vec{X}_1, b-2)$, and $F(0, b, \vec{X}_1, b-1)$. Thus, $\vec{X}_1$ is determined uniquely. Diagram 3 illustrates this procedure in case $b = 5$.

5					
4	$\lambda_0^3 - 2\lambda_0$	$\ldots$			
3	$\lambda_0^2 - 1$	$\lambda_0^3 - \lambda_0$	$\ldots$		
2	λ_0	λ_0^2	λ_0^3	$\ldots$	
1	1	λ_0	λ_0^2	λ_0^3	$\ldots$
	0	1	2	3	

Diagram 3

Now we can fill in all of P.A. $(b, \vec{X}_1)$. By Corollary 6, its bottom row satisfies the linear recurrence relation with auxiliary equation $P_b(\lambda) = 0$. By the remark following Corollary 6 and Lemma 7, that row must be $\{1, \lambda_0, \lambda_0^2, \lambda_0^3, \ldots\}$. □

Next we show that $\{1, \lambda_0, \lambda_0^2, \ldots\}$, and indeed all of P.A. $(\vec{X}_1, b)$, can be embedded in a Pascal array of height bc for any integer $c > 0$. It then follows easily that λ_0 is also a root of $P_{bc}(\lambda) = 0$.

If $\vec{X} = \alpha_1\vec{e}_1 + \cdots + \alpha_k\vec{e}_k$, then the *palindrome* $\vec{X}^p$ is defined to be

$$\alpha_k\vec{e}_1 + \cdots + \alpha_1\vec{e}_k.$$

We construct an arbitrary array $G(n, i)$, $n \geq 0$ and $0 < i < bc$, as follows:

$$G(n, i) = \begin{cases} F(n, b, \vec{X}_1, i - 2db) \text{ if } 2db < i < (2d+1)b \text{ and } 0 \leq d \leq \left[\dfrac{c-1}{2}\right]. \\ 0 \text{ if } i \text{ is a multiple of } b, \\ F(n, b, -\vec{X}_1^p, i - (2d-1)b) \text{ if } (2d-1)b < i < 2db \text{ and } 1 \leq d \leq \left[\dfrac{c}{2}\right]. \end{cases}$$

In Diagram 4, we illustrate this construction in the case $b = 5$, $c = 2$, $\vec{X}_1 = \vec{e}_3$.

								$i = bc$
9	0	0	-1	0	-3	0	-8	0
8	0	-1	0	-3	0	-8	0	-21
7	-1	0	-2	0	-5	0	-13	0
6	0	-1	0	-2	0	-5	0	-13
5	0	0	0	0	0	0	0	0
4	0	1	0	2	0	5	0	13
3	1	0	2	0	5	0	13	0
2	0	1	0	3	0	8	0	21
1	0	0	1	0	3	0	8	0
	0	1	2	3	4	5	6	7

Diagram 4

LEMMA 9: $G(n, i)$ is a Pascal array of height bc with initial vector

$$\vec{X}_2 = (\vec{X}_1, 0, -\vec{X}_1^p, 0, \ldots).$$

PROOF: This follows by checking definition (1) in the five cases of an entry in an all-zero row, an entry next to an all-zero row, an entry in the interior of one of the copies of P.A. $(\vec{X}_1, b)$ or P.A. $(-\vec{X}_1^p, b)$, and an entry in the top or bottom row of the whole array. □

THEOREM 10: Any root of $P_b(\lambda) = 0$ is a root of $P_{bc}(\lambda) = 0$.

PROOF: We have just seen that if λ_0 is any root of $P_b(\lambda) = 0$, then $\{1, \lambda_0, \lambda_0^2, \ldots\}$ is the bottom row of a Pascal array of height bc. By Corollary 6, the sequence satisfies the linear recurrence relation with auxiliary equation $P_{bc}(\lambda) = 0$. Applying this fact to the subsequence $\{1, \lambda_0, \ldots, \lambda_0^{bc-1}\}$, we have that

$$P_{bc}(\lambda_0) = 0. \ \square$$

THEOREM 11: $P_b(\lambda)$ divides $P_{bc}(\lambda)$, with quotient a polynomial $Q(\lambda)$ with integer coefficients and leading coefficient ±1.

PROOF: By Theorem 10, $P_b(\lambda)$ divides $P_{bc}(\lambda)$. Let the quotient be $Q(\lambda)$. Define

$$Q(\lambda) = \sum_{k=0} \alpha_k \lambda^k \ , \ P_b(\lambda) = \sum_{k=0}^{b-2} \beta_k \lambda^k \pm \lambda^{b-1}, \text{ and } P_{bc}(\lambda) = \sum_{k=0} \gamma_k \lambda^k \pm \lambda^{bc-1}.$$

The form of these last two expressions is dictated by Lemma 4. By multiplication of leading coefficients, $(\pm 1)\alpha_{bc-b} = \pm 1$, which implies that $\alpha_{bc-b} = \pm 1$. Suppose we have proved that $\alpha_{bc-b}, \alpha_{bc-b-1}, \ldots, \alpha_{bc-b-k}$ are integers. Then by polynomial multiplication,

$$\alpha_{bc-b-(k+1)}(\pm 1) + \alpha_{bc-b-k}\beta_{bc-2} + \cdots + = \gamma_{bc-(k+2)}.$$

Thus $\alpha_{bc-b-(k+1)}$ must also be an integer. This completes the proof by induction. □

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GENERALIZED FIBONACCI NUMBERS BY MATRIX METHODS

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In [7], Silvester shows that a number of the properties of the Fibonacci sequence can be derived from a matrix representation. In so doing, he shows that if $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ then

$$A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix}, \tag{1}$$

where u_k represents the kth Fibonacci number. This is a special case of a more general phenomenon. Suppose the $(n+k)$th term of a sequence is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1} \tag{2}$$

($c_0, \ldots, c_{k-1}$ are constants). Given values for the first k terms, $a_0, a_1, \ldots, a_{k-1}$, (2) uniquely determines a sequence $\{a_n\}$. In this context, the Fibonacci sequence $\{u_n\}$ may be viewed as the solution to

$$a_{n+2} = a_n + a_{n+1}$$

which has initial terms $u_0 = 0$ and $u_1 = 1$.

Difference equations of the form (2) are expressible in a matrix form analogous to (1). This formulation is unfortunately absent in some general works on difference equations (e.g. [2], [4]), although it has been used extensively by Bernstein (e.g. [1]) and Shannon (e.g. [6]). Define the matrix A by

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then, by an inductive argument, we reach the generalization of (1):

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}. \tag{3}$$

Just as Silvester derived many interesting properties of the Fibonacci numbers from a matrix representation, it also is possible to learn a good deal about $\{a_n\}$ from (3). We will confine ourselves to deriving a general formula for a_n as a function of n valid for a large class of equations (2). The reader is invited to generalize our results and explore further consequences of (3).

Following Shannon [5], we define a generalized Fibonacci sequence as a solution to (2) with the initial terms $[a_0, \ldots, a_{k-1}] = [0, 0, \ldots, 0, 1]$. Equation (3) then becomes

$$\begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k} \end{bmatrix} = A^n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

More specifically, a formula for a_n is given by

$$a_n = [1 \quad 0 \quad 0 \ \ldots \ 0]A^n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{4}$$

When A can be brought to diagonal form, (4) is easily evaluated to provide the desired formula for a_n.

As many readers have doubtless recognized, A is the companion matrix for the polynomial

$$p(t) = t^k - c_{k-1}t^{k-1} - c_{k-2}t^{k-2} - \cdots - c_0. \tag{5}$$

In consequence, $p(t)$ is both the characteristic and minimal polynomial for A, and A can be diagonalized precisely when p has k distinct roots. In this case we have

$$p(t) = (t - r_1)(t - r_2) \ldots (t - r_k) \tag{6}$$

and the numbers $r_1, r_2, \ldots, r_k$ are the eigenvalues of A.

To determine an eigenvector for A corresponding to the eigenvalue r_i we consider the system

$$(A - r_iI)X = 0. \tag{7}$$

As there are k eigenvalues, each must have geometric multiplicity one, and so the rank of $(A - r_iI)$ is $k - 1$. The general solution to (7) is readily preceived as

$$X = x_1 \begin{bmatrix} 1 \\ r_i \\ r_i^2 \\ \vdots \\ r_i^{k-1} \end{bmatrix}$$

where x_1 may be any scalar. For convenience, we take $x_1 = 1$.

Following the conventional procedure for diagonalizing A, we invoke the factorization

$$A = SDS^{-1},$$

where S is a matrix with eigenvectors of A for columns and D is a diagonal matrix. Interestingly, the previous discussion shows that for a polynomial p with distinct roots $r_1, r_2, \ldots, r_k$, the companion matrix A can be diagonalized by choosing S to be the Vandermonde array

$$V(r_1, r_2, \ldots, r_k) = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ r_1 & r_2 & r_3 & \ldots & r_k \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_k^2 \\ & & \vdots & & \\ r_1^{k-1} & r_2^{k-1} & r_3^{k-1} & \ldots & r_k^{k-1} \end{bmatrix}.$$

Related results have been previously discussed in Jarden [3].

To make use of the diagonal form, we substitute for A in (4) and derive the following:

$$a_n = [1 \quad 0 \quad 0 \ \ldots \ 0]V(r_1, r_2, \ldots, r_k)D^nV^{-1}(r_1, r_2, \ldots, r_k)\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}.$$

Noting that the product of the first three matrices at right is $[r_1^n \quad r_2^n \ \ldots \ r_k^n]$, we represent the product of the remaining matrices by

$$\begin{bmatrix}y_1\\y_2\\\vdots\\y_k\end{bmatrix}$$

and a much simpler formula for a_n results:

$$a_n = \sum_{i=1}^{k} r_i^n y_i. \tag{8}$$

Now, to determine the values $y_1, \ldots, y_k$, we solve

$$V(r_1, r_2, \ldots, r_k)\begin{bmatrix}y_1\\y_2\\\vdots\\y_k\end{bmatrix} = \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}.$$

By Cramer's rule, y_m is given by the ratio of two determinants. In the numerator, after expanding by minors in column m, the result is

$$(-1)^{m+k}\det V(r_1, \ldots, r_{m-1}, r_{m+1}, \ldots, r_k),$$

while the denominator is $\det V(r_1, \ldots, r_k)$. Thus, the ratio simplifies to

$$y_m = \frac{(-1)^{m+k}}{(-1)^{k-m}\prod_{i \neq m}(r_m - r_i)}.$$

The final form of the formula is derived by utilizing the notation of (6) and recognizing the last product above as $p'(r_m)$. Substitution in (8), and elimination of the factors of (-1) complete the computations and produce a simple formula for a_n:

$$a_n = \sum_{i=1}^{k} \frac{r_i^n}{p'(r_i)} \tag{9}$$

We conclude with a few examples and comments that pertain to the case $k = 2$. Taking $c_0 = c_1 = 1$, the sequence $\{a_n\}$ is the Fibonacci sequence. Here

$$p(t) = t^2 - t - 1 = \left(t - \frac{1 + \sqrt{5}}{2}\right)\left(t - \frac{1 - \sqrt{5}}{2}\right)$$

and $p'(t) = 2t - 1$. By using (9), we derive the familiar formula:

$$a_n = \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{\left(\frac{1 - \sqrt{5}}{2}\right)^n}{-\sqrt{5}}.$$

Consider next the case $c_0 = c_1 = 1/2$, in which each term in the sequence is the average of the two preceding terms. Now,

$$p(t) = t^2 - \frac{1}{2}t - \frac{1}{2} = (t - 1)\left(t + \frac{1}{2}\right).$$

This time, (9) leads to

$$a_n = \frac{1}{3}\left[2 + \left(-\frac{1}{2}\right)^{n-1}\right].$$

More generally for $k = 2$, the discriminant of $p(t)$ will be $D = c_1^2 + 4c_0$ and (9) produces the formula

$$a_n = \frac{(c_1 + \sqrt{D})^n - (c_1 - \sqrt{D})^n}{2^n\sqrt{D}}.$$

If D is negative, we may express the complex number $c_1 + \sqrt{D}$ in polar form as

$$R(\cos\theta + i\sin\theta).$$

Then the formula for a_n simplifies to

$$a_n = \left(\frac{R}{2}\right)^{n-1}\frac{\sin n\theta}{\sin\theta}.$$

Thus, for example, with $c_1 = c_0 = -1$, we obtain

$$a_n = (-1)^{n-1}\frac{2}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

This sequence $\{a_n\}$ is periodic, repeating 0, 1, -1, as may be verified inductively from the original difference equation

$$a_{n+2} = -a_n - a_{n+1};\ a_0 = 0;\ a_1 = 1.$$

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EXPLICIT DESCRIPTIONS OF SOME CONTINUED FRACTIONS

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In a previous paper [1], the author proved the following theorem.

THEOREM 1: Let

$$B(u, v) = \sum_{k=0}^{v} \left(\frac{1}{u}\right)^{2^k} \qquad (u \geq 3, \text{ an integer}).$$

Then we have

(a) $B(u, 0) = [0, u]$,
$B(u, 1) = [0, u - 1, u + 1]$.

(b) Suppose $B(u, v) = [a_0, a_1, \ldots, a_n]$. Then

$$B(u, v + 1) = [a_0, a_1, \ldots, a_{n-1}, a_n + 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1].$$

Repeated application of this theorem generates the continued fraction for

$$B(u, \infty) = \sum_{k=0}^{\infty} \left(\frac{1}{u}\right)^{2^k}.$$

For example, we find

(1) $B(3, \infty) = [0, 2, 5, 3, 3, 1, 3, 5, 3, 1, 5, 3, 1, \ldots]$,
(2) $B(u, \infty) = [0, u - 1, u + 2, u, u, u - 2, u, u + 2, u, u - 2, \ldots]$.

Recently, Bergman [2] provided an explicit, nonrecursive description of the partial quotients in (1), and by implication, in (2). (This description is our Theorem 3.) The purpose of this paper is to prove Bergman's result, and to provide similar results for the continued fractions given in [3] and [4].

We start off with some terminology about "strings." By a *string*, we mean a (finite or infinite) ordered sequence of symbols. Thus, for example, we may consider the partial quotients

$$[a_0, a_1, \ldots, a_n]$$

of a continued fraction to be a string. If w and x are strings, then by wx, the *concatenation* of w with x, we mean the juxtaposition of the elements of w with those of x. By $|w|$, we mean the *length* of w, i.e., the number of symbols in w. Note that $|w|$ may be either 0 or ∞. If w is a finite string, then by w^R, the *reversal* of w, we mean the symbols of w taken in reverse order. Finally, by the symbol w^n, we mean the string

$$\underbrace{www \ldots w}_{n \text{ times}}$$

By w^0, we mean the *empty string*, denoted by $\emptyset$, with the property that $w\emptyset = \emptyset w = w$. Note that $(wx)^R = x^R w^R$, and so $(w^R)^n = (w^n)^R$.

THEOREM 2: Let A_0 and B be finite strings. Define $A_{n+1} = A_n B A_n^R$. Let the symbol A_∞ stand for the unique infinite string of which $A_0, A_1, \ldots$ are all prefixes.

Then $A_\infty = X_1 Y_1 X_2 Y_2 X_3 Y_3 \ldots$ where

(a) $X_k = \begin{cases} A_0 & \text{if } k \text{ is odd} \\ A_0^R & \text{if } k \text{ is even,} \end{cases}$

(b) $Y_k = \begin{cases} B^R & \text{if } k \in S \\ B & \text{if } k \notin S. \end{cases}$

and

$$S = \{n \geq 1\colon n = 2^i(1 + 2j),\ i,\ j \text{ integers} \geq 0 \text{ and } j \text{ is odd}\}$$
$$= \{3, 6, 7, 11, 12, 14, 15, \ldots\}.$$

To prove this result, we need a lemma.

LEMMA 1: Let A_0, A_n, and B be as in Theorem 2. Then

$$A_{n+1} = (A_0B^*A_0^RB^*)^{2^n-1}A_0B^*A_0^R$$

where by the symbol B^* we mean *either* B or B^R.

PROOF: We use induction on n. Clearly, the lemma is true for $n = 0$. Assume true for n. Then we find

$$A_{n+2} = A_{n+1}BA_{n+1}^R = (A_0B^*A_0^RB^*)^{2^n-1}A_0B^*A_0^RBA_0B^*A_0^R(B^*A_0B^*A_0^R)^{2^n-1}$$
$$= (A_0B^*A_0^RB^*)^{2^{n+1}-1}A_0B^*A_0^R$$

and the proof of the lemma is complete.

We can now prove Theorem 2. Part (a) follows immediately from the lemma. To prove part (b) we will prove, by induction on n, that the theorem is true for all $k \leq 2^n$.

Clearly, part (b) is true for $n = 0$. Assume true for all $k \leq 2^n$. Then we wish to show part (b) is true for all k such that $2^n < k \leq 2^{n+1}$.

Assume $2^n < k < 2^{n+1}$. Since $A_{n+1} = A_nBA_n^R$, we see that if $Y_k = B^R$ then $Y_{2^{n+1}-k} = B$; similarly, if $Y_k = B$ then $Y_{2^{n+1}-k} = B^R$.

We note that every positive integer can be written uniquely in the form

$$2^i(1 + 2j),$$

where i and j are nonnegative integers. Thus, it suffices to show that (for $2^n < k < 2^{n+1}$) if $k = 2^i(1 + 2j)$, then $2^{n+1} - k = 2^{i'}(1 + 2j')$, where j and j' are of opposite parity.

If $2^n < k < 2^{n+1}$, then the largest power of 2 dividing k is 2^{n-1}; hence, $0 \leq i \leq n - 1$. Therefore,

$$2^{n+1} - k = 2^{n+1} - 2^i(1 + 2j) = 2^i(2^{n+1-i} - 1 - 2j)$$
$$= 2^i(1 + 2(2^{n-i} - j - 1)) = 2^{i'}(1 + 2j').$$

But $n - i \geq 1$; hence, j and j' are indeed of opposite parity.

Finally, we must examine the case $k = 2^{n+1}$. But it is easy to see from Lemma 1 that $Y_k = B$ if k is a power of 2.

Now that we have built up some machinery, we can state and prove the explicit description of the continued fraction for $B(u, \infty)$.

THEOREM 3 (Bergman):

$$B(u, \infty) = [0, u - 1, U_1, V_1, U_2, V_2, U_3, V_3, \ldots]$$

where

$$U_k = \begin{cases} (u + 2, u) & \text{if } k \text{ is odd} \\ (u, u + 2) & \text{if } k \text{ is even,} \end{cases}$$

$$V_k = \begin{cases} (u, u - 2) & \text{if } k \notin S \\ (u - 2, u) & \text{if } k \in S. \end{cases}$$

S is as in Theorem 2.

Bergman's result follows immediately from Theorem 2 and the following lemma.

LEMMA 2: Let $A_0 = (u + 2, u)$; $B = (u, u - 2)$; let A_n and A_∞ be as in Theorem 2. Then

(a) $B(u, v + 2) = [0, u - 1, A_v, u - 1]$,

(b) $B(u, \infty) = [0, u - 1, A_\infty]$.

PROOF: To prove part (a), we use induction on v. From Theorem 1, we have

$$B(u, 2) = [0, u - 1, u + 2, u, u - 1] = [0, u - 1, A_0, u - 1].$$

Hence, the lemma is true for $v = 0$.

Now assume true for v. We have

$$B(u, v + 2) = [0, u - 1, A_v, u - 1].$$

But by Theorem 1,

$$\begin{aligned} B(u, v + 3) &= [0, u - 1, A_v, u, u - 2, A_v^R, u - 1] \\ B(u, v + 3) &= [0, u - 1, A_{v+1}, u - 1]. \end{aligned} \tag{3}$$

This proves part (a) of the lemma. To prove part (b), we simply let v approach ∞ in both sides of (3).

Note: Lemma 2 was independently discovered by M. Kmošek [5].

These results provide a different proof of the fact, proved in [1], that $B(u, \infty)$ is not a quadratic irrational. This is an implication of the following more general result.

THEOREM 4: Let A_0, B, A_n, and A_∞ be as in Theorem 2, A_0 and B not both empty. Then A_∞ is eventually periodic if and only if B is a palindrome.

PROOF: We say the infinite string w is *eventually periodic* if and only if $w = xy^\infty$ where, by the symbol y^∞, we mean the infinite string $yyyy\ldots$. The string y is called the *repeating portion*, or the *period*.

Suppose B is a palindrome. Then by Lemma 1,

$$A_{n+1} = (A_0B^*A_0^RB^*)^{2^n-1}A_0B^*A_0^R.$$

But $B = B^R$; so B^* always equals B. Hence,

$$A_\infty = (A_0BA_0^RB)^\infty.$$

Now assume A_∞ is eventually periodic, i.e., $A_\infty = xy^\infty$. Since

$$|A_n| = 2^n(|A_0| + |B|) - |B|,$$

we may choose n such that $|x| \le |A_n|$. Then since $A_nBA_n^R$ is a prefix of A_∞, we may assume (by renaming x and y, if necessary) that $x = A_n$.

Now let $z = y^{|A_n| + |B|}$. Clearly, $A_\infty = xy^\infty = xz^\infty$. If y is a repeating portion, then so is z. The string z consists of groups of $B^*A_n^*$'s; hence, if we can show that groups of $B^*A_n^*$'s repeat only if $B = B^R$, we will be done.

By renaming "A_n" to be "A_0," we may use the result given in Theorem 2 to describe the positions of the B's in A_∞. We will show that for all integers $i \ge 1$, there exist $c_i \in S$ and $d_i \notin S$ such that $c_i - d_i = i$. This shows that in A_∞ there exists a B and a B^R exactly $|z|$ symbols apart; hence, if z really is a repeating portion, we must have $B = B^R$.

Let i be written in base 2 as a string of ones and zeros. Then, clearly, for some $m \ge 0$, $n \ge 1$, this expression has the form

$$z\ 0\ 1^n\ 0^m,$$

where z is an arbitrary string of ones and zeros.

Let c_i be the number represented by the binary string $z\ 1\ 1^n\ 0^m$ and let d_i be the number represented by $1\ 0^n\ 0^m$. Then, clearly, $c_i - d_i = i$, and it is easily verified that $c_i \in S$ and $d_i \notin S$.

Thus, $B = B^R$ and Theorem 4 is proved.

Note: Theorem 4 was stated without proof in [6].

COROLLARY 1: $B(u, \infty)$ is not a quadratic irrational.

PROOF: From Lemma 2, we have $B(u, \infty) = [0, u - 1, A_\infty]$, where $A_0 = (u + 2, u)$ and $B = (u, u - 2)$. Since $B \neq B^R$, A_∞ cannot be eventually periodic. Hence, by a well-known theorem (see Hardy & Wright [7]), $B(u, \infty)$ is not a quadratic irrational.

COROLLARY 2: Suppose each element x of the strings A_0 and B satisfies $0 \leq x < b$, where b is an integer ≥ 2. Then we may consider A_∞ to be the base b representation of a number between 0 and 1. Then Theorem 4 implies that this number is irrational if and only if $B = B^R$.

As the last result of this paper, we state a theorem giving a description similar to that in Theorem 3 for another type of continued fraction.

In [3] and [4], the following result is proved.

THEOREM 5: Let $\{c(k)\}_{k=0}^{\infty}$ be a sequence of positive integers such that $c(v + 1) \geq 2c(v)$ for all $v \geq v'$. Let $d(v) = c(v + 1) - 2c(v)$. Define $S(u, v)$ as follows:

$$S(u, v) = \sum_{k=0}^{v} u^{-c(k)} \qquad (u \geq 2, \text{ an integer}).$$

Then, if $v \geq v'$ and $S(u, v) = [a_0, a_1, \ldots, a_n]$ and n is even,

$$S(u, v + 1) = [a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1].$$

It is possible to use the techniques above to get an explicit description of the continued fraction for $S(u, \infty)$ similar to that for $B(u, \infty)$. This description is somewhat more complicated due to the extra terms given in Theorem 5. If we assume that $v' = 0$, $c(v + 1) > 2c(v)$ and $u \geq 3$, then the description becomes somewhat more manageable.

THEOREM 6: Let $S(u, \infty) = \lim_{v \to \infty} S(u, v)$. Let us write $n = 2^{i_n}(1 + 2j_n)$ where i_n and j_n are nonnegative integers; however, put $j_n = -1$ for $n = 0$. Define $p(n)$, the *parity* of an integer n as 0 if n is even, and 1 if n is odd. Then under the simplifying assumptions of the previous paragraph,

$$S(u, \infty) = [0, A_0, B_1, C_1, B_2, A_1, B_3, C_2, B_4, A_2, B_5, \ldots],$$

where

$$A_n = (u^{c(0)} + p(j_n) - 2, 1, u^{d(0)} - 1, u^{c(0)} - p(n))$$

$$C_n = (u^{c(0)} - p(n), \quad u^{d(0)} - 1, 1, u^{c(0)} - 1 - p(j_n))$$

$$B_n = \begin{cases} (u^{d(1+i_n)} - 1, 1) \text{ if } j_n \text{ is even,} \\ (1, u^{d(1+i_n)} - 1) \text{ if } j_n \text{ is odd.} \end{cases}$$

PROOF: The proof is a straightforward (though tedious) application of previous techniques, and is omitted here.

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THE NONEXISTENCE OF QUASIPERFECT NUMBERS OF CERTAIN FORMS

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1. INTRODUCTION

A natural number n is called perfect, multiperfect, or quasiperfect according as $\sigma(n) = 2n$, $\sigma(n) = kn$ ($k \geq 2$, an integer), or $\sigma(n) = 2n+1$, respectively, where $\sigma(n)$ is the sum of the positive divisors of n.

No odd multiperfect numbers are known. In many papers concerned with odd perfect numbers (summarized in McDaniel & Hagis [5]), values have been obtained which cannot be taken by the even exponents on the prime factors of such numbers, if all those exponents are equal. McDaniel [4] has given results of a similar nature for odd multiperfect numbers.

No quasiperfect numbers have been found. It is known [Cattaneo [1]) that if there are any they must be odd perfect squares, and it has recently been shown by Hagis & Cohen [3] that such a number must have at least seven distinct prime factors and must exceed 10^{35}. In this paper we shall give results analogous to those described for odd multiperfect numbers, but with extra generality. In particular, we shall show that no perfect fourth power is quasiperfect, and no perfect sixth power, prime to 3, is quasiperfect. We are unable to prove the nonexistence of quasiperfect numbers of the form m^2, where m is squarefree, but will show that any such numbers must have more than 230,000 distinct prime factors, so the chance of finding any is slight!

All italicized letters here denote nonnegative integers, with p and q primes, $p > 2$.

2. SOME LEMMAS

The following result is due to Cattaneo [1].

LEMMA 1: If n is quasiperfect and $r|\sigma(n)$, then $r \equiv 1$ or $3 \pmod 8$.

We shall need

LEMMA 2: Suppose n is quasiperfect and $p^{2a} \| n$. If $q|2a+1$, then

$$(q-1)(p+1) \equiv 0 \text{ or } 4 \pmod{16}.$$

PROOF: Notice first that if b is odd, then, modulo 8,

$$\sigma(p^{b-1}) = 1 + p + p^2 + \cdots + p^{b-1} \equiv 1 + (p+1) + \cdots + (p+1) = 1 + \tfrac{1}{2}(b-1)(p+1). \tag{1}$$

Let $F_d(\xi)$ denote the cyclotomic polynomial of order d. It is well known that

$$\xi^m - 1 = \prod_{d|m} F_d(\xi) \qquad (m > 0),$$

so

$$\sigma(p^{2\ }) = \prod_{\substack{d|2a+1 \\ d>1}} F_d(p). \tag{2}$$

Hence $\sigma(p^{q-1}) = F_q(p)\,|\,\sigma(p^{2a})\,|\,\sigma(n)$, since $\sigma(n)$ is multiplicative. From (1) and Lemma 1, $1 + \frac{1}{2}(q - 1)(p + 1) \equiv 1$ or $3 \pmod 8$ and the result follows.

LEMMA 3: If n is quasiperfect and $p^{2a}\|n$, where $a \equiv 1 \pmod 3$, then $p \not\equiv 3$ or $5 \pmod 8$ and $p \not\equiv b$ or $c \pmod q$ for b, c, q in Table 1.

Table 1

b	c	q	b	c	q	b	c	q	b	c	q
2	4	7	48	132	181	171	267	439	72	678	751
3	9	13	92	106	199	21	441	463	27	729	757
5	25	31	39	183	223	232	254	487	174	648	823
10	26	37	94	134	229	129	411	541	125	703	829
13	47	61	28	242	271	210	396	607	220	632	853
23	55	79	116	160	277	65	547	613	282	594	877
46	56	103	122	226	349	43	587	631	52	866	919
45	63	109	83	283	367	296	364	661	142	824	967
19	107	127	88	284	373	227	481	709	113	877	991
32	118	151	34	362	397	281	445	727	304	692	997
12	144	157	20	400	421	307	425	733			

PROOF: Since $a \equiv 1 \pmod 3$, we take $q = 3$ in Lemma 2 to see that $p \equiv 1$ or $7 \pmod 8$. If $p \equiv b$ or $c \pmod q$, for any triple (b, c, q) in Table 1, then

$$\sigma(p^2) \equiv 0 \pmod q.$$

From (2), we have

$$\sigma(p^{2a}) = F_3(p) \prod_{\substack{d|2a+1 \\ d \geq 5}} F_d(p),$$

so

$$q\,|\,\sigma(p^2) = F_3(p)\,|\,\alpha(p^{2a})\,|\,\sigma(n).$$

But $q \equiv 5$ or $7 \pmod 8$, so Lemma 1 is contradicted. Hence $p \not\equiv b$ or $c \pmod q$.

Note: The primes q in Table 1 are all primes less than 1000 that are congruent to 5 or 7 (mod 8) and to 1 (mod 3), and b and $c \equiv b^2 \pmod q$ are the positive integers belonging to the exponent 3 (mod q). Lemma 3 provides a useful screening of primes p such that p^2 [or p^{2a} where $a \equiv 1 \pmod 3$] can exactly divide a quasiperfect number: the three smallest primes p such that $\sigma(p^2)$ has a divisor congruent to 5 or 7 (mod 8) and not eliminated by Lemma 3 are 2351, 3767, and 5431.

3. THE THEOREMS

THEOREM 1: (i) No number of the form m^4 is quasiperfect.

(ii) No number of the form m^6, where $(m, 3) = 1$, is quasiperfect.

PROOF: (i) Suppose

$$n = \prod_{i=1}^{t} p_i^{4a_i}$$

and that n is quasiperfect. It is easy to see that $\sigma(p_i^{4a_i}) \equiv 1 \pmod 4$ so, by Lemma 1, $\sigma(p_i^{4a_i}) \equiv 1 \pmod 8$ and hence

$$\sigma(n) = \prod_{i=1}^{t} \sigma(p_i^{4a_i}) \equiv 1 \pmod 8.$$

But n is an odd square, so $\sigma(n) = 2n + 1 \equiv 3 \pmod 8$, a contradiction.

(ii) Suppose

$$n = \prod_{i=1}^{t} p_i^{6a_i},\ p_i \geq 5,$$

and that n is quasiperfect. We have $\sigma(p_i^{6a_i}) \equiv 1 \pmod 3$, so

$$\sigma(n) = \prod_{i=1}^{t} \sigma(p_i^{6a_i}) \equiv 1 \pmod 3.$$

However, since n is a square and $3 \nmid n$, we have $\sigma(n) = 2n + 1 \equiv 0 \pmod 3$, another contradiction.

THEOREM 2: If a number of the form $\prod_{i=1}^{t} p_i^{24a_i+2b}$ is quasiperfect, then $b = 1$, 5, or 11, $p_i \not\equiv 3 \pmod 8$ for any i, and $t \geq 10$.

PROOF: Suppose

$$n = \prod_{i=1}^{t} p_i^{24a_i+2b}$$

is quasiperfect. From Theorem 1(i), $b \neq 0$, 2, 4, 6, 8, or 10. It then follows, using (1), that $p_i \not\equiv 3 \pmod 8$ for any i (or see Cattaneo [1]). In particular, $3 \nmid n$, so from Theorem 1(ii), $b \neq 3$ or 9. Suppose $b = 7$. Then by Lemma 3 we have $p_i \equiv 1$ or $7 \pmod 8$ for all i. If $p_i \equiv 1 \pmod 8$ for some i, then

$$\sigma(p_i^{24a_i+14}) \equiv 24a_i + 15 \equiv 7 \pmod 8,$$

and this contradicts Lemma 1. Hence, for all i, $p_i \equiv 7 \pmod 8$, so

$$\sigma(p_i^{24a_i+14}) \equiv 1 \pmod 8 \quad \text{and} \quad \sigma(n) \equiv 1 \pmod 8.$$

As in the proof of Theorem 1(i), this is a contradiction. Thus $b \neq 7$. Since $p_i \not\equiv 3 \pmod 8$ for any i, we have, finally, if $t \leq 9$,

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{t} \frac{p_i}{p_i - 1} \leq \frac{5}{4}\,\frac{7}{6}\,\frac{13}{12}\,\frac{17}{16}\,\frac{23}{22}\,\frac{29}{28}\,\frac{31}{30}\,\frac{37}{36}\,\frac{41}{40} < 2.$$

This contradicts the fact that $\sigma(n)/n = 2 + 1/n > 2$. Hence $t \geq 10$.

We are unable to establish in particular that there are no quasiperfect numbers of the form

$$\prod_{i=1}^{t} p_i^{2b} \text{ for } b = 1,\ 5, \text{ or } 11.$$

The next theorem includes information on the case $b = 1$, and the final theorem is related to the cases $b = 5$ and $b = 11$.

THEOREM 3: If a number of the form $\prod_{i=1}^{t} p_i^{6a_i+2}$ is quasiperfect, then $t \geq 230876$.

PROOF: Let q_i be the ith prime ($q_1 = 2$), and let Π' denote a product over primes not congruent to 3 or 5 (mod 8) or to b or c (mod q), with b, c, q as in Lemma 3. The 5000th such prime is $P = 309769 = q_{26775}$ and the 225876th prime greater than P is $Q = 3538411$. We have computed that

$$\prod_{q_i \leq P}{}' \frac{q_i}{q_i - 1} < 1.6768 = \alpha, \text{ say.}$$

Suppose $n = \prod_{i=1}^{t} p^{6a_i+2}$ is quasiperfect and that $t < s = 230876$. From Lemma 3, $p_i \not\equiv$ 3 or 5 (mod 8) and $p_i \not\equiv b$ or c (mod q) for any i and any triple (b, c, q) in Lemma 3. Thus

$$\frac{\sigma(n)}{n} \le \prod_{q_i \le P}' \frac{\sigma(q_i^{6a_i+2})}{q_i^{6a_i+2}} \prod_{i=5001}^{t} \frac{\sigma(p_i^{6a_i+2})}{p_i^{6a_i+2}} < \prod_{q_i \le P}' \frac{q_i}{q_i - 1} \prod_{i=26776}^{s+21775} \frac{q_i}{q_i - 1} < \alpha \prod_{P < q \le Q} \frac{q}{q-1}$$

$$= \alpha \prod_{q \le Q} \frac{q}{q-1} \prod_{q \le P} \frac{q-1}{q} < \frac{\alpha}{\log P} \log\left(Q + \frac{2}{\sqrt{Q}}\right) < 2,$$

using Theorem 23 in Rosser & Schoenfeld [6]. But n is quasiperfect, so $\sigma(n)/n > 2$, and we have a contradiction. Hence $t \ge s$.

THEOREM 4: No number of the form $3^{2a}m^{2b}$, where $3 \nmid m$, $a \equiv 2 \pmod 5$, and either $b \equiv 0 \pmod 5$ or $b \equiv 0 \pmod{11}$ is quasiperfect.

PROOF: Suppose $n = 3^{2a}m^{2b}$ $(3 \nmid m)$, with a, b as given, is quasiperfect. Since

$$\sigma(3^{2a}) = \prod_{\substack{d \mid 2a+1 \\ d > 1}} F_d(3) = F_5(3) \prod_{\substack{d \mid 2a+1 \\ d > 1,\, d \ne 5}} F_d(3),$$

we have $11^2 = F_5(3) \mid \sigma(3^{2a}) \mid \sigma(n)$. Since $3^{10} \equiv 1 \pmod{121}$,

$$\sigma(n) = 2n + 1 \equiv 2 \cdot 3^4 m^{2b} + 1 \equiv 0 \pmod{121}.$$

From this, $m^{2b} \equiv 59 \pmod{121}$, and thus

$$m^b \equiv 46 \text{ or } 75 \pmod{121}. \tag{3}$$

For each possible value of $c = \phi(121)/(b, \phi(121))$ it is not the case that $46^c \equiv 1 \pmod{121}$ or $75^c \equiv 1 \pmod{121}$ (ϕ is Euler's function). Euler's criterion for the existence of power residues (Griffin [2, p. 129]) shows the congruences (3) to be insolvable. This contradiction proves the theorem.

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REPRESENTATIONS OF EVERY INTEGER AS THE DIFFERENCE OF POWERFUL NUMBERS

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(Submitted May 1981)

1. INTRODUCTION

A powerful number has been defined by Golomb [3] to be a positive integer with the property that whenever the prime p divides r, p^2 divides r. In this paper, we show that every nonzero integer can be written as the difference of two relatively prime powerful numbers in infinitely many ways.

Let $P(m_1, m_2) = m_1 - m_2$, where m_1 and m_2 are powerful numbers. $k = m_1 - m_2$ is said to be a proper representation of k by P if $(m_1, m_2) = 1$, and an improper representation if $(m_1, m_2) > 1$.

It has been shown that there exist infinitely many proper representations of k by P if $k = 1$ or 4 [3], if $k = 2$ [6], or if k is a prime congruent to 1 modulo 8 [4]. It is also known [3] that there is at least one proper representation of each odd integer and each multiple of 8 by P. Golomb has conjectured that there are infinitely many integers that cannot be written as the difference of two powerful numbers. Our principal result in this paper disproves this conjecture, showing that, in fact, every integer $\neq 0$ has infinitely many representations as the difference of relatively prime powerful numbers.

2. THE DIOPHANTINE EQUATION $x^2 - Dy^2 = n$

Our approach involves showing that corresponding to a given positive integer $n \not\equiv 2 \pmod 4$ there exists an integer D such that

$$x^2 - Dy^2 = n \tag{1}$$

has an infinitude of solutions x, y for which $D|y$.

(1) has been extensively studied (see [5] or [7]), and it is well known that if p, q is a solution of (1), where D is not a square, and u, v is a solution of the Pell equation $x^2 - Dy^2 = 1$, then $pu + Dqv$, $pv + qu$ is also a solution of (1). It follows that (1) has an infinitude of solutions when one solution exists, since the Pell equation has infinitely many solutions: if u, v is a solution of $x^2 - Dy^2 = 1$, then so is x_j, y_j, where

$$x_j + y_j\sqrt{D} = (u + v\sqrt{D})^j, \quad j = 1, 2, 3, \ldots,$$

that is, where

$$x_j = u^j + \sum_{k=2}^{j} \binom{j}{k} u^{j-k} v^k D^{k/2}, \tag{2}$$

and

$$y_j = ju^{j-1}v + \sum_{k=3}^{j} \binom{j}{k} u^{j-k} v^k D^{(k-1)/2}. \tag{3}$$

[The index ranges over even values of k in (2) and odd values in (3).]

We will find it convenient to make the following definition.

DEFINITION: If p, q is a solution of (1) and u, v is a solution of $x^2 - Dy^2 = 1$, $pu + Dqv$, $pv + qu$ is a Type A solution of (1) if u is odd, v is even, and $D|pv + qu$.

THEOREM 1: Let p, q be a solution of (1) and x_0, y_0 be a solution of $x^2 - Dy^2 = \pm 1$. Then (1) has infinitely many Type A solutions if $d = (2py_0, D)$ implies $d|q$.

PROOF: Let $x_1 = x_0^2 + Dy_0^2$ and $y_1 = 2x_0y_0$. Then $x_1^2 - Dy_1^2 = 1$, y_1 is even, and x_1 is odd. Replacing u by x_1 and v by y_1 in (2) and (3) yields solutions

$$X_j = px_j + Dqy_j,\ Y_j = py_j + qx_j$$

of (1) with x_j odd and y_j even for $j = 1, 2, 3, \ldots$. Now, $D|Y_j$ if

$$0 \equiv Y_j \equiv py_j + qx_j \equiv pjx_1^{j-1}y_1 + qx_1^j \equiv x_1^{j-1}(py_1j + qx_1) \pmod{D}.$$

Since $x_1^2 - Dy_1^2 = 1$, $(x_1, D) = 1$, so $D|Y_j$ if

$$0 \equiv (py_1)j + qx_1 \equiv (2x_0y_0p)j + q(x_0^2 + Dy_0^2) \equiv (2x_0y_0p)j + qx_0^2 \pmod{D}.$$

Solutions of this linear congruence exist if and only if $(2x_0y_0p, D)$ divides qx_0^2. Since $x_0^2 - Dy_0^2 = \pm 1$ implies $(x_0, D) = 1$, it follows that if $(2y_0p, D)$ divides q, $j \equiv b \pmod{D}$ for some integer b and X_{b+tD}, Y_{b+tD} is a Type A solution of (1) for $t = 1, 2, 3, \ldots$.

We observe at this point that if u, v is a solution of $x^2 - Dy^2 = 1$, and p and q are relatively prime integers, then $pu + Dqv$ and $pv + qu$ are relatively prime integers, for if $d = (pu + Dqv, pv + qu)$, then d divides $u(pv + qu) - v(pu + Dqv) = q$ and d divides $u(pu + Dqv) - vD(pv + qu) = p$, which implies that $d = 1$.

THEOREM 2: If $n \equiv -1$, 0, or 1 (mod 4), there exists an odd integer D such that $x^2 - Dy^2 = n$ has infinitely many relatively prime Type A solutions.

PROOF: The proof involves making a judicious choice for D in each of the three cases. In each case, we identify a solution p, q of (1) and a solution x_0, y_0 of $x^2 - Dy^2 = \pm 1$. D is odd in each of the three cases and is clearly not a square; it is then shown that $(py_0, D) = 1$, assuring, by Theorem 1, that (1) has infinitely many Type A solutions, and that $(p, q) = 1$, making the solutions relatively prime.

Case 1. $n = 4k - 1$, $k = 1, 2, 3, \ldots$. We choose $D = 16k^2 - 8k + 5$. If p, q, x_0, and y_0 are chosen, respectively, to be $8k^2 - 6k + 2$, $2k - 1$, $32k^3 - 24k^2 + 12k - 2$, and $8k^2 - 4k + 1$, then $p^2 - Dq^2 = 4k - 1$, and $x_0^2 - Dy_0^2 = -1$. Let $d_0 = (p, D)$. We find that d_0 divides $4(D - 2p) - [(D - 2p)^2 - D] = 8$, so $d_0 = 1$. Let $d_1 = (y_0, D)$. Since d_1 divides $D - 2y_0 = 3$, and $D \not\equiv 0 \pmod 3$ for any k, $d_1 = 1$. Let $d_2 = (p, q)$. Since d_2 divides $(4k - 1)q - p = -1$, $d_2 = 1$.

Case 2. $n = 4k + 1$, $k = 2, 3, 4, \ldots$. We choose $D = 4k^2 - 4k - 1$. If p, q, x_0, and y_0 are chosen, respectively, to be $2k$, 1, $4k^2 - 4k$, and $2k - 1$, then $p^2 - Dq^2 = 4k + 1$ and $x_0^2 - Dy_0^2 = 1$. Let $d_0 = (p, D)$. Since d_0 divides $p^2 - 2p - D = 1$, $d_0 - 1$. (y_0, D) and (p, q) are obviously equal to 1.

Because $D = 4k^2 - 4k - 1$ is negative when $k = 0$ or 1, we treat $n = 1$ and $n = 5$ separately, by considering $x^2 - 3y^2 = 1$ and $x^2 - 11y^2 = 5$. $x^2 - 3y^2 = 1$ is satisfied by p, q and x_0, y_0 if $p = x_0 = 2$ and $q = y_0 = 1$. If $p = 4$, $q = 1$, $x_0 = 10$, and $y_0 = 3$, then $p^2 - 11q^2 = 5$ and $x_0^2 - 11y_0^2 = 1$. Clearly, in both cases, (py_0, D) and (p, q) equal 1.

Case 3. $n = 4k$, $k = 1, 2, 3, \ldots$. We choose $D = 4k^2 + 1$. If p, q, x_0, and y_0 are chosen, respectively, to be $2k + 1$, 1, $2k$, and 1, then $p^2 - Dq^2 = 4k$ and $x_0^2 - Dy_0^2 = -1$. Let $d = (py_0, D)$. Since d divides $D - (2k - 1)py_0 = 2$, $d = 1$. Obviously, $(p, q) = 1$.

Since the proof gives no clue as to how the polynomials D were found, it might be helpful to mention that they were discovered, essentially, as a result of a process which began in an examination of the continued fraction expansion of $\sqrt{m}$, where m is a polynomial whose continued fraction has a relatively small period (≤ 10). The interested reader might consult Chrystal's *Algebra* [2] and the paper by Boutin [1].

3. APPLICATION TO POWERFUL NUMBERS

THEOREM 3: If n is any integer $\neq 0$, there exist infinitely many relatively prime pairs m_1 and m_2 of powerful numbers such that $n = m_1 - m_2$.

PROOF: If X_j, Y_j is a Type A solution of $x^2 - Dy^2 = n$, $n \neq 2 \pmod 4$, then $m_1 = X_j^2$ and $m_2 = DY_j^2$ are powerful numbers whose difference is n. Since in each of the three cases of Theorem 2 p and q were shown to be relatively prime, X_j and Y_j and, hence, m_1 and m_2 are relatively prime.

If $n \equiv 2 \pmod 4$, we let $n = 2 + 4t$ and consider the equation of Case 2 of Theorem 2: $x^2 - Dy^2 = 4k + 1$. Since $n^2/4 = 4(t^2 + t) + 1$, there exist infinitely many relatively prime Type A solutions X_j, Y_j of $x^2 - Dy^2 = n^2/4$, where $D = 3$, if $t = 0$, and

$$D = 4(t^2 + t)^2 - 4(t^2 + t) - 1, \text{ if } t \geq 1.$$

Let $m_1 = X_j + n/2$ and $m_2 = X_j - n/2$. We observe that since, for all k in Case 2, p is even and q is odd, and since X_j, Y_j is a Type A solution, X_j is even and Y_j is odd. Thus m_1 and m_2 are odd. It follows immediately that $(m_1, m_2) = 1$: any common divisor of m_1 and m_2 must be odd and must divide $m_1 + m_2 = 2X_j$ and $m_1 - m_2 = n$, but $(X_j, Y_j) = 1$ and $X_j^2 - DY_j^2 = n^2/4$ imply that $(X_j, n) = 1$. Since $m_1m_2 = DY_j^2$ is a powerful number, so is each of m_1 and m_2. Hence $n = m_1 - m_2$ is the difference of two relatively prime powerful numbers.

The theorem is obviously true when n is negative, since $n = m_1 - m_2$ implies $-n = m_2 - m_1$.

COROLLARY: Let S denote the set of all squarefree integers and n be any integer. n has infinitely many improper representations by P if $n \notin S$. If $n \in S$, n has no improper representations by P.

PROOF: Assume $n \notin S$. If $n = 0$, the result is obvious. If $n \neq 0$, there exists a prime p and an integer $m \neq 0$ such that $n = mp^2$. By Theorem 3, there exist infinitely many pairs of powerful numbers m_1 and m_2 such that $m = m_1 - m_2$. Then, $n = m_1p^2 - m_2p^2$, the difference of two powerful numbers. Conversely, if n has an improper representation by P, then n is divisible by the square of an integer and is not in S.

Example 1. $9 = m_1 - m_2$. The equation $x^2 - 7y^2 = 9$ has Type A solutions

$$X_{2+7t},\ Y_{2+7t}.$$

For $t = 0$, we obtain

$$m_1 = X_2^2 = (214372)^2 = 2^4 \cdot 53593^2 \quad \text{and} \quad m_2 = 7Y_2^2 = 7(81025)^2 = 5^4 \cdot 7^3 \cdot 463^2.$$

Example 2. $6 = m_1 - m_2$. Since $6 \equiv 2 \pmod 4$ and $6^2/4 = 9$, we again use $x^2 - 7y^2 = 9$. Letting $m_1 = X_2 + 3 = 214375 = 5^4 \cdot 7^3$ and $m_2 = X_2 - 3 = 214369 = 463^2$, we have

$$5^4 \cdot 7^3 - 463^2 = 6.$$

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ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of

$$x^2 - x - 1 = 0.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-466 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $A_n = 1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \cdots + (-1)^{n-1}n(n+1)$.

(a) Determine the values of n for which $2A_n$ is a perfect square.
(b) Determine the values of n for which $|A_n|/2$ is the product of two consecutive positive integers.

B-467 *Proposed by Herta T. Freitag, Roanoke, VA*

Let A_n be as in B-466 and let $B_n = \sum_{i=1}^{n}\sum_{k=1}^{i} k$. For which positive integers n is $|A_n|$ an integral divisor of B_n?

B-468 *Proposed by Miha'ly Bencze, Brasov, Romania*

Find a closed form for the nth term a_n of the sequence for which a_1 and a_2 are arbitrary real numbers in the open interval (0, 1) and

$$a_{n+2} = a_{n+1}\sqrt{1 - a_n^2} + a_n\sqrt{1 - a_{n+1}^2}.$$

The formula for a_n should involve Fibonacci numbers if possible.

B-469 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

Describe the appearance in base F_n notation of:

(a) $1/F_{n-1}$ for $n \geq 5$; (b) $1/F_{n+1}$ for $n \geq 3$.

B-470 *Proposed by Larry Taylor, Rego Park, NY*

Find positive integers a, b, c, r, and s and choose each of G_n, H_n, I_n to be F_n or L_n so that aG_n, bH_{n+r}, cI_{n+s} are in arithmetic progression for $n \geq 0$ and this progression is 6, 6, 6 for some n.

B-471 Proposed by Larry Taylor, Rego Park, NY

Do there exist positive integers d and t such that aG_n, bH_{n+r}, cI_{n+s}, dJ_{n+t} are in arithmetic progression, with J_n equal to F_n or L_n and everything else as in B-470?

SOLUTIONS

Lucas Analogue of Cosine Identity

B-442 Proposed by P. L. Mana, Albuquerque, NM

The identity $2\cos^2\theta = 1 + \cos(2\theta)$ leads to the identity

$$8\cos^4\theta = 3 + 4\cos(2\theta) + \cos(4\theta).$$

Are there corresponding identities on Lucas numbers?

Solution by Sahib Singh, Clarion State College, Clarion, PA

Yes; $L_{2n} = a^{2n} + b^{2n} = (a^n + b^n)^2 - 2(-1)^n = L_n^2 - 2(-1)^n$. Hence

(1) $$L_n^2 = L_{2n} + 2(-1)^n.$$

Using (1), $L_{2n}^2 = L_{4n} + 2$. Again using (1), the above equation reduces to

$$(L_n^2 - 2(-1)^n)^2 = L_{4n} + 2,$$

which yields

(2) $$L_n^4 = 6 + 4(-1)^n L_{2n} + L_{4n}.$$

Equations (1) and (2) are the required identities.

Also solved by Paul S. Bruckman, Paul F. Byrd, Herta T. Freitag, Calvin L. Gardner, Bob Prielipp, M. Wachtel, Gregory Wulczyn, and the proposer.

Lucas Products Identity

B-443 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

For all integers n and w with w odd, establish the following:

$$L_{n+2w}L_{n+w} - 2L_wL_{n+w}L_{n-w} - L_{n-w}L_{n-2w} = L_n^2(L_{3w} - 2L_w).$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

The given equation is equivalent to:

$$L_{n+2w}L_{n+w} - L_{n-w}L_{n-2w} - L_n^2L_{3w} = 2L_w(L_{n+w}L_{n-w} - L_n^2).$$

Using the identity $L_{n+w}L_{n-w} - L_n^2 = 5(-1)^{n+w}F_w^2$, the above equation becomes:

(1) $$L_{n+2w}L_{n+w} - L_{n-w}L_{n-2w} - L_n^2L_{3w} = 10(-1)^{n+w}L_wF_w^2.$$

Using $L_n = a^n + b^n$, the left side of (1) becomes

$$2(-1)^{n+w}(L_w + L_{3w}).$$

Thus (1) reduces to $L_w + L_{3w} = 5L_wF_w^2$. Since w is odd, by using $L_n = a^n + b^n$, the above equation is true and we are done.

Also solved by Paul S. Bruckman, Herta T Freitag, Calvin L. Gardner, Bob Prielipp, M. Wachtel, and the proposer.

Generating Palindromes

B-444 Proposed by Herta T. Freitag, Roanoke, VA

In base 10, the palindromes (i.e., numbers reading the same forward or backward) 12321 and 112232211 are converted into new palindromes using

$$99[10^3 + 9(12321)] = 11077011 \quad \text{and} \quad 99[10^5 + 9(112232211)] = 100008800001.$$

Generalize on these to obtain a method or methods for converting certain palindromes in a general base b to other palindromes in base b.

Solution by Paul S. Bruckman, Concord CA

Let $\mathcal{P}_b$ denote the set of palindromes in base b. We will prove the following theorem.

THEOREM: If $m \geq 1$, let $P \in \mathcal{P}_b$ be given by

$$P \equiv \sum_{k=0}^{m-1} (b^k + b^{2m-k})\theta_k + b^m\theta_m \equiv (\theta_0\theta_1\theta_2 \ldots \theta_{m-1}\theta_m\theta_{m-1} \ldots \theta_1\theta_0) . \tag{1}$$

Moreover, suppose the digits θ_k satisfy the following conditions:

$$1 \leq \theta_0 \leq \theta_1 \leq b-1, \text{ if } m = 1; \; 1 \leq \theta_0 \leq \theta_1 \leq \theta_0 + \theta_1 \leq \theta_2 \leq b-1, \text{ if } m \geq 2; \tag{2}$$

$$0 \leq \theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3} \leq b - 1, \text{ if } 3 \leq k \leq m; \tag{3}$$

$$0 \leq \theta_{m-2} \leq \theta_m \leq b - 1, \text{ if } m \geq 3. \tag{4}$$

Let

$$Q \equiv (b^2 - 1)(b^{m+1} + (b - 1)P). \tag{5}$$

Then $Q \in \mathcal{P}_b$.

PROOF: $Q = (b - 1)(b^{m+2} + b^{m+1}) + (b^3 - b^2 - b + 1)P$

$$= (b - 1)(b^{m+2} + b^{m+1}) + \sum_{k=0}^{m-1} (b^{k+3} + b^{2m+3-k})\theta_k - \sum_{k=0}^{m-1} (b^{k+2} + b^{2m+2-k})\theta_k$$

$$- \sum_{k=0}^{m-1} (b^{k+1} + b^{2m+1-k})\theta_k + \sum_{k=0}^{m-1} (b^k + b^{2m-k})\theta_k + (b^{m+3} - b^{m+2} - b^{m+1} + b^m)\theta_m$$

$$= (b - 1)(b^{m+2} + b^{m+1}) + \sum_{k=3}^{m+2} (b^k + b^{2m+3-k})\theta_{k-3} - \sum_{k=2}^{m+1} (b^k + b^{2m+3-k})\theta_{k-2}$$

$$- \sum_{k=1}^{m} (b^k + b^{2m+3-k})\theta_{k-1} + \sum_{k=0}^{m-1} (b^k + b^{2m+3-k})\theta_k + (b^{m+3} - b^{m+2} - b^{m+1} + b^m)\theta_m.$$

After some manipulation, this last expression simplifies to the following:

$$Q = (b^0 + b^{2m+3})\theta_0 + (b^1 + b^{2m+2})(\theta_1 - \theta_0) + (b^2 + b^{2m+1})(\theta_2 - \theta_1 - \theta_0)$$

$$+ \sum_{k=3}^{m} (b^k + b^{2m+3-k})(\theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3}) + (b^{m+1} + b^{m+2})(b - 1 - \theta_m + \theta_{m-2}).$$

If we make the following definitions

$$c_0 \equiv \theta_0, \tag{6}$$

$$c_1 \equiv \theta_1 - \theta_0, \tag{7}$$

$$c_2 \equiv \theta_2 - \theta_1 - \theta_0, \tag{8}$$

$$c_k \equiv \theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3}, \; 3 \leq k \leq m, \tag{9}$$

and

(10) $$c_{m+1} \equiv b - 1 - \theta_m + \theta_{m-2},$$

we may express Q as follows:

(11) $$Q = \sum_{k=0}^{m+1} (b^k + b^{2m+3-k})c_k.$$

Moreover, we see from conditions (2), (3), and (4) that, for $0 \le k \le m+1$, the inequalities $0 \le c_k \le b-1$ (with $c_0 \ge 1$) are satisfied, i.e., the c_k's are digits in base b. Therefore, $Q = (c_0c_1c_2 \ldots c_mc_{m+1}c_{m+1} \ldots c_1c_0)_b \in \mathcal{P}_b$. Q.E.D.

This result is readily specialized to decimal numbers by setting $b = 10$ in the theorem. Thus, if $P \in \mathcal{P}_{10}$ is given by (1)-(4), with $b = 10$, then

$$Q = 99(10^{m+1} + 9P)$$

is a palindrome in base 10.

Also solved by the proposer.

Simple Form

B-445 *Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, PA*

Show that

$$5F_{2n+2}^2 + 2L_{2n}^2 + 5F_{2n-2}^2 = L_{2n+2}^2 + 10F_{2n}^2 + L_{2n-2}^2,$$

and find a simpler form for these equal expressions.

Solution by F. D. Parker, St. Lawrence University, Canton, NY

Using the identities $L_n = a^n + b^n$, $F_n = (a^n - b^n)/\sqrt{5}$, and $ab = -1$, both sides reduce to

$$a^{4n+4} + b^{4n+4} + 2a^{4n} + 2b^{4n} + a^{4n-4} + b^{4n-4},$$

which can be written

$$L_{4n+4} + 2L_{4n} + L_{4n-4}.$$

Using $L_n = L_{n-1} + L_{n-2}$, we see that this is equal to $9L_{4n}$.

Also solved by Paul S. Bruckman, Herta T. Freitag, Calvin L. Gardner, Graham Lord, Bob Prielipp, Sahib Singh, M. Wachtel, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. Preference will be given to solutions that are submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-335 Proposed by Paul S. Bruckman, Concord, CA

Find the roots, in exact radicals, of the polynomial equation:

$$p(x) = x^5 - 5x^3 + 5x - 1 = 0. \tag{1}$$

H-336 Proposed by Lawrence Somer, Washington, D.C.

Let p be an odd prime.

(a) Prove that if $p \equiv 3$ or $7 \pmod{20}$, then

$$5F^2_{(p-1)/2} \equiv -4 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv -1 \pmod{p}.$$

(b) Prove that if $p \equiv 11$ or $19 \pmod{20}$, then

$$5F^2_{(p-1)/2} \equiv 4 \pmod{p} \text{ and } 5F^2_{(p+1)/2} \equiv 1 \pmod{p}.$$

(c) Prove that if $p \equiv 13$ or $17 \pmod{20}$, then

$$F^2_{(p-1)/2} \equiv -1 \pmod{p} \text{ and } F_{(p+1)/2} \equiv 0 \pmod{p}.$$

(d) Prove that if $p \equiv 21$ or $29 \pmod{40}$, then

$$F_{(p-1)/2} \equiv 0 \pmod{p} \text{ and } F_{(p+1)/2} \equiv 1 \pmod{p}.$$

(e) Prove that if $p \equiv 1$ or $9 \pmod{40}$, then

$$F_{(p-1)/2} \equiv 0 \pmod{p} \text{ and } F_{(p+1)/2} \equiv \pm 1 \pmod{p}.$$

Show that both the cases $F_{(p+1)/2} \equiv -1 \pmod{p}$ and $F_{(p+1)/2} \equiv 1 \pmod{p}$ do in fact occur.

H-337 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

(a) Evaluate the determinant:

$$\det A = \begin{vmatrix} 1 & -4L_{2r} & 6L_{4r}+16 & -(4L_{6r}+24L_{2r}) & L_{8r}+16L_{4r}+36 \\ L_{2r} & -(3L_{4r}+10) & 3L_{6r}+25L_{2r} & -(L_{8r}+25L_{4r}+60) & 10L_{6r}+60L_{2r} \\ L_{4r} & -(2L_{6r}+6L_{2r}) & L_{8r}+12L_{4r}+30 & -(6L_{6r}+50L_{2r}) & 30L_{4r}+80 \\ L_{6r} & -(L_{8r}+7L_{4r}) & 7L_{6r}+21L_{2r} & -(21L_{4r}+70) & 70L_{2r} \\ L_{8r} & -8L_{6r} & 28L_{4r} & -56L_{2r} & 140 \end{vmatrix} \begin{matrix} \text{(e)} \\ \text{(d)} \\ \text{(c)} \\ \text{(b)} \\ \text{(a)} \end{matrix}$$

(b) Show that:

$$\begin{aligned}625F_{2r}^2 &= L_{8r}^2 - 8L_{6r}^2 + 28L_{4r}^2 - 56L_{2r}^2 + 140\\ &= 8L_{6r}^2 + (L_{8r} + 7L_{4r})^2 - 14(L_{6r} + 3L_{2r})^2 + 7(3L_{4r+10})^2 - 280L_{2r}^2\\ &= 28L_{4r}^2 - 14(L_{6r} + 3L_{2r})^2 + (L_{8r} + 12L_{4r} + 30)^2 - 2(3L_{6r} + 25L_{2r})^2\\ &\qquad + 20(3L_{4r} + 8)^2\\ &= -56L_{2r}^2 + 7(3L_{4r} + 10)^2 - 2(3L_{6r} + 25L_{2r})^2 + (L_{8r} + 25L_{4r} + 60)^2\\ &\qquad - 40(L_{6r} + 6L_{2r})^2.\end{aligned}$$

Grace Note: If the elements of this determinant are the coefficients of a 5 X 5 linear homogeneous system, then the solution to the 4 X 5 system represented by equations (b), (c), (d), (e) is given by the elements of the first column. The solution to (a), (c), (d), (e) is given by the elements of the second column; and so on.

H-338 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC

An integer n is abundant if $\sigma(n) > 2n$, where $\sigma(n)$ is the sum of the divisors of n. Show that there is a probability of at least:

(a) 0.15 that a Fibonacci number is abundant;
(b) 0.10 that a Lucas number is abundant.

SOLUTIONS

Sum Enumerator

H-316 Proposed by B. R. Myers, Univ. of British Columbia, Vancouver, Canada (Vol. 18, no. 2, April 1980)

The enumerator of compositions with exactly k parts is $(x+x^2+\cdots)^k$, so that

$$[W(x)]^k = (w_1x + w_2x^2 + \cdots)^k \tag{1}$$

is then the enumerator of weighted k-part compositions. After Hoggatt and Lind ["Compositions and Fibonacci Numbers," *The Fibonacci Quarterly* 7 (1969):253-66], the number of weighted compositions of n can be expressed in the form

$$C_n(w) = \sum_{\nu(n)} w_{a_1} \ldots w_{a_k} \qquad (n > 0), \tag{2}$$

where $w = \{w_1, w_2, \ldots\}$ and where the sum is over all compositions $a_1 + \cdots + a_k$ of n (k variable). In particular (*ibid.*),

$$\sum_{\nu(n)} a_1 \ldots a_k = F_{2n}(1, 1), \tag{3}$$

where $F_k(p, q)$ is the kth number in the Fibonacci sequence

$$\begin{aligned}F_1(p, q) &= p \qquad (\geq 0)\\ F_2(p, q) &= q \qquad (\geq p)\\ F_{n+2}(p, q) &= F_{n+1}(p, q) + F_n(p, q) \qquad (n \geq 1).\end{aligned} \tag{4}$$

Show that

$$\sum_{\nu(n)} (a_1 \pm 1)a_1 \ldots a_k = 2[F_{2n-1}(1, 1) - 1] \tag{5}$$

and, hence, that

$$\sum_{\nu(n)} (a_1 - 1)a_1 \ldots a_k + \sum_{\nu(n)} a_1 \ldots a_k = F_{2n}(1, 1 + 2m) - 2m \qquad (m \geq 0). \tag{6}$$

Solution by the proposer.

As in the example $C_4 = 1(3 + 21 + 12 + 111) + 2(2 + 11) + 3(1) + 4$, the compositions C_n of n are given by

$$C_n = 1C_{n-1} + 2C_{n-2} + \cdots + nC_0, \tag{7}$$

where C_0 is the identity element ($nC_0 \equiv n$). Equation (7) implies that

$$\begin{aligned}\sum_{\nu(n)} a_1(a_1a_2 \ldots a_k) = \sum_{\nu(n-1)} 1^2a_1a_2 \ldots a_k + \sum_{\nu(n-2)} 2^2a_1a_2 \ldots a_k \\ + \cdots + \sum_{\nu(1)} n^2a_1a_2 \ldots a_k,\end{aligned} \tag{8}$$

so that, by (3),

$$\sum_{\nu(n)} a_1(a_1a_2 \ldots a_k) = n^2F_1(1, 1) + \sum_{i=1}^{n-1} i^2F_{2(n-i)}(1, 1). \tag{9}$$

It is not difficult to show (for example, from Problems P.36-P.37 on p. 9 of Bro. U. Alfred's "An Introduction to Fibonacci Discovery," The Fibonacci Association, 1965) that

$$F_{2n}(1, 3) - 2 = n^2F_1(1, 1) + \sum_{i=1}^{n-1} i^2F_{2(n-i)}(1, 1), \tag{10}$$

so that, from (9) and (10)

$$\sum_{\nu(n)} a_1(a_1a_2 \ldots a_k) = F_{2n}(1, 3) - 2. \tag{11}$$

Equations (5) and (6) follow routinely from manipulation of (11) in conjunction with the identity

$$F_k(1, 1 + 2m) = (m - 1)F_{k-1}(1, 1) + F_{k+1}(1, 1) \tag{12}$$

for $m \geq 0$, $k > 1$.

Prime Time

H-317 Proposed by Lawrence Somer, Washington, D.C.
(Vol. 18, no. 3, April 1980)

Let $\{G_n\}_{n=0}^{\infty}$ be any generalized Fibonacci sequence such that $G_{n+2} = G_{n+1} + G_n$, $(G_0, G_1) = 1$, and $\{G_n\}$ is not a translation of the Fibonacci sequence. Show that there exists at least one prime p such that both

$$G_n + G_{n+1} \equiv G_{n+2} \pmod{p} \quad \text{and} \quad G_{n+1} \equiv rG_n \pmod{p}$$

for a fixed $r \not\equiv 0 \pmod{p}$ and for all $n \geq 0$.

Solution by Paul S. Bruckman, Concord, CA.

We define the discriminant of $\{G_n\}_{n=0}^{\infty}$ by

$$D(G_n) \equiv G_n^2 - G_{n-1}G_{n+1}, \quad n = 1, 2, \ldots . \tag{1}$$

This satisfies the invariance relation

$$D(G_n) = (-1)^{n-1}D(G_1). \tag{2}$$

Since $\{G_n\}_{n=0}^{\infty}$ is not a translation of the Fibonacci sequence, thus $|D(G_1)| > 1$. Hence $D(G_1)$ is divisible by a prime p. If we were to have $p|G_0$, then we would also have $p|G_1$, and conversely, since $D(G_1) = G_1^2 - G_0G_1 - G_0^2$. This, however, would contradict the condition $(G_0, G_1) = 1$; hence $p\nmid G_0$, $p\nmid G_1$.

The congruence

$$G_n + G_{n+1} \equiv G_{n+2} \pmod{p} \tag{3}$$

is a trivial consequence of the recursion satisfied by $\{G_n\}_{n=0}^{\infty}$. If the congruence

(4) $$rG_n \equiv G_{n+1} \pmod p$$

is to hold for all $n \geq 0$, it must in particular hold for $n = 0$, for some fixed r, so that $rG_0 \equiv G_1 \pmod p$. Since $G_0 \not\equiv 0 \pmod p$, this uniquely determines $r \pmod p$:

(5) $$r \equiv G_1(G_0)^{-1} \pmod p, \text{ where } p \text{ is any prime divisor of } D(G_1).$$

Note that $r \not\equiv 0 \pmod p$. To show that this r satisfies (4) for all $n \geq 0$, we proceed by induction on n. Let S denote the set of nonnegative integers n such that (4) holds, where r is given by (5). Clearly, $0 \in S$. Also, $rG_1 \equiv G_1^2(G_0)^{-1} \equiv G_0G_2(G_0)^{-1} \equiv G_2 \pmod p$, which shows that $1 \in S$. Suppose $k \in S$, $k = 0, 1, \ldots, m$. Then $rG_{m+1} \equiv r(G_m + G_{m-1}) \equiv G_{m+1} + G_m \equiv G_{m+2} \pmod p$. Hence $k \in S \Rightarrow (k + 1) \in S$. By induction, (4) is proved.

Also solved by the proposer.

Canonical Möbius

H-318 Proposed by James Propp, Harvard College Cambridge, MA (Vol. 18, no. 3, April 1980)

Define the sequence operator M so that for any infinite sequence $\{u_i\}$,

$$M(u_n) = M(u_n) - \sum_{i|n} M(u_i)\mu\left(\frac{n}{i}\right),$$

where μ is the Möbius function. Let the "Möbinacci Sequence" S be defined so that $S_1 = 1$ and $S_n = M(S_n) + M(M(S_n))$ for $n > 1$. Find a formula for S_n in terms of the prime factorization of n.

Remarks: I've been unable to solve this problem, but some special cases were easier. Let p be a prime. $S(1) = 1$, $S(p) = 1$. For $a \geq 2$,

$$M(S_{p^a}) = S_{p^{a-1}} \quad \text{and} \quad M(M(S_{p^a})) = S_{p^{a-2}},$$

so that $S_{p^a} = S_{p^{a-1}} + S_{p^{a-2}}$. Solving this difference equation, we get

$$S_{p^a} = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^a - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^a$$

I have found no explicit formula for the case $n = p^a q^b$, but if one holds b fixed and finds S_n in terms of a, the characteristic equations seem to have only the the roots ϕ and $-1/\phi$, where $\phi = \frac{1}{2}(1 + \sqrt{5})$.

Fibonacci Never More

H-319 Proposed by Verner E. Hoggatt, Jr., San Jose State Univ., San Jose, CA (Vol. 18, no. 3, April 1980)

If $F_n < x < F_{n+1} < y < F_{n+2}$, then $x + y$ is never a Fibonacci number.

Solution by M. J. DeLeon, Florida Atlantic Univ., Boca Raton, FL.

Assume that $F_n < x < F_{n+1} < y < F_{n+2}$. Since $F_n < x$ and $F_{n+1} < y$, $F_{n+2} = F_n + F_{n+1} < x + y$. Since $x < F_{n+1}$ and $y < F_{n+2}$, $x + y < F_{n+1} + F_{n+2} = F_{n+3}$. Therefore,

$$F_{n+2} < x + y < F_{n+3}.$$

Since $F_n < F_{n+1} < F_{n+2}$, $n \geq 0$. Since $n \geq 0$, there is no Fibonacci number between F_{n+2} and F_{n+3}. Therefore $x + y$ is not a Fibonacci number.

Also solved by P. Bruckman, R. Giuli, G. Lord, F. D. Parker, B. Prielipp, S. Singh, L. Somer, M. Wachtel, R. Whitney, and the proposer.

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Introduction to Fibonacci Discovery by Brother Alfred Brousseau. Fibonacci Association (FA), 1965.

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A Primer For the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972

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A Collection of Manuscripts Related to the Fibonacci Sequence — 18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980.

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