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CONTENTS

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THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ON m-TH ORDER LINEAR RECURRENCES

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(Submitted April 1983)

1. INTRODUCTION

Fix numbers $u_0, u_1, \ldots, u_{-1}$, and for every $n \geqslant 0$, define u_{m+n} by means of the m preceding terms with the rule

$$u_{m+n} - k_1 u_{m+n-1} - \cdots - k_m u_n = 0, \text{ with } k_m \neq 0. \tag{1.1}$$

In this note, we wish to present two formulas for these numbers u_n satisfying the above m-th order linear recurrence (Sections 2 and 3).

These results are probably known to some readers; however, since from time to time we happen to see in the literature special cases of these formulas, it may be worthwhile to present them once and for all.

Note that for $m = 2$, $k_1 = k_2 = 1$, $u_0 = u_1 = 1$, one is dealing with the Fibonacci numbers, which have been extensively studied by many authors (see, for instance, [13], [5], and [3]), and which were used by Matijasevič [9] in his notorious proof that Hilbert's tenth problem is recursively unsolvable.

2. GENERATING FUNCTION AND BINET'S FORMULA

Using the m-th order linear recurrence

$$u_{m+n} = k_1 u_{m+n-1} + k_2 u_{m+n-2} + \cdots + k_m u_n, \quad k_m \neq 0, \tag{2.1}$$

(with the k_i's in $\mathbb{Z}$ for instance, or in a given field), we easily obtain

$$\left(\sum_{n=0}^{\infty} u_n X^n\right)(1 - k_1 X - \cdots - k_m X^m) = \sum_{i=0}^{m-1} v_i X^i,$$

where the v_i's, functions of the initial conditions on $u_0, u_1, \ldots, u_{m-1}$, are defined by

$$v_i = -\sum_{j=0}^{i} u_{i-j} k_j, \tag{2.2}$$

(with $k_0 = -1$ throughout this article). Associated with that recursive sequence is the following polynomial,

$$f(X) = X^m = k_1 X^{m-1} - \cdots - k_{m-1} X - k_m = (X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_m),$$

whose roots we assume distinct (and nonzero, since $k_m \neq 0$).

Then we have

$$\sum_{n=0}^{\infty} u_n X^n = \frac{v_0 + v_1 X + \cdots + v_{m-1} X^{m-1}}{1 - k_1 X - k_2 X^2 - \cdots - k_m X^m}$$

(continued)

$$= \frac{v_0 + v_1X + \cdots + v_{m-1}X^{m-1}}{(1-\alpha_1X)(1-\alpha_2X)\cdot\cdots\cdot(1-\alpha_mX)}$$

$$= (v_0 + v_1X + \cdots + v_{m-1}X^{m-1})(1 + p_1X + \cdots + p_jX^j + \cdots),$$

where p_j stands for the sum of all symmetric functions of weight j in $\alpha_1, \alpha_2, \ldots, \alpha_m$; in other words [2], p_j is the "sum of the homogeneous products of j dimensions" of the m symbols $\alpha_1, \ldots, \alpha_m$.

Let us recall from Volume 1 of [2, p. 178] that

$$p_j = \sum_{i=1}^{m} \frac{\alpha_i^{m-1+j}}{f'(\alpha_i)} \quad \text{with } p_0 = 1,$$

and that $p_{-1} = p_{-2} = \cdots = p_{-m+1} = 0$ (which follows from Example 4 of p. 172). We therefore obtain for the m-th number u_n what can be called

BINET'S FORMULA: $\boxed{u_n = \sum_{j=0}^{m-1} v_j p_{n-j}}$.

<u>EXAMPLES</u>: (1) Let $v_0 = v_1 = \cdots = v_{m-2} = 0$, $v_{m-1} = 1$; then, as in Formula 9 of [7],

$$u_n = p_{n-m+1} = \frac{\alpha_1^n}{f'(\alpha_1)} + \frac{\alpha_2^n}{f'(\alpha_2)} + \cdots + \frac{\alpha_m^n}{f'(\alpha_m)}.$$

(2) For $m = 2$, $m = 3$, we recover Binet's formulas of [3] and [11].

(3) For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, define s_n by

$$s_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n.$$

As is well known (see [2]), Newton's formulas state that:

$$s_n = \begin{cases} m & \text{if } n = 0 \\ k_1 & \text{if } n = 1 \\ k_1s_{n-1} + k_2s_{n-2} + \cdots + k_{n-1}s_1 + nk_n & \text{if } 2 \leqslant n \leqslant m-1 \\ k_1s_{n-1} + k_2s_{n-2} + \cdots + k_ms_{n-m} & \text{if } n \geqslant m. \end{cases}$$

In particular, if $u_n = s_n$, then $\{u_n\}$ satisfies (2.1).

Thus, using the fact that $k_0 = -1$, we find that (2.2) gives:

$$\begin{cases} v_0 = m = -mk_0 \\ v_1 = s_1 - mk_1 = -(m-1)k_1 \\ v_2 = s_2 - s_1k_1 - mk_2 = -(m-2)k_2 \\ \quad\vdots \\ v_{m-1} = s_{m-1} - s_{m-2}k_1 - s_{m-3}k_2 - \cdots - s_1k_{m-2} - mk_{m-1} = -1k_{m-1}. \end{cases}$$

In short, for $j = 0, 1, \ldots, m-1$, $v_j = -(m-j)k_j$, and Binet's formula becomes

$$s_n = -\sum_{j=0}^{m-1}(m-j)k_jp_{n-j}.$$

3. ANOTHER FORMULA

We can also use the multinomial theorem to obtain a formula for u_n that is a function of (i.e., the elementary symmetric functions of) $k_1, k_2, \ldots, k_m$. Here we no longer require that the roots of f be distinct. Within a certain radius of convergence, we find that

$$\sum_{n=0}^{\infty} u_n X^n = \frac{v_0 + v_1 X + \cdots + v_{m-1}X^{m-1}}{1 - (k_1 X + k_2 X^2 + \cdots + k_m X^m)}$$

$$= (v_0 + v_1 X + \cdots + v_{m-1}X^{m-1})\left[\sum_{j=0}^{\infty}(k_1 X + \cdots + k_{m-1}X^{m-1} + k_m X^m)^j\right]$$

$$= (v_0 + v_1 X + \cdots + v_{m-1}X^{m-1})\left(\sum_{i=0}^{\infty} A(i)X^i\right),$$

where

$$A(i) = \sum \frac{(t_1 + t_2 + \cdots + t_m)!}{t_1!t_2!\ldots t_m!} k_1^{t_1} k_2^{t_2} \ldots k_m^{t_m},$$

the last sum being taken over all m-tuples $(t_1, t_2, \ldots, t_m)$ of $\mathbb{N}^m$ such that

$$t_1 + 2t_2 + \cdots + mt_m = i.$$

Defining $A(i)$ to be 0 for $i < 0$, we therefore conclude:

$$\boxed{u_n = \sum_{j=0}^{m-1} v_j A(n-j)}. \tag{3.1}$$

EXAMPLES: (1) If $v_0 = 1$, $v_1 = v_2 = \cdots = v_{m-1} = 0$, then

$$u_n = A(n).$$

(2) Let $s_n = \alpha_1^n + \alpha_2^n + \cdots + \alpha_m^n$. Replacing v_j by $-(m-j)k_j$, and making in the j-th summation ($j = 1, \ldots, m-1$) of (3.1) the change of variable $t_j \to t_j - 1$, we obtain after a few calculations what is called in [2] Waring's formula for s_n:

$$s_n = \sum_{t_1 + 2t_2 + \cdots + mt_m = n} \frac{n(t_1 + t_2 + \cdots + t_m - 1)!}{t_1!t_2! \ldots t_m!} k_1^{t_1} k_2^{t_2} \ldots k_m^{t_m}.$$

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GEGENBAUER POLYNOMIALS REVISITED

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(Submitted June 1983)

1. INTRODUCTION

The *Gegenbauer* (or *ultraspherical*) *polynomials* $C_n^\lambda(x)$ $(\lambda > -\frac{1}{2},\ |x| \leqslant 1)$ are defined by

$$C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x \tag{1.1}$$

with the recurrence relation

$$nC_n^\lambda(x) = 2x(\lambda + n - 1)C_{n-1}^\lambda(x) - (2\lambda + n - 2)C_{n-2}^\lambda(x) \qquad (n \geqslant 2). \tag{1.2}$$

Gegenbauer polynomials are related to $T_n(x)$, the *Chebyshev polynomials of the first kind*, and to $U_n(x)$, the *Chebyshev polynomials of the second kind*, by the relations

$$T_n(x) = \frac{n}{2}\lim_{\lambda \to 0}\left(\frac{C_n^\lambda(x)}{\lambda}\right) \qquad (n \geqslant 1), \tag{1.3}$$

and

$$U_n(x) = C_n^1(x). \tag{1.4}$$

Properties of the rising and descending diagonals of the Pascal-type arrays of $\{T_n(x)\}$ and $\{U_n(x)\}$ were investigated in [2], [3], and [5], while in [4] the rising diagonals of the similar array for $C_n^\lambda(x)$ were examined.

Here, we consider the descending diagonals in the Pascal-type array for $\{C_n^\lambda(x)\}$, with a backward glance at some of the material in [4].

As it turns out, the descending diagonal polynomials have less complicated computational aspects than the polynomials generated by the rising diagonals.

Brief mention will also be made of the generalized Humbert polynomial, of which the Gegenbauer polynomials and, consequently, the Chebyshev polynomials, are special cases.

2. DESCENDING DIAGONALS FOR THE GEGENBAUER POLYNOMIAL ARRAY

Table 1 sets out the first few Gegenbauer polynomials (with $y = 2x$):

TABLE 1. Descending Diagonals for Gegenbauer Polynomials

$$\left\{\begin{array}{ll}
C_0^\lambda(x) = 1 & \quad d_0^\lambda(x) \\
C_1^\lambda(x) = \lambda y & \\
C_2^\lambda(x) = \dfrac{(\lambda)_2 y^2}{2!} - \lambda & \\
 & \quad d_1^\lambda(x) \\
C_3^\lambda(x) = \dfrac{(\lambda)_3 y^3}{3!} - \dfrac{(\lambda)_2 y}{1!} & \\
C_4^\lambda(x) = \dfrac{(\lambda)_4 y^4}{4!} - \dfrac{(\lambda)_3 y^2}{2!} + \dfrac{(\lambda)_2}{2!} & \\
 & \quad d_2^\lambda(x) \\
C_5^\lambda(x) = \dfrac{(\lambda)_5 y^5}{5!} - \dfrac{(\lambda)_4 y^3}{3!} + \dfrac{(\lambda)_3 y}{2!} & \\
C_6^\lambda(x) = \dfrac{(\lambda)_6 y^6}{6!} - \dfrac{(\lambda)_5 y^4}{4!} + \dfrac{(\lambda)_4 y^2}{2!2!} - \dfrac{(\lambda)_3}{3!} & \\
 & \quad d_3^\lambda(x) \\
C_7^\lambda(x) = \dfrac{(\lambda)_7 y^7}{7!} - \dfrac{(\lambda)_6 y^5}{5!} + \dfrac{(\lambda)_5 y^3}{2!3!} - \dfrac{(\lambda)_4 y}{3!} & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &
\end{array}\right. \tag{2.1}$$

wherein we have written

$$(\lambda)_n = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1). \tag{2.2}$$

Polynomials (2.1) may be obtained either from the generating recurrence (1.2) together with the initial values (1.1), or directly from the known explicit summation representation

$$C_n^\lambda(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m} (2x)^{n-2m}}{m!(n-2m)!}, \quad \lambda \text{ an integer and } n \geqslant 2, \tag{2.3}$$

where, as usual, $[n/2]$ symbolizes the integer part of $n/2$.

The generating function for the Gegenbauer polynomials is

$$\sum_{n=0}^{\infty} C_n^\lambda(x) t^n = (1 - 2xt + t^2)^{-\lambda} \qquad (|t| < 1). \tag{2.4}$$

Designate the descending diagonals in Table 1, indicated by lines, by the symbols $d_j^\lambda(x)$ $(j = 0, 1, 2, \ldots)$.

Then we have

$$\begin{aligned}
&d_0^\lambda(x) = 1, \quad d_1^\lambda(x) = \lambda(2x - 1), \quad d_2^\lambda(x) = \frac{(\lambda)_2(2x - 1)^2}{2!} \\
&d_3^\lambda(x) = \frac{(\lambda)_3(2x - 1)^3}{3!}, \quad d_4^\lambda(x) = \frac{(\lambda)_4(2x - 1)^4}{4!}, \ \ldots\ .
\end{aligned} \tag{2.5}$$

From the emerging pattern in (2.5), one can confidently expect that

$$d_n^\lambda(x) = \frac{(\lambda)_n (2x - 1)^n}{n!} = \binom{\lambda + n - 1}{n}(2x - 1)^n, \tag{2.6}$$

a result which we now proceed to prove.

Proof of (2.6): Suppose we represent the pairs of values of m and n which give rise to $d_n^\lambda(x)$ by the couplet (m, n).

Then, at successive "levels" of the descending diagonal $d_n^\lambda(x)$ in Table 1, we have the couplets

$$(0, n),\ (1, n + 1),\ (2, n + 2),\ \ldots,\ (n - 1, 2n - 1),\ (n, 2n),$$

so that corresponding values of $n - 2m$ are $n, n - 1, n - 2, \ldots, 1, 0$, while $n - m$ always has the value n. [It is important to note that the maximum value for m in the couplets must be n.]

Consequently, from Table 1 and (2.3), with $y = 2x$ for convenience, we have

$$\begin{aligned} d_n^\lambda(x) &= \frac{(\lambda)_n y^n}{0!n!} - \frac{(\lambda)_n y^{n-1}}{1!(n-1)!} + \frac{(\lambda)_n y^{n-2}}{2!(n-2)!} - \cdots + (-1)^n \frac{(\lambda)_n y^0}{n!0!} \\ &= \frac{(\lambda)_n}{n!}\left\{\binom{n}{0}y^n - \binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2} - \cdots + (-1)^n\binom{n}{n}y^0\right\} \\ &= \frac{(\lambda)_n}{n!}(y - 1)^n = \frac{(\lambda)_n}{n!}(2x - 1)^n = \binom{\lambda + n - 1}{n}(2x - 1)^n. \end{aligned}$$

From (2.6) it follows immediately that

$$\frac{d_n^\lambda(x)}{d_{n-1}^\lambda(x)} = \frac{\lambda + n - 1}{n}(2x - 1), \quad n \geqslant 1. \tag{2.7}$$

Moreover,

$$(2x - 1)\frac{d}{dx}(d_n^\lambda(x) - 2nd_n^\lambda(x) = 0, \quad n \geqslant 0, \tag{2.8}$$

readily follows.

Putting

$$g \equiv d(\lambda, x, t) = \sum_{n=0}^{\infty} d_n^\lambda(x)t^n, \tag{2.9}$$

we find that the generating function for $\{d_n^\lambda(x)\}$ is

$$g = [1 - (2x - 1)^t]^{-\lambda}. \tag{2.10}$$

Furthermore,

$$2t\frac{\partial g}{\partial t} - (2x - 1)\log(2x - 1)\frac{\partial g}{\partial x} = 0, \tag{2.11}$$

which is independent of λ.

Additionally, we easily establish that

$$2\lambda^2 t(2x - 1)^{t-1}\frac{\partial g}{\partial x} - g^{-\lambda^{-1}} \log g \frac{\partial g}{\partial x} = 0 \tag{2.12}$$

and

$$\lambda^2(2x - 1)^t \log(2x - 1)\frac{\partial g}{\partial \lambda} - g^{-\lambda^{-1}} \log g \frac{\partial g}{\partial t} = 0 \tag{2.13}$$

if we allow λ to vary.

Differentiating (2.8) w.r.t. x and substituting from (2.8), we obtain

$$(2x - 1)^2 \frac{d^2}{dx^2}(d_n^\lambda(x) - 2^2 n(n - 1)d_n^\lambda(x) = 0. \tag{2.14}$$

Continued repetition of this process, with substitution from the previous steps, ultimately yields

$$(2x - 1)^r \frac{d^r}{dx^r}(d_n^\lambda(x)) - 2^r r!\binom{n}{r} d_n^\lambda(x) = 0 \tag{2.15}$$

for the rth derivative of the descending diagonal polynomial. If we write $z = d_0^\lambda(x)$ for simplified symbolism, result (2.15) appears in a more attractive visual form as, when $r = n$,

$$(2x - 1)^n z^{(n)} - 2^n n! z = 0 \tag{2.15$'$}$$

or by (2.6),

$$z^{(n)} - 2^n(\lambda)_n = 0. \tag{2.15$''$}$$

Observe that (2.15) can also be expressed as

$$\frac{d^r}{dx^r}(d_n^\lambda(x)) = 2^r(\lambda)_r d_{n-r}^{\lambda+r}(x), \quad r = 1, 2, \ldots, n, \tag{2.15$'''$}$$

on using $(\lambda)_n(\lambda + r)_{n-r} = (\lambda)_n$ and (2.6).

Note the formal equivalence of (2.15)$'''$ and the known differential equation for Gegenbauer polynomials

$$\frac{d^r}{dx^r} C_n^\lambda(x) = 2^r(\lambda)_r C_{n-r}^{\lambda+r}(x). \tag{2.16}$$

Elementary calculations yield, additionally, by using (2.8) and (2.7),

$$(2x - 1)\frac{d}{dx}(d_n^\lambda(x)) = \frac{n}{\lambda + n}\frac{d}{dx}(d_{n+1}^\lambda(x)), \tag{2.17}$$

which differs in form from the corresponding result involving Gegenbauer polynomials.

3. SPECIAL CASES: CHEBYSHEV POLYNOMIALS

If we substitute $\lambda = 1$ in the relevant results of the preceding section we obtain corresponding results already given in [3] for the special case (1.4) of the Chebyshev polynomials $U_n(x)$. [Allowance must be made for a small variation in notation, namely $d_n^1(x) = b_{n+1}(x)$ in [3]; e.g., $d_4^1(x) = (2x - 1)^4 = b_5(x)$.]

Coming now to the similar results for the Chebyshev polynomials $T_n(x)$, we appreciate that the limiting process (1.3) requires a less obvious approach.

Let us write

$$\mathscr{D}_n^\lambda(x) = (n + 1)d_{n+1}^\lambda(x) - nd_n^\lambda(x) \qquad (n \geqslant 0). \tag{3.1}$$

By careful analogy with the forms of (1.3), we may then define

$$D_n(x) = \frac{1}{2}\lim_{\lambda \to 0}\left(\frac{\mathscr{D}_n^\lambda(x)}{\lambda}\right), \tag{3.2}$$

where $D_n(x)$ is the nth descending diagonal polynomial for $T_n(x)$.

Calculation yields

$$D_n(x) = \frac{1}{2}(2x - 2)(2x - 1)^n = (x - 1)(2x - 1)^n. \tag{3.3}$$

Comparison should not be made with corresponding results produced in [3] where, it should be noted, each Chebyshev function is *twice* the corresponding Chebyshev polynomial in this paper. Accordingly, we have $2O_m(x) = a_{n+2}(x)$ in [3]; e.g., $D_4(x) = (x - 1)(2x - 1)^4 = (1/2)a_6(x)$.

Thus, we have shown that our results for the descending diagonals in the Pascal-type array of Gegenbauer polynomials are generalizations of corresponding results for the specialized Chebyshev polynomials, as expected.

4. GENERALIZED HUMBERT POLYNOMIALS

Along with many other polynomials, the Gegenbauer polynomials (and consequently the Chebyshev polynomials) are special cases of the *generalized Humbert polynomial* (see Gould [1]).

Generalized Humbert polynomials, which are represented by the symbol

$$P_n(m, x, y, p, C)$$

are defined by the generating function

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n, \tag{4.1}$$

where $m \geqslant 1$ is an integer and the other parameters are in general unrestricted.

Particular cases of the generalized Humbert polynomial are:

$$\begin{cases} P_n(2, x, 1, -\frac{1}{2}, 1) = P_n(x) & (\textit{Legendre}, 1784) \\ P_n(2, x, 1, -1, 1) = U_n(x) & (\textit{Chebyshev}, 1859) \\ P_n(2, x, 1, -\lambda, 1) = C_n^{\lambda}(x) & (\textit{Gegenbauer}, 1874) \\ P_n(3, x, 1, -\frac{1}{2}, 1) = \mathscr{P}_n(x) & (\textit{Pincherle}, 1890) \\ P_n(m, x, 1, -\nu, 1) = \Pi_{n,m}^{\nu}(x) & (\textit{Humbert}, 1921) \\ P_n(2, x, -1, -1, 1) = \phi_{n+1}(x) & (\textit{Byrd}, 1963) \\ P_n\left(m, x, 1, -\frac{1}{m}, 1\right) = P_n(m, x) & (\textit{Kinney}, 1963) \end{cases} \tag{4.2}$$

The recurrence relation for the generalized Humbert polynomial is

$$CnP_n - m(n - 1 - p)xP_{n-1} + (n - m - mp)yP_{n-m} = 0 \quad (n \geqslant m \geqslant 1), \tag{4.3}$$

where we have written $P_n = P_n(m, x, y, p, C)$ for brevity.

Suitable substitution of the parameters in (4.2) for Gegenbauer polynomials reduces (4.3) to (1.2).

In passing, it might be noted in (4.2) that Legendre polynomials are special cases of Gegenbauer polynomials occurring when $\lambda = \frac{1}{2}$. Hence, results for Gegenbauer polynomials $C_n^{\lambda}(x)$ in [4] and in this article may be specialized for the Legendre polynomials $C_n^{\frac{1}{2}}(x)$. Moreover, Gegenbauer polynomials are closely

related to Jacobi polynomials, and they may also be expressed in terms of hypergeometric functions.

Using the generating function for Fibonacci numbers F_n, namely

$$(1 - x - x^2)^{-1} = \sum_{n=1}^{\infty} F_n x^{n-1}, \tag{4.4}$$

we readily see that

$$P_n(2, \tfrac{1}{2}, -1, -1, 1) = F_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k}, \tag{4.5}$$

whence the recurrence relation (4.3) simplifies to the defining recurrence relation

$$F_{n+1} - F_n - F_{n-1} = 0. \tag{4.6}$$

Gould [1] remarks that Eq. (4.5) is better than the usual device of using Chebyshev or other polynomials with imaginary exponent for expressing Fibonacci numbers.

While it is not the purpose of this paper to pursue the properties of the generalized Humbert polynomial, it is thought that publicizing their connection with the polynomials under discussion—Gegenbauer and Chebyshev—may be a useful service.

To whet the appetite of the interested reader for further knowledge of the generalized Humbert polynomial, we append the explicit form given in [1]:

$$P_n(m, x, y, p, C) = \sum_{k=0}^{[n/m]} \binom{p}{k}\binom{p-k}{n-m-k} C^{p-n+(m-1)k} y^k (-mx)^{n-mk}, \tag{4.7}$$

from which one may obtain the explicit forms of the special cases given in (4.2).

Likewise, the first few terms of the polynomials in (4.2) may be checked against the generalized terms given in [1]:

$$\begin{cases} P_0 = C^p \\ P_1 = -pmxC^{p-1} + p\binom{p-1}{1-m} y(-mx)^{1-m} C^{p+m-2} \\ P_2 = \binom{p}{2} C^{p-2} m^2 x^2 + p\binom{p-1}{2-m} C^{p+m-3} y(-mx)^{2-m} \end{cases} \tag{4.8}$$

with

$$P_n = \binom{p}{n} C^{p-n}(-mx)^n \qquad (m > n). \tag{4.9}$$

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(Please turn to page 307)

FIBONACCI AND LUCAS NUMBERS OF THE FORM $3z^2 \pm 1$

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1. INTRODUCTION

The purpose of this paper, which is a continuation of [1], is to report some results regarding the generalized Fibonacci and Lucas numbers of the form $3z^2 \pm \mu$.

In particular, we show for the Fibonacci and Lucas numbers that the following relations hold:

$F_m = 3z^2 + 1$ if and only if $m = \pm 1, 2, \pm 7$

$F_m = 3z^2 - 1$ if and only if $m = -2, \pm 3, \pm 5$

$L_m = 3z^2 + 1$ if and only if $m = 1, 3, 9$

$L_m = 3z^2 - 1$ if and only if $m = -1, 0, 5, \pm 8$

This author tried to show similar properties for other recursive sequences while working on class number problems for his Dissertation.

Throughout this paper we will make frequent use of the relations developed in [1]; thus, the numbering of the relations in this paper continues from that of [1].

Also, as in [1], d will always be a rational integer of the first kind and $\frac{a+b\sqrt{d}}{2}$ will be the fundamental solution of the Pellian equation $x^2 - dy^2 = -4$. The sequences $\{U_m\}$ and $\{V_m\}$ are as defined in [1].

2. PRELIMINARIES

Lemma 1: i) Let $ab \not\equiv 0 \pmod 3$. Then the equation $U_m = 3z^2$ has

(a) the solutions $m = 0, 4$ if $d = 5$,
(b) only the solution $m = 0$ in all other cases.

ii) Let $b = 1$ and $a \not\equiv 0 \pmod 3$. Then the equation $U_m = 3z^2$ has

(a) the solutions $m = 0, 4$ if $d = 5$,
(b) the solutions $m = 0, 2$ if $a = 3z^2$,
(c) only the solution $m = 0$ in all other cases.

Proof of i): According to our assumptions $(U_m)_{m \in \mathbf{Z}}$ is periodic mod 3 with length of period 8 and 3 divides U if and only if 4 divides m. Hence, $U_m = 3z^2$ implies $U_{2n}V_{2n} = 3z^2$, by (5). Since $n = 0$ is an obvious solution, we assume $n \neq 0$.

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Case 1. Let $n \not\equiv 0 \pmod 3$. Then $(U_{2n}, V_{2n}) = 1$, by (10), and we obtain

$$(U_{2n} = 3z_1^2,\ V_{2n} = z_2^2) \quad \text{or} \quad (U_{2n} = z_1^2,\ V_{2n} = 3z_1^2).$$

The first subcase is impossible, by (28). For the second, it is sufficient, by (30), to check only the value $n = 1$ in case a and b are both perfect squares ($n = 6$, $d = 5$, $L_{12} = 322 \neq 3z_2^2$).

For $n = 1$, we have

$$V_2 = a^2 + 2 = 3z_2^2,\ a = t^2,\ b = r^2.$$

That is, $t^4 + 2 = 3z_2^2$. Using [3], the last diophantine equation has at most one solution, $(t, z_2) = (\pm 1, \pm 1)$, which corresponds to the value $d = 5$.

Case 2. Let $n \equiv 0 \pmod 3$, $n \neq 0$. Equation (10) implies $(U_{2n}, V_{2n}) = 2$, so we must have

$$(U_{2n} = 2z_1^2,\ V_{2n} = 6z_2^2) \quad \text{or} \quad (U_{2n} = 6z_1^2,\ V_{2n} = 2z_2^2).$$

The first subcase is impossible because, by (31), the only possible value of n for which $U_{2n} = 2z_1^2$ is $n = 3$ ($d = 5$) for which $L_6 = 18 \neq 6z_2^2$. The second subcase has, by (29) and direct computation, no solution for $n = \pm 3$, $d = 5, 29$. The proof of (ii) follows along the same lines as the proof of (i); hence, the details are omitted.

Lemma 2: Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 3z^2$ has the solutions $m = \pm 2$ if $a^2 + 2 = 3z^2$ and no solution in all other cases.

Proof: Since $a \not\equiv 0 \pmod 3$, $(V_m)_{m \in \mathbf{Z}}$ is periodic mod 3 with length of period 8 and 3 divides V_m if and only if $m \equiv \pm 2 \pmod 8$.

Case 1. Let $m \equiv \pm 2 \pmod{16}$. Then, $a^2 + 2 = 3z^2$. The solutions of this equation are given in [4] by

$$3z + a\sqrt{3} = (3 + \sqrt{3})(2 + \sqrt{3})^n \quad \text{for } n = 0, 1, 2, \ldots .$$

If $m \neq \pm 2$, then (4) says we only have to consider the case $m \equiv 2 \pmod{16}$. We write $m = 2 + 2 \cdot 3^s \cdot n$ where $8 | n$ and $3 \nmid n$. Then, by (22), $V_m \equiv -V_2 \pmod{V_n}$. If $V_m = 3z^2$, we have $(3z)^2 \equiv -3V_2 \pmod{V_n}$ where $8 | n$ and $3 \nmid n$, which is impossible since $V_n \equiv 2 \pmod 3$, $(V_n, 3) = 1$, and $(V_m, V_2) = (2, V_2) = 1$ imply $(-3V_2/V_n) = 1$ by (33).

Case 2. Let $m \equiv \pm 6 \pmod{16}$. If $m = \pm 6$, then $a^6 + 6a^4 + 9a^2 + 2 = 3z^2$ or $(a^2 + 2)(a^4 + 4a^2 + 1) = 3z^2$ so that $c(c^2 - 3) = 3z$ where $c = a^2 + 2 \equiv 0 \pmod 3$ by our assumption on a. Since $(c, c^2 - 3) = 3$, we need only check the following two subcases:

$$\text{(i)} \quad c = 3z_1^2,\quad c^2 - 3 = (3z_2)^2,$$

$$\text{(ii)} \quad c = (3z_1)^2,\quad c^2 - 3 = 3z_2^2.$$

By (i) we have $3z_2 = \pm 1$, which is impossible. By (ii), $3^3 z_1^2 - 1 = z_2^2$, which is impossible mod 3. Now let $m \equiv 6 \pmod{16}$ with $m \neq 6$. We write

$$m = 6 + 2 \cdot 3^s \cdot n, \text{ where } 8 | n,\ 3 \nmid n.$$

Then, by (22), $V_m \equiv -V_6 \pmod{V_n}$. If $V_m = 3z^2$, we have

$$(3z)^2 \equiv -3V_6 \pmod{V_n} \text{ with } 8 | n,\ 3 \nmid n. \tag{62}$$

By using (13) repeatedly with (4), we obtain

$$2V_n = 2V_{8\lambda} \equiv \cdots \equiv \pm 2V_4 \pmod{V_6}. \tag{63}$$

We now note that $(V_6, V_4) = (V_4, V_2) = 1$ and $V_6 \equiv 2 \pmod 8$, since $a \equiv 1 \pmod 2$. However,

$$2V_4 \equiv -2V_2 \pmod{V_4} \tag{64}$$

and, by (22),

$$2V_4 \equiv -2V_0 \equiv -4 \pmod{V_2}. \tag{65}$$

Applying the Jacobi symbol, we now have:

$$\begin{aligned}
\left(\frac{-3V_6}{V_n}\right) &= (-1)\left(\frac{V_{6/2}}{V_n}\right) = -\left(\frac{V_n}{V_{6/2}}\right) \\
&= -\left(\frac{\pm V_4}{V_{6/2}}\right), && \text{by (63);} \\
&= -\left(\frac{V_{6/2}}{V_4}\right), && \text{since } V_6 \equiv 2 \pmod 8; \\
&= -\left(\frac{-2V_2}{V_4}\right), && \text{by (64);} \\
&= \left(\frac{V_2}{V_4}\right), && \text{by (19);} \\
&= -1, && \text{by (65).}
\end{aligned}$$

Therefore, (62) has no solution and the Lemma follows.

Lemma 3: For the generalized Fibonacci numbers U_n the following identity holds:

$$U_{4n\pm 1} = U_{2n}V_{2n\pm 1} + b. \tag{66}$$

Proof: This is like the relation (36) of Lemma 2 in [1].

Lemma 4: Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 6z^2$ has no solution.

Proof: Since a is odd and $a \not\equiv 0 \pmod 3$, we have $a \equiv \pm 1 \pmod 6$ or $a^2 \equiv 1 \pmod{12}$. In this case the generalized Lucas numbers are periodic mod 6 with period 24 as are the usual Lucas numbers. Hence, it still holds that

$$V_m \equiv 0 \pmod 6 \text{ if and only if } m \equiv 6 \pmod{12},$$

and

$$V_m \equiv 18 \pmod{24} \text{ if } m \equiv 6 \pmod 2.$$

With $V_m = 6z^2$, we now have $z^2 \equiv 3 \pmod 4$, which has no solution.

3. FIBONACCI NUMBERS OF THE FORM $3z^2 \pm 1$

From now on b will always be 1; that is, $d = a^2 + 4$.

Theorem 1: The equation $U_m = 3z^2 + 1$, $m \equiv 1 \pmod 2$ has

(a) the solutions $m = \pm 1, \pm 7$, if $d = 5$,
(b) the solutions $m = \pm 1, \pm 5$, if $d = 13$,
(c) only the solutions $m = \pm 1$ in all other cases.

Proof: For $m = 4n \pm 1$, (66) implies that $U_{2n}V_{2n\pm1} = 3z^2$. If $n = 0$, then $U_{2n} = 0$, so that $z = 0$ is a solution which gives us $m = \pm 1$. Now assume that $n \neq 0$, then $U_{2n}V_{2n\pm1} \neq 0$. Corollary 9 of [1] implies $(U_{2n}, V_{2n\pm1}) = 1$. Hence, we must have

$$U_{2n} = 3z_1^2, \quad V_{2n\pm1} = z_2^2, \tag{67}$$

or

$$U_{2n} = z_1^2, \quad V_{2n\pm1} = 3z_2^2. \tag{68}$$

By using (28) and a direct computation of U_{2n}, we find that (67) has a solution for $m = 5$ if $d = 13$, and one for $m = 7$ if $d = 5$. By using (30), we find that the possible values of n for (68) to have a solution are $n = 6$ if $d = 5$, and $n = 1$ if $a = t^2$.

When $n = 6$, $d = 5$, we have $L_{13} = 521 \neq 3z^2$ and $L_{11} = 199 \neq 3z^2$ so (68) has no solution in this case. When $n = 1$, $a = t^2$, we have $V_1 = 3z_2^2 = a$, which is impossible. Furthermore, $V_3 = a^3 + 3a = 3z_2^2$, which implies that $a^2 + 3 = 3w^2$ or $t^4 + 3 = 3w^2$ or $27t^4 + 1 = w^2$. The last equation, by [2], has no solutions, so (68) is impossible.

Note that $m = -5$, $d = 13$ and $m = -7$, $d = 5$ are also solutions, by (3).

Theorem 1′: The equation $U_m = 3z^2 - 1$, $m \equiv 1 \pmod 2$ has only the solutions

$$m = \pm 3,\ 15 \text{ if } a^2 + 2 = 3z^2$$

and no solutions in all other cases.

Proof: This follows the arguments of Theorem 1 by using (36), Corollary 9, (28) and (29) from [1].

Theorem 2: Let $a^2 + 2 = p$ where p is a prime. The equation $U_m = 3z^2 + a$, $m \equiv 0 \pmod 2$ has only the solution $m = 2$.

Proof: Case 1. Let $m = 4n$. By (38), we have $U_{2n+1}V_{2n-1} = 3z^2$. But, Lemma 3 of [1] implies $(U_{2n+1}, V_{2n-1}) = V_2 = p$, so the following possibilities must be checked:

$$U_{2n+1} = 3z_1^2, \quad V_{2n-1} = z_2^2 \tag{69}$$

$$U_{2n+1} = z_1^2, \quad V_{2n-1} = 3z_2^2 \tag{70}$$

$$U_{2n+1} = 3pz_1^2, \quad V_{2n-1} = pz_2^2 \tag{71}$$

$$U_{2n+1} = pz_1^2, \quad V_{2n-1} = 3pz_2^2 \tag{72}$$

Equation (69) has no solutions, since the possible values for which V_{2n-1} is a perfect square are given by (28) in [1] and none of them gives a solution to $U_{2n+1} = 3z_1^2$.

Equation (70) has no solution either, because the values of n for which $U_{2n+1} = z^2$ are $n = 0, -1$, which gives $V_{-1} = -a \neq 3z_2^2$ and $V_{-3} = -(a^3 + 3a) \neq 3z_2^2$.

If we write $2n - 1 = 4\lambda \pm 1$ and apply (13) repeatedly, we find that

$$2V_{2n-1} \equiv -2V_{4\lambda-4\pm1} \equiv \cdots \equiv \pm 2V_{\ 1} \pmod{V_2}.$$

Hence, if $V_{2\ -1} = pz_2^2 = V_2z_2^2$, we have V_2 divides $\pm 2V_{\pm1}$ or $a^2 + 2$ divides $\pm 2a$, which is impossible. Hence, (71) has no solution.

If we write $2n + 1 = 4\lambda \pm 1$ and apply (13 repeatedly, we find that

$$2U_{2n+1} \equiv -2U_{4\lambda-4\pm1} \equiv \cdots \equiv \pm 2U_1 \pmod{V_2}.$$

Hence, if $U_{2n+1} = pz_1^2 = V_2 z_1^2$, we have $V_2 | \pm 2$, which is impossible. Thus, (72) has no solution.

Case 2. Let $m = 4n - 2$. Equation (40) implies $U_{2n-2}V_{2n} + a = 3z^2 + a$, so $U_{2n-2}V_{2n} = 3z^2$.

If $n = 1$, then $U_{2n-2} = 0$ and $z = 0$ which is a solution giving $m = 2$. When $n \neq 1$, then $U_{2n-2} \neq 0$. Recalling Lemma 3 of [1], we see that (U_{2n-2}, V_{2n}) divides $V_2 = p$. Hence, we must check the following four possibilities:

$$U_{2n-2} = 3z_1^2, \quad V_{2n} = z_2^2 \tag{73}$$

$$U_{2n-2} = z_1^2, \quad V_{2n} = 3z_2^2 \tag{74}$$

$$U_{2n-2} = 3pz_1^2, \quad V_{2n} = pz_2^2 \tag{75}$$

$$U_{2n-2} = pz_1^2, \quad V_{2n} = 3pz_2^2 \tag{76}$$

Equation (73) has no solution by (28).

The solutions of $U_{2n-2} = z_1^2$ are ($n = 7$, $d = 5$), ($n = 2$ if $a = t^2$), and $n = 1$. For $n = 7$, we have $L_{14} = 843 \neq 3z_2^2$. For $n = 2$, $V_4 \neq 3z_2^2$ by Lemma 2 if $a \not\equiv 0 \pmod 3$, while $V_4 = 3z_2^2$ if $a \equiv 0 \pmod 3$ is obviously impossible. Since $n = 1$ is also impossible, (74) has no solutions.

If $n \equiv 0 \pmod 2$, then we can see that $V_{2n} \neq pz_2^2$ by the same argument given for Case 1.

Now let $n \equiv 1 \pmod 2$, $n \neq 1$. Since $V_{-2n} = V_{2n}$, it is sufficient to consider only the case $n \equiv 1 \pmod 4$, that is, $2n \equiv 2 \pmod 8$. We write $2n = 2 + 2t \cdot 3^s$ with $4 | t$ and $3 \nmid t$ so that $V_{2n} \equiv -V_2 \pmod{V_t}$. Applying (13) repeatedly, and taking into account that $t = 4\lambda$, we obtain

$$2V_t \equiv \pm 2V_0 \equiv \pm 4 \pmod{V_2}, \text{ that is, } (V_t, V_2) = 1,$$

which implies $p \nmid V_t$. Hence, $V_{2n} = pz^2$ implies $(pz)^2 \equiv -p^2 \pmod{V_t}$, which is impossible since $(-p^2/V_t) = -1$ by (19). Therefore, (75) has no solution.

Now let $U_{2n-2} = pz_1^2$. Equation (5) implies that $U_{n-1}V_{n-1} = pz_1^2$. If $n \not\equiv 1 \pmod 3$, then $(U_{n-1}, V_{n-1}) = 1$ by (10), and we have

$$(U_{n-1} = pz_3^2,\ V_{n-1} = z_4^2) \quad \text{or} \quad (U_{n-1} = z_3^2,\ V_{n-1} = pz_4^2).$$

By using (28) and (30), we see that both are impossible.

If $n \equiv 1 \pmod 3$, then (10) implies $(U_{n-1}, V_{n-1}) = 2$, and we have

$$(U_{n-1} = 2pz_3^2,\ V_{n-1} = 2z_4^2) \quad \text{or} \quad (U_{n-1} = 2z_3^2,\ V_{n-1} = 2pz_4^2).$$

The first is impossible by (29) and a direct computation of U_{n-1}; the second is impossible by (31) and a direct computation of V_{n-1}. [For the second case, with $n = 4$, we should have $V_3 = 2pz_4^2$, which is impossible since, otherwise, we would have $p = a^2 + 2$ dividing $V_3 = a(a^2 + 3)$].

Theorem 2′: Let $a^2 + 2 = p$ where p is a prime. The equation $U_m = 3z^2 - a$, $m \equiv 0 \pmod 2$ has

(a) the solutions $m = -2, 0, 6$, if $a = 3t^2$,
(b) only the solution $m = -2$ in all other cases.

Proof: The proof of this theorem follows that of Theorem 1′ with the exception of the case

$$U_{2n-1} = z_1^2, \quad V_{2n+1} = 3z_2^2, \text{ when } n = 1.$$

Under these conditions, we have $V_3 = 3z_2^2$, which can be transformed by simple reasoning into $27\mu^4 + 1 = v^2$, which has no solution by [2].

Corollary 1: (a) $F_m = 3z^2 + 1$ if and only if $m = \pm 1, 2, \pm 7$.

(b) $F_m = 3z^2 - 1$ if and only if $m = -2, \pm 3, \pm 5$.

4. LUCAS NUMBERS OF THE FORM $3z^2 \pm 1$

Theorem 3: Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 3z^2 + a$, $m \equiv 1 \pmod 2$ has

(a) the solutions $m = 1, 3, 9$ if $d = 5$,
(b) only the solution $m = 1$ in all other cases.

Proof: Case 1. Let $m = 4n - 1$. Equation (42) implies that $V_{2n-1}V_{2n} = 3z^2$. However, $(V_{2n-1}, V_{2n}) = 1$, so we have

$$(V_{2n-1} = z_1^2,\ V_{2n} = 3z_2^2) \quad \text{or} \quad (V_{2n-1} = 3z_1^2,\ V_{2n} = z_2^2).$$

For the first subcase, (28) implies

$$\begin{array}{lll} & n = 1 & \text{if } a = t^2,\ d \neq 5, \\ & n = 1, 2 & \text{if } \quad d = 5, \\ \text{or} & n = 2 & \text{if } \quad d = 13. \end{array}$$

When $n = 1$ and $a = t^2$, $V_{2n} = 3z_2^2$ if and only if $3z_2^2 - t^4 = 2$. Ljunggren [3] has proved that this equation possesses only the solution $(z_2, t) = (\pm 1, \pm 1)$, which gives $a = 1$ and so $d = 5$.

For $n = 2$, $d = 5$, we have $L_4 = 7 \neq 3z_2^2$, while for $n = 2$, $d = 13$, we obtain $L_{14} = 119 \neq 3z_2^2$.

By using (28) once more, we see that the second subcase has no solution.

Case 2. Let $m = 4n + 1 = 2(2n) + 1$. Equation (42) implies that $V_{2n}V_{2n+1} - 2a = 3z^2$. By (8) and (42), we see that

$$\{V_{2n}^2 - 2(-1)^n\}\{V\ V_{n+1} - (-1)^n a\} - 2a = 3z^2$$

or

$$(V_n^3 V_{n+1} - (-1)^n a V_n^2 - 2(-1)^n V_n V_{n+1} = 3z^2.$$

Hence, $V_n M_n = 3z^2$ with

$$M_n = V_n^2 V_{n+1} - (-1)^n a V_n - 2(-1)^n V_n.$$

Let p be an odd prime not equal to 3 with $p^e \| V_n$. Since $p \nmid M_n$, we have $e \equiv 0 \pmod 2$. This implies that $V_n = w^2$ or $V_n = 2w^2$ or $V_n = 3w^2$ or $V_n = 6w^2$.

When $V_n = w^2$, (28) implies

$$\begin{array}{lll} & n = 1 & \text{if } a = t^2,\ d \neq 5, \\ & n = 1, 3 & \text{if } \quad d = 5, \\ \text{or} & n = 3 & \text{if } \quad d = 13. \end{array}$$

When $n = 1$ and $a = t^2$, we have $m = 5$. Hence, we must examine the equation

$$a^5 + 5a^3 + 5a = 3z^2 + a$$

for solutions. According to our assumptions, this equation can be written as

$$(a^2 + 2)^2 + a^2 = 3f^2.$$

However, $a^2 \equiv 1 \pmod{12}$ and $3f^2 \equiv 10 \pmod{12}$, so the equation is unsolvable.

By direct calculation, we can show that for all other possible values of n no solutions exist.

Let $V_n = 2w^2$. Using (29) and direct calculation, we find that the unique solution in this case is $n = 0$ or $m = 1$.

Let $V_n = 3w^2$. In this case, Lemma 2 implies that solutions exist only for $n = \pm 2$ if $a^2 + 2 = 3w_1^2$.

When $n = -2$, we have $m = -7$. Since $V_{-7} < 0$, we know that $V_{-7} \neq 3z + a$. Hence, we have only to check the case for $n = 2$ or $m = 9$, that is, the possible solutions of the equation $a^9 + 9a^7 + 27a^5 + 30a^3 + 9a = 3z^2 + a$. Factoring, we have $a(a^2 + 2)(a^6 + 7a^4 + 13a^2 + 4) = 3z^2$ which, by replacing $a^2 + 2$ with $3w_1^2$, becomes $a(a^6 + 7a^4 + 13a^2 + 4) = w_2^2$.

However, $(a, a^6 + 7a^4 + 13a^2 + 4) = (a, 4) = 1$, so it follows that

$$a^6 + 7a^4 + 13a^2 + 4 = s^2 \quad \text{or} \quad (a^2 + 4)(a^4 + 3a^2 + 1) = s^2.$$

Now, the greatest common divisor tells us that

$$\begin{aligned}(a^2 + 4, a^4 + 3a^2 + 1) &= (a^2 + 4, (a^2 + 4) - 5(a^2 + 3)) \\ &= (a^2 + 4, 5(a^2 + 3)) \\ &= (a^2 + 4, 5) = 1 \text{ or } 5.\end{aligned}$$

If $(a^2 + 4, a^4 + 3a^2 + 1) = 1$, it follows that $a^2 + 4 = \lambda^2$ with $a = t^2$. This implies $a = 0$, which is impossible since $a > 0$.

Now let $(a^2+4, a^4+3a^2+1) = 5$. Then $a^2 + 4 = 5\lambda_1^2$ and $a^4 + 3a^2 + 1 = 5\lambda_2^2$ with $a = t^2 \equiv 1 \pmod 6$. Recall that $t^4 - 5\lambda_1^2 = -4$ has the solutions $t = 1$ and $t = 2$ by (28). When $t = 1$, $a = 1$ and $d = 5$. When $t = 2$, $a = 4$, which is impossible since $a \equiv 1 \pmod 2$.

Therefore, in this case, we have only the solution $m = 9$, $d = 5$.

By Lemma 4, $V_n = 6w^2$ has no solutions.

Following the arguments of Theorem 3, we can also show

Theorem 4: Let $a \not\equiv 0 \pmod 3$. Then the equation $V_m = 3z^2 - a$, $m \equiv 1 \pmod 2$ has

(a) the solutions $m = -1, 5$ if $d = 5$,
(b) only the solution $m = -1$ in all other cases.

Theorem 5: The equation $L_m = 3z^2 + 1$, $m \equiv 0 \pmod 2$ has no solution.

Proof: **Case 1.** Let $m = 4n$. Equation (8) implies that $L_{2n}^2 = L_{4n} + 2$, which is the same as $3z^2 + 1 = L_{2n}^2 - 2$. Hence, $3(z^2 + 1) = L_{2n}^2$, so that $3 | L_{2n}$. Therefore, $2n \equiv 2 \pmod 4$ or $m \equiv 4 \pmod 8$. Since for even m, $L_{-m} = L_m$, it is sufficient to consider only the case $m \equiv 4(16)$.

If $m = 4$, then $L_4 = 7 \neq 3z^2 + 1$.

Let $m \neq 4$. We write $m = 4 + 2n3$ with $8 | n$, $3 \nmid n$. Then $V_m \equiv -V \pmod{V_n}$ by (22). If $V_m = 3z^2 + 1$, we have $(3z)^2 \equiv -24 \pmod{V_n}$, where $8 | n$ and $3 \nmid n$. Since for $8 | n$, $V_n \equiv 2 \pmod 3$, we can now apply the Jacobi symbol which is calculated to be -1, by (19) and (20). Hence, no solution exists.

Case 2. Let $m = 4n + 2$. Equation (8) gives $L_{2n+1}^2 = L_{4n+2} - 2$ or $L_{2n+1}^2 = 3z^2 - 1$. But $L_{2n+1}^2 - 5F_{2n+1}^2 = -4$ and so $5F_{2n+1}^2 = 3(z^2 + 1)$. This implies that

$3 | F_{2n+1}$, which is impossible since 3 divides F_m if and only if 4 divides m. Hence, in this case also, there are no solutions.

Theorem 6: The equation $L_m = 3z^2 - 1$, $m \equiv 0 \pmod 2$ has only the solutions $m = 0, \pm 8$.

Proof: The proof is the same as that of Theorem 3, where we take into account the fact that $L_m \equiv -1 \pmod{23}$ if 16 divides n.

Corollary 2: (a) $L_m = 3z^2 + 1$ if and only if $m = 1, 3, 9$.

(b) $L_m = 3z^2 - 1$ if and only if $m = -1, 0, 5, 18$.

Remark: We can apply (26) and (27) as in [1] in order to obtain some statements about the solutions of diophantine equations of the form

$$DY^2 = AX^4 + BX^2 + C.$$

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Note: All the particular cases listed in (4.2) are referenced in Gould [1] except P. F. Byrd, "Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers," *The Fibonacci Quarterly 1*, no. 1 (1963):16-24.

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GENERALIZED WYTHOFF NUMBERS FROM SIMULTANEOUS FIBONACCI REPRESENTATIONS

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1. INTRODUCTION

In developing a Zeckendorf theorem for double-ended sequences, Hoggatt and Bicknell-Johnson [1] found a remarkable pattern arising from applying Klarner's theorem [2],[3] on simultaneous representations using Fibonacci numbers. Here we study the properties of the array generated, after first providing enough background information to make this paper self-contained. We shall show relationships with the Lucas numbers, the Wythoff pair sequences, and generalized Wythoff numbers [7].

David Klarner [2] has proved

Klarner's Theorem: Given nonnegative integers A and B, there exists a unique set of integers $\{k_1, k_2, k_3, \ldots, k_r\}$ such that

$$A = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \quad B = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1},$$

for $|k_i - k_j| \geqslant 2$, $i \neq j$, where each F_i is an element of the sequence $\{F_i\}_{-\infty}^{\infty}$, $F_{i+1} = F_i + F_{i-1}$, $F_1 = 1$, $F_2 = 1$.

Thus, to represent a single integer $m > 0$, we merely solve

$$A = 0 = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}, \quad B = m = F_{k_1} + F_{k_2} + \cdots + F_{k_r},$$

which has a unique solution by Klarner's Theorem. A constructive method of solution is given in [3], and we will soon use this idea to generate a most interesting array.

We shall also need some properties of Wythoff pairs (a_n, b_n), which are formed by letting $a_1 = 1$ and taking a_n as the smallest positive integer not yet used, and letting $b_n = a_n + n$. Wythoff pairs have been discussed, among other sources, in [4], [5], [6], [7], and [8]. Early values are shown below.

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_n:	1	3	4	6	8	9	11	12	14	16	17	19	21	22
b_n:	2	5	7	10	13	15	18	20	23	26	28	31	34	36

We list the following properties:

$$a_k + k = b_k \tag{1.1}$$

$$a_{b_n} = a_n + b_n \quad \text{and} \quad b_{b_n} = a_n + 2b_n \tag{1.2}$$

$$a_{a_n} = b_n - 1 \quad \text{and} \quad b_{a_n} = a_n + b_n - 1 \tag{1.3}$$

$$a_{k+1} - a_k = \begin{cases} 2, & k = a_n \\ 1, & k = b_n \end{cases} \tag{1.4}$$

$$b_{k+1} - b_k = \begin{cases} 3, & k = a_n \\ 2, & k = b_n \end{cases} \tag{1.5}$$

Further, (a_n, b_n) are related to the Fibonacci numbers in several ways, one being that, if $A = \{a_n\}$ and $B = \{b_n\}$, then A and B are the sets of positive integers for which the smallest Fibonacci number used in the unique Zeckendorf representation has respectively an even or an odd subscript [9].

Also, the Wythoff pairs are related to the Golden Section Ratio

$$\alpha = (1 + \sqrt{5})/2,$$

and recall that $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where $\beta = 1/\alpha$, as

$$a_n = [n\alpha], \qquad b_n = [n\alpha^2], \tag{1.6}$$

where $[x]$ is the greatest integer in x.

Lastly, we recall the generalized Wythoff numbers A_n, B_n, and C_n of [7] with beginning values

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_n:	1	4	5	8	11	12	15	16	19	22	23	26	29	30
B_n:	3	7	10	14	18	21	25	28	32	36	39	43	47	50
C_n:	2	6	9	13	17	20	24	27	31	35	38	42	46	49

and the following properties useful in this paper:

$$A_n = 2a_n - n \tag{1.7}$$

$$B_n = a_n + 2n = b_n + n \tag{1.8}$$

$$C_n = a_n + 2n - 1 = b_n + n - 1 = a_{a_n} + n \tag{1.9}$$

$$C_n + 1 = B_n \quad \text{and} \quad C_n - 1 = A_{a_n} \tag{1.10}$$

$$A_{n+1} - A_n = \begin{cases} 1, & n = b_k \\ 3, & n = a_k \end{cases} \tag{1.11}$$

$$B_{n+1} - B_n = \begin{cases} 3, & n = b_k \\ 4, & n = a_k \end{cases} \tag{1.12}$$

$$C_{n+1} - C_n = \begin{cases} 3, & n = b_k \\ 4, & n = a_k \end{cases} \tag{1.13}$$

$$A_{a_n} = a_n + 2n - 2 \quad \text{and} \quad B_{a_n} = 3a_n + n - 1 \tag{1.14}$$

$$A_{a_{b_n}} = A_{b_{a_n}} + 1 = A_{b_{a_n}+1} \tag{1.15}$$

The sequences A_n, B_n, and C_n divide the positive integers into three disjoint subsets, classified by Zeckendorf representation using Lucas numbers [9].

2. AN ARRAY ARISING FROM KLARNER'S DUAL ZECKENDORF REPRESENTATION

Recall the Klarner dual Zeckendorf representation given in §1, where

$$\begin{cases} A = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} = 0 \\ B = F_{k_1} + F_{k_2} + \cdots + F_{k_r} = n, \end{cases} \tag{2.1}$$

where $n = 1, 2, 3, \ldots, |k_i - k_j| \geqslant 2, i \neq j$, and the Fibonacci number F_j comes from the double-ended sequence $\{F_j\}_{-\infty}^{\infty}$. The constructive method described in our earlier work [3] for solving for the subscripts k_j to represent A and B leads to a symbolic display with a generous sprinkling of Lucas numbers L_n ($L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1} + L_n$) and Wythoff pairs.

Here we use only two basic formulas,

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad 2F_n = F_{n+1} + F_{n-2}, \tag{2.2}$$

to push both right and left in forming successive lines of the array. The display is for expressions for B only; A is a translation of one space to the right. At each step, $B = n$ and $A = 0$.

The basic column centers under F_{-1}. We continue to add $F_{-1} = 1$ at each step, using the rules given in (2.2) to simplify the result. For example, for $n = 1$, we have $F_{-1} = 1$. For $n = 2$, $F_{-1} + F_{-1} = 2F_{-1} = F_0 + F_{-3} = 2$. For $n = 3$, $F_{-1} + F_0 + F_{-3}$ becomes $F_1 + F_{-3} = 1 + 2 = 3$. We display Table 2.1 on the following page.

Many patterns are discernible from Table 2.1. There are always the same number of successive entries in a given column. Under F_{-2} there are L_1; under F_{-3}, L_2; under F_{-4}, L_3; and under F_{-k}, L_{k+1} successive entries. The columns to the right of F_{-1} (under F_0, for instance) have $L_n \pm 1$ alternately successive entries, but the same number of successive entries always appears in a given column. Also, we notice that once we have all spaces cleared except the extreme edges in the pattern being built, we start again in the middle, as in lines 4, 8, 19, 48, ..., $L_{2k} + 1$, ...

Reading down the columns, we write the sequence of numbers first using that F_k is its representation. For example, the sequence of numbers using F_{-1} is 1, 4, 8, 11, 15, 19, ..., with first difference $\Delta_1 = 3$ and second difference $\Delta_2 = 4$. We want only the numbers first used when reading down the columns, so for F_{-3} we would use 2, 9, 20, 27, ..., and ignore 3, 4, 10, 11, 21, 22, ... We list sequences appearing beneath F_k in Table 2.1 along with first and second differences:

F_0:	2, 6, 9, 13, 17, 20, 24, 27, ...	$\Delta_1 = 3$, $\Delta_2 = 4$
F_1:	3, 10, 14, 21, 28, 32, 39, 43, ...	$\Delta_1 = 7$, $\Delta_2 = 4$
F_2:	5, 16, 23, 34, 45, 52, ...	$\Delta_1 = 11$, $\Delta_2 = 7$
F_3:	7, 25, 36, 54, 72, ...	$\Delta_1 = 18$, $\Delta_2 = 11$
F_4:	12, 41, 59, 88, ...	$\Delta_1 = 29$, $\Delta_2 = 18$
F_{-1}:	1, 4, 8, 11, 15, 19, 22, 26, ...	$\Delta_1 = 3$, $\Delta_2 = 4$
F_{-2}:	5, 12, 16, 23, 30, 34, 41, 45, ...	$\Delta_1 = 7$, $\Delta_2 = 4$

(continued)

Table 2.1 F_n Used To Represent B from Klarner's Theorem

Subscript n:

B	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1									x								
2							x			x							
3							x				x						
4							x		x		x						
5					x			x				x					
6					x					x		x					
7					x								x				
8					x				x				x				
9					x		x			x			x				
10					x		x				x		x				
11					x		x		x		x		x				
12			x			x		x						x			
13			x			x				x				x			
14			x			x					x			x			
15			x			x			x		x			x			
16			x					x				x		x			
17			x							x		x		x			
18			x												x		
19			x						x						x		
20			x				x			x					x		
21			x				x				x				x		
22			x				x		x		x				x		
23			x		x			x				x			x		
24			x		x					x		x			x		
25			x		x								x		x		
26			x		x				x				x		x		
27			x		x		x			x			x		x		
28			x		x		x				x		x		x		
29			x		x		x		x		x		x		x		
30	x			x		x		x								x	
31	x			x		x				x						x	
32	x			x		x					x					x	
33	x			x		x			x		x					x	
34	x			x				x				x				x	
35	x			x						x		x				x	
36	x			x									x			x	
37	x			x					x				x			x	
38	x			x			x			x			x			x	
39	x			x			x				x		x			x	
40	x			x			x		x		x		x			x	
41	x					x		x						x		x	
42	x					x				x				x		x	
43	x					x					x			x		x	
44	x					x			x		x			x		x	
45	x							x				x		x		x	
46	x									x		x		x		x	
47	x																x
48	x								x								x
49	x						x			x							x
50	x						x				x						x
51	x						x		x		x						x
52	x				x			x				x					x
53	x				x					x		x					x
54	x				x								x				x
55	x				x				x				x				x
56	x				x		x			x			x				x
57	x				x		x				x		x				x
58	x				x		x		x		x		x				x
59	x		x			x		x						x			x
60	x		x			x				x				x			x

F_{-3}:	2, 9, 20, 27, 38, 49, ...	$\Delta_1 = 7$, $\Delta_2 = 11$
F_{-4}:	12, 30, 41, 59, 77, 88, ...	$\Delta_1 = 18$, $\Delta_2 = 11$
F_{-5}:	5, 23, 52, 70, 99, ...	$\Delta_1 = 18$, $\Delta_2 = 29$

Surely the reader sees the Lucas numbers 1, 3, 4, 7, 11, 18, 29, ..., as the first and second differences. In the next section, we write formulas for each term in the sequences given, and find both Lucas numbers and the Wythoff pair numbers.

As a final observation, notice that the sequences associated with F_k when k is a negative odd integer have different behavior than all the others listed. For those sequences, $\Delta_2 > \Delta_1$, and successive differences follow the pattern $\Delta_1, \Delta_2, \Delta_1, \Delta_2, \Delta_2, \ldots$, while all the others have $\Delta_2 < \Delta_1$ and a pattern of successive differences that begins $\Delta_1, \Delta_2, \Delta_1, \Delta_1, \Delta_2, \ldots$.

3. LUCAS NUMBERS AND THE WYTHOFF PAIRS

We write the general term u_n for the sequence of numbers first using F_k in its representation as observed from Table 2.1 for $k \geqslant 0$.

F_0: $u_n = 2n + a_n - 1$

F_1: $u_n = n + 3a_n - 1$

F_2: $u_n = 3n + 4a_n - 2$

F_3: $u_n = 4n + 7a_n - 4$

F_4: $u_n = 7n + 11a_n - 6$

F_5: $u_n = 11n + 18a_n - 11$

F_6: $u_n = 18n + 29a_n - 17$

Again we see the Lucas numbers L_n, defined by

$$L_1 = 1,\ L_2 = 3, \text{ and } L_{n+1} = L_n + L_{n-1}.$$

Observe that the last terms are either L_n or one less than L_n, and the pattern of general terms seems to be

$$F_k:\quad u_n = L_k n + L_{k+1}a_n - [L_k - (1 + (-1)^k)/2],$$

where a_n is the first member of a Wythoff pair.

Theorem 3.1: The sequence of numbers first using F_k, $k \geqslant 0$, in its representation arising from Klarner's theorem is given by

$$F_k:\quad u_n = nL_k + a_nL_{k+1} - [L_k - (1 + (-1)^k)/2].$$

Proof: From [8], all Fibonacci representations can be put in the form

$$u_n = (2n - 1 - a_n)\Delta_2 + (a_n - n)\Delta_1 + u_1, \tag{3.1}$$

where Δ_1 and Δ_2 are the first and second differences and u_1 is the beginning term of the sequence. By the method of generation of the array,

$$u_1 = \begin{cases} L_{k+1}, & k \text{ even} \\ L_{k+1} + 1, & k \text{ odd} \end{cases}$$

and $\Delta_2 = L_{k+2}$, $\Delta_1 = L_{k+3}$ for $k \geqslant 1$. Substitution of these values into (3.1) yields the result quite quickly.

For $k = 0$, we note that the sequence for F_0 can be written from (3.1) by letting $\Delta_1 = 3$, $\Delta_2 = 4$, and $u_1 = 2$.

The sequence of general terms for the sequences using F_k when k is negative gives us a different story. First, take k negative and even:

$$F_{-2}: \quad u_n = n + 3a_n + 1 \qquad \Delta_1 = 7,\ \Delta_2 = 4$$

$$F_{-4}: \quad u_n = 4n + 7a_n + 1 \qquad \Delta_1 = 18,\ \Delta_2 = 11$$

suggesting

$$F_{-k}: \quad u_n = nL_{k-1} + a_nL_k + 1 \qquad \Delta_1 = L_{k+2},\ \Delta_2 = L_{k+1}$$

When k is negative and odd, we let $m = n - 1$ and list

$$F_{-1}: \quad u_n = 2m + a_m + 1$$

$$F_{-3}: \quad u_n = 3m + 4a_m + 2$$

$$F_{-5}: \quad u_n = 7m + 11a_m + 5$$

suggesting

$$F_{-k}: \quad u_n = mL_{k-1} + a_mL_k + L_{k-2} + 1.$$

Theorem 3.2: The sequence of numbers first using F_{-k} in its representation is given by

(i) F_{-2j}: $u_n = nL_{2j-1} + a_nL_{2j} + 1$;

(ii) F_{-1}: $u_n = 2(n - 1) + a_{n-1} + 1$;

(iii) $F_{-(2j+1)}$, $j > 0$: $u_n = (n - 1)L_{2j} + a_{n-1}L_{2j+1} + L_{2j-1} + 1$.

Proof: (i) follows readily from (3.1) by taking

$$\Delta_1 = L_{2j+2},\ \Delta_2 = L_{2j+3}, \text{ and } u_1 = L_{2j+1} + 1.$$

(ii) is proved by mathematical induction. Note that (ii) is true for early values. Study the pattern of successive differences $\Delta_1 = 3$, $\Delta_2 = 4$, and by the rules for generation of the array, we have

$$u_{n+1} - u_n = \begin{cases} 3, & n - 1 = b_i \\ 4, & n - 1 = a_j \end{cases}$$

Assume $u_k = 2(k - 1) + a_{k-1} + 1$. Then, when $k - 1 = b_i$, (1.4) lets us write

$$\begin{aligned} u_{k+1} = 3 + u_k &= 3 + 2(k - 1) + a_{k-1} + 1 \\ &= 3 + 2(k - 1) + a_k \\ &= 2k + a_k + 1. \end{aligned}$$

When $k - 1 = a_j$, we again apply (1.4), and

$$\begin{aligned} u_{k+1} = 4 + u_k &= 4 + 2(k - 1) + a_{k-1} + 1 \\ &= 2k + (a_{k-1} + 2) + 1 \\ &= 2k + a_k + 1, \end{aligned}$$

so that u_{k+1} again has the form of (ii), establishing (ii) by mathematical induction.

The general case (iii) can be proved by mathematical induction in a similar way by using (1.4), if we take $\Delta_1 = L_{2j+2}$, $\Delta_2 = L_{2j+3}$, and $u_1 = L_{2j-1} + 1$. We again have Δ_1 when $n - 1 = b_i$ and Δ_2 when $n - 1 = a_j$.

Corollary 3.2: A second formula for the sequence of numbers first using F_{-k} in its representation is given by

$$\begin{aligned} F_{-2j}&: \quad u_n = b_n L_{2j-1} + a_n L_{2j-2} + 1, \ j > 0; \\ F_{-1}&: \quad u_n = n + b_{n-1} \\ F_{-(2j+1)}&: \quad u_n = b_{n-1} L_{2j} + (a_{n-1} + 1) L_{2j-1} + 1, \ j > 0. \end{aligned}$$

Proof: Change the form of the sequence for F_{2j} given in Theorem 3.2 by applying (1.1):

$$\begin{aligned} u_n = nL_{2j-1} + a_n L_{2j} + 1 &= nL_{2j-1} + a_n L_{2j-1} + a_n L_{2j-2} + 1 \\ &= b_n L_{2j-1} + a_n L_{2j-2} + 1. \end{aligned}$$

Again apply (1.1) to F_{-1}:

$$\begin{aligned} u_n = 2(n-1) + a_{n-1} + 1 &= (n-1) + (n - 1 + a_{n-1}) + 1 \\ &= n + b_{n-1}. \end{aligned}$$

The proof for $F_{-(2j+1)}$ is similar.

If we take k negative and odd, and apply (3.1) to write the terms of the sequences, we observe

$$\begin{aligned} F_{-1}&: \quad u_n = 5n - a_n - 3 \\ F_{-3}&: \quad u_n = 15n - 4a_n - 9 \\ F_{-5}&: \quad u_n = 40n - 11a_n - 24 \end{aligned}$$

leading us to

Theorem 2.3: If k is odd and greater than 1, then the sequence of numbers first using F_{-k} in its representation arising from Klarner's Theorem is given by

$$F_{-k}: \quad u_n = 5nF_{k+1} - L_k a_n - 5F_k + 1.$$

Proof: Let $u_1 = L_{k-2} + 1$, $\Delta_2 = L_{k+2}$, and $\Delta_1 = L_{k+1}$ in (3.1) and simplify using $L_{k+2} + L_k = 5F_{k+1}$.

4. THE GENERALIZED WYTHOFF NUMBERS

The generalized Wythoff numbers A_n, B_n, and C_n of [7] provide another description of the general term of the sequences arising from using F_k in the representation from Klarner's theorem. Observe that, for F_0,

$$u_n = 2n + a_n - 1 = C_n$$

by (1.9). Each sequence we have generated is a subsequence of the sequence for A_n, B_n, or C_n. The sequences for F_0 and F_{-3} contain only C_i's, while the sequences for F_{2k+1} contain only B_i's, $k \geqslant 0$. All of the other sequences contain A_i's exclusively.

Theorem 4.1: The sequences arising from first using F_{2k+1}, $k \geqslant 0$, in the representation from Klarner's Theorem are

$$F_1: \quad u_n = B_{a_n}$$

$$F_3: \quad u_n = B_{b_{a_n}}$$

$$F_5: \quad u_n = B_{b_{b_{a_n}}}$$

$$F_{2k+1},\ k \geqslant 0: \quad u_n = B_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k}}$$

Proof: We simplify the form B_i to demonstrate that u_n has the form given by Theorem 3.1. For F_1, observe (1.14).

For F_3, we apply (1.8) and then (1.2) and (1.3) in sequence finishing with (1.1) to obtain

$$B_{b_{a_n}} = a_{b_{a_n}} + 2b_{a_n} = (a_{a_n} + b_{a_n}) + 2b_{a_n} = (b_n - 1) + 3(a_n + b_n - 1)$$

$$= 4b_n + 3a_n - 4 = 4(n + a_n) + 3a_n - 4 = 4n + 7a_n - 4.$$

For F_5, we apply the same sequence of steps repeatedly to reduce the subscripted subscripts. For F_{2k+1}, the reduction of subscripted subscripts will always follow the same steps repeatedly. We show Lemma 4.1 to demonstrate one step of the subscript-reduction process and to show that we will end with the required form in terms of Lucas numbers.

Lemma 4.1: $$L_{i+1} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k}} + L_i \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k} = L_{i+3} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}} + L_{i+2} \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}$$

Proof: Apply (1.2) followed by (1.1).

$$L_{i+1} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k}} + L_i \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k} = L_{i+1} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}} + L_{i+1} \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k} + L_i \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k}$$

$$= L_{i+1} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}} + L_{i+2} \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k}$$

$$= L_{i+1} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}} + L_{i+2} \left(\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1} + a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}} \right)$$

$$= L_{i+3} a_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}} + L_{i+2} \underbrace{b_{b_{\cdot^{\cdot^{\cdot^{b_{a_n}}}}}}}_{k-1}$$

Theorem 4.2: F_0: $u_n = C_n$, F_{-1}: $u_n = A_{a_{n-1}+1}$, and F_{-3}: $u_n = C_{a_{a_{n-1}+1}}$

Proof: The form for F_0 follows by comparing (1.9) and Theorem 3.1. For F_{-1}, we apply (1.11) and (1.14) and then compare with Theorem 3.2:

$$A_{a_{n-1}+1} = A_{a_{n-1}} + 3 = (a_{n-1} + 2(n - 1) - 2) + 3 = 2(n - 1) + a_{n-1} + 1$$

For F_{-3}, by (1.9) followed by (1.3), (1.4), and (1.5):

$$C_{a_{a_{n-1}+1}} = a_{a_{a_{n-1}+1}} + 2a_{a_{n-1}+1} - 1 = (b_{a_{n-1}+1} - 1) + 2a_{a_{n-1}+1} - 1$$

$$= (b_{a_{n-1}} + 3) - 1 + 2(a_{a_{n-1}} + 2) - 1 = b_{a_{n-1}} + 2a_{a_{n-1}} + 5$$

Next, use (1.3) finished by (1.1),

$$C_{a_{a_{n-1}+1}} = (a_{n-1} + b_{n-1} - 1) + 2(b_{n-1} - 1) + 5 = a_{n-1} + 2b_{n-1} + 2$$

$$= a_{n-1} + 3(a_{n-1} + (n - 1)) + 2 = 3(n - 1) + 4a_{n-1} + 2,$$

and compare with Theorem 3.2.

Theorem 4.3: F_2: $u_n = A_{a_{b_n}} = A_{b_{a_n}+1}$

F_4: $u_n = A_{b_{b_{a_n}}+1}$

F_{2k}, $k > 0$: $u_n = A_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot}}}}}_{k}+1}$ b_{a_n}

Proof: For F_2, use (1.15) and (1.14), followed by (1.2) and (1.1):

$$A_{a_{b_n}} = a_{b_n} + 2b_n - 2 = (b_n + a_n) + 2b_n - 2$$

$$= 3(n + a_n) + a_n - 2 = 3n + 4a_n - 2$$

Then compare with u_n as given in Theorem 3.1.

For F_4, first apply (1.15) and then (1.7). After than, use (1.2) followed by (1.1) repeatedly to reduce the subscripted subscripts.

$$A_{b_{b_{a_n}}+1} = A_{b_{b_{a_n}}} + 1 = 2a_{b_{b_{a_n}}} - b_{b_{a_n}} + 1 = 2(a_{b_{a_n}} + b_{b_{a_n}}) - b_{b_{a_n}} + 1$$

$$= 2a_{b_{a_n}} + (a_{b_{a_n}} + b_{a_n}) + 1 = 3(a_{a_n} + b_{a_n}) + b_{a_n} + 1$$

$$= 3a_{a_n} + 4(a_{a_n} + a_n) + 1 = 7(b_n - 1) + 4a_n + 1$$

$$= 7(a_n + n) + 4a_n - 6 = 7n + 11a_n - 6$$

Now compare with Theorem 3.1.

For F_{2k}, the steps are always the same as for F_4, except for more repetitions.

Theorem 4.4: F_{-2}: $u_n = A_{b_n+1}$

F_{-4}: $u_n = A_{b_{b_n}+1}$

F_{-2k}, $k > 0$: $u_n = A_{\underbrace{b_{b_{\cdot^{\cdot^{\cdot}}}}}_{k}+1}$ b_n

Proof: Use (1.15) and (1.7). Then reduce the subscripted subscripts repeatedly by applying (1.2) followed by (1.1), and compare with Theorem 3.2. Because the proof is so much like that for F_4 and F_{2k} in Theorem 4.3, we show only F_{-2}.

$$A_{b_n+1} = A_{b_n} + 1 = 2a_{b_n} - b_n + 1 = 2(a_n + b_n) - b_n + 1$$
$$= 2a_n + (a_n + n) + 1 = 3a_n + n + 1.$$

Theorem 4.5: F_{-5}: $u_n = A_{a_{b_{a_{a_{n-1}+1}}}}$

F_{-7}: $u_n = A_{a_{a_{a_{b_{a_{a_{n-1}+1}}}}}}$

$F_{-(2k+1)}$, $k \geqslant 2$: $u_n = A_{\underbrace{a_{a_{\cdot^{\cdot^{\cdot^{a}}}}}}_{2k-3}{}_{b_{a_{a_{n-1}+1}}}}$

Proof: In a manner similar to the proofs of Theorem 4.2 and Theorem 4.3, the subscripted subscripts can be painfully reduced, eventually, to match the form of Theorem 3.2. But, we almost have subscripted subscripts using the Wythoff pairs numbers a_n and b_n, except for the last subscript.

We apply results of [8]. Let $U = \{u_n\}_{n=1}^{\infty}$ be a sequence of integers. If U^* is a subsequence of U such that the general term is formed by subscripted subscripts taken from the Wythoff pair numbers, then we give each a-subscript weight 1 and each b-subscript weight 2. Then, U^* has first and second differences Δ_1^* and Δ_2^* given by

$$\Delta_2^* = F_{w+1}\Delta_2 + F_w\Delta_1 \quad \text{and} \quad \Delta_1^* = F_w\Delta_2 + F_{w-1}\Delta_1,$$

where w is the weight of the sequence and Δ_1 and Δ_2 are the first and second differences of U, the original sequence.

Notice that F_{-5} has weight 4 because the last subscript could be either a_i or b_j. $A_{a_{n-1}+1}$ is the original sequence, so we have $\Delta_1 = 3$, $\Delta_2 = 4$ because, by Theorem 4.2, we are looking at the sequence for F_{-1}. Then

$$\Delta_2^* = 4F_5 + 3F_4 = 4 \cdot 5 + 3 \cdot 3 = 29 = L_7$$
$$\Delta_1^* = 4F_4 + 3F_3 = 4 \cdot 3 + 3 \cdot 2 = 18 = L_6$$

where these are the known value for F_{-5}. Since we know u_1 for F_{-5}, we must have the same sequence.

For $F_{-(2k+1)}$, $k \geqslant 2$, the weight is $2k$, and

$$\Delta_2^* = 4F_{2k+1} + 3F_{2k} = L_{2k+3}$$
$$\Delta_1^* = 4F_{2k} + 3F_{2k-1} = L_{2k+2},$$

which we recognize from earlier sections.

Discussion: The weights for all of the other sequences for F_k are easier to calculate. For example, F_{2k} in Theorem 4.3 has weight $2k + 1$ and we can use A_n as the original sequence, with $\Delta_1 = 3$, $\Delta_2 = 1$, so that

$$\Delta_1^* = 3F_{2k+2} + F_{2k+1} = L_{2k+3} \quad \text{and} \quad \Delta_2^* = 3F_{2k+1} + F_{2k} = L_{2k+2}$$

which we recognize. From Theorem 4.1, the weight of F_{2k+1} is also $2k + 1$, and B_n gives the original sequence, so that $\Delta_1 = 4$, $\Delta_2 = 3$,

$$\Delta_1^* = 4F_{2k+2} + 3F_{2k+1} = L_{2k+4} \quad \text{and} \quad \Delta_2^* = 4F_{2k+1} + 3F_{2k} = L_{2k+3},$$

which again are known from earlier work.

Notice that we can use original sequences to relate all of the sequences of this paper to the sequences for F_0, F_1, F_{-1}, and F_{-2}, by looking at the next to last subscript in u_n. The original sequence related to $F_{-(2k+1)}$ then is

$$A_{a_{n-1}+1},$$

the sequence for F_{-1}. Even the sequence for F_{-3} is so related, because

$$C_{a_{n-1}+1} = A_{a_{n-1}+1} + 1.$$

Now, F_{2k+1} has original sequence B_{a_n}, which is F_1, while F_{-2k} goes with A_{b_n}, which gives F_{-2}. Lastly, F_{2k} has original sequence A_{a_n}, which is related to F_0, since $C_n = A_{a_n} + 1$.

Further, all of the sequences are related to the sequences for F_{-1}, F_0, or F_1. All of the sequences for F_{2k+1} are subsequences of B_n and thus are related to F_1; F_{-3} and F_0 have sequences that are subsequences of C_n. All of the other sequences are subsequences of A_n, making them related to the sequence for F_{-1}.

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INVERSE TRIGONOMETRICAL SUMMATION FORMULAS INVOLVING PELL POLYNOMIALS

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1. INTRODUCTION

Pell polynomials $P_n(x)$ and *Pell-Lucas Polynomials* $Q_n(x)$ are defined in [3] by the recurrence relation and initial conditions

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \qquad P_0(x) = 0,\ P_1(x) = 1 \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \qquad Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2}$$

Properties of these polynomials are also set out in [3]. Among these, the most important for our current purposes are the following:

$$\left.\begin{aligned} P_n(x) &= \frac{\alpha^n - \beta^n}{\alpha - \beta} && (1.3)\\ \text{and}\quad Q_n(x) &= \alpha^n + \beta^n && (1.4)\end{aligned}\right\}\ \textit{Binet forms}$$

where

$$\alpha = x + \sqrt{x^2+1}, \qquad \beta = x - \sqrt{x^2+1} \tag{1.5}$$

are the roots of the characteristic equation

$$\lambda^2 - 2x\lambda - 1 = 0 \tag{1.6}$$

of the recurrences (1.1) and (1.2), so that:

$$\alpha + \beta = 2x, \quad \alpha - \beta = 2\sqrt{x^2+1}, \quad \alpha\beta = -1; \tag{1.7}$$

$$\left.\begin{aligned} P_{n+1}(x)P_{n-1}(x) - P_n^2(x) &= (-1)^n, && (1.8)\\ \text{and}\quad Q_{n+1}(x)Q_{n-1}(x) - Q_n^2(x) &= (-1)^{n-1}\cdot 4(x^2+1); && (1.9)\end{aligned}\right\}\ \textit{Simson's formulas}$$

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x); \tag{1.10}$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2+1)P_n(x); \tag{1.11}$$

$$P_n(x)Q_n(x) = P_{2n}(x). \tag{1.12}$$

When $x = 1$, $P_n(1) = P_n$ and $Q_n(1) = Q_n$ reduce to the Pell numbers and the "Pell-Lucas" numbers, respectively. On the other hand, $x = \frac{1}{2}$ leads to $P_n(\frac{1}{2}) = F_n$ and $Q_n(\frac{1}{2}) = L_n$, the Fibonacci and Lucas numbers, respectively.

Analogous results to some of those obtained below occur in [2], which provided the stimulus for this article.

2. INVERSE TANGENT AND COTANGENT FORMULAS

Calculation using (1.3) yields

$$P_{2n+1}(x)P_{2n+2}(x) - P_{2n}(x)P_{2n+3}(x) = 2x \qquad (n \geqslant 0). \tag{2.1}$$

Substituting for $P_{2n+3}(x)$ from (1.1) and rearranging, we obtain

$$P_{2n}(x) = \frac{(P_{2n+1}(x)/2x)P_{2n+2}(x) - 1}{(P_{2n+1}(x)/2x) + P_{2n+2}(x)}, \tag{2.2}$$

which can be expressed trigonometrically as

$$\cot^{-1}P_{2n}(x) = \cot^{-1}(P_{2n+1}(x)/2x) + \cot^{-1}P_{2n+2}(x). \tag{2.3}$$

Summing, we derive

$$\sum_{r=0}^{n} \cot^{-1}(P_{2r+1}(x)/2x) = \frac{\pi}{2} - \cot^{-1}P_{2n+2}(x), \tag{2.4}$$

since $\cot^{-1}0 = \pi/2$. Setting $x = 1$ and letting $n \to \infty$, we have a result about Pell numbers P_n:

$$\sum_{r=0}^{\infty} \cot^{-1}(P_{2r+1}/2) = \frac{\pi}{2}, \tag{2.5}$$

while putting $x = \frac{1}{2}$ leads to the known summation formula involving Fibonacci numbers F_n:

$$\sum_{r=0}^{\infty} \cot^{-1}F_{2r+1} = \frac{\pi}{2}. \tag{2.6}$$

Next,

$$\tan^{-1}\left(\frac{1}{P_{2r-2}(x)}\right) - \tan^{-1}\left(\frac{1}{P_{2r}(x)}\right) = \tan^{-1}\left(\frac{P_{2r}(x) - P_{2r-2}(x)}{1 + P_{2r}(x)P_{2r-2}(x)}\right). \tag{2.7}$$

$$= \tan^{-1}\left(\frac{2x}{P_{2r-1}(x)}\right)$$

using (1.1) and (1.8) and simplifying.

Consequently, summation of (2.7) produces

$$\sum_{r=1}^{n} \tan^{-1}(2x/P_{2r-1}(x)) = \frac{\pi}{2} - \tan^{-1}(1/P_{2n}(x)) \tag{2.8}$$

since $P_0(x) = 0$ and $\tan x$ is undefined for $x = \pi/2$.

Alternatively, (2.8) is a direct consequence of (2.4).

As above, the special cases $x = \frac{1}{2}$, $x = 1$ reduce (2.8) to information about the Fibonacci numbers and the Pell numbers, respectively.

In particular, when $x = \frac{1}{2}$, (2.8) leads to the limiting summation

$$\sum_{r=1}^{\infty} \tan^{-1}\left[\frac{1}{F_{2r-1}}\right] = \frac{\pi}{2},$$

which, like (2.6), is a slight variation of the D. H. Lehmer summation result for Fibonacci numbers (given in [2] as Theorem 5).

When $x = 1$ in (2.8), we obtain another form of (2.5) for Pell numbers.

Furthermore, using (1.9) and (1.11), one may obtain

$$\tan^{-1}\left(\frac{1}{Q_{2r-2}(x)}\right) + \tan^{-1}\left(\frac{1}{Q_{2r}(x)}\right) = \tan^{-1}\left(\frac{4(x^2+1)P_{2r-1}(x)}{Q_{2r-1}^2(x) + 4(x^2+1) - 1}\right). \tag{2.7a}$$

Unfortunately, the right-hand side does not simplify any further as we should have desired, by comparison with (2.7). However, if we choose $x = \frac{1}{2}$ [so that $P_r(\frac{1}{2}) = F_r$, $Q_r(\frac{1}{2}) = L_r$], then the equation reduces to

$$\begin{aligned}\tan^{-1}\left(\frac{1}{L_{2r-2}}\right) + \tan^{-1}\left(\frac{1}{L_{2r}}\right) &= \tan^{-1}\left(\frac{1}{F_{2r-1}}\right)\\ &= \tan^{-1}\left(\frac{1}{F_{2r-2}}\right) - \tan^{-1}\left(\frac{1}{F_{2r}}\right) \quad \text{from (2.7)},\end{aligned}$$

both of which are given in [2] (as Theorems 3 and 4), in a slightly varied form.

Proceeding to the limiting summation in the first of these equations (with r replaced by $r + 1$) produces the result for Lucas numbers given in [2] as Theorem 6, namely

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{L_{2r}}\right) = \frac{1}{2}\tan^{-1}2 = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right).$$

Furthermore, for Pell-Lucas polynomials,

$$\begin{aligned}\tan^{-1}\left(\frac{Q_{r-1}(x)}{Q_r(x)}\right) - \tan^{-1}\left(\frac{Q_r(x)}{Q_{r+1}(x)}\right) &= \tan^{-1}\left(\frac{Q_{r+1}(x)Q_{r-1}(x) - Q_r^2(x)}{Q_r(x)(Q_{r+1}(x) + Q_{r-1}(x))}\right) \qquad (2.9)\\ &= \tan^{-1}\left(\frac{(-1)^{r-1}}{P_{2r}(x)}\right)\end{aligned}$$

by (1.9), (1.11), and (1.12).

Hence,

$$\begin{aligned}\sum_{r=1}^{n} \tan^{-1}\left(\frac{(-1)^{r-1}}{P_{2r}(x)}\right) &= \tan^{-1}\left(\frac{Q_0(x)}{Q_1(x)}\right) - \tan^{-1}\left(\frac{Q_n(x)}{Q_{n+1}(x)}\right) \qquad (2.10)\\ &= \tan^{-1}\left(\frac{P_n(x)}{P_{n+1}(x)}\right)\end{aligned}$$

by (1.3), (1.4), and (1.5), since $1 + x\alpha = \alpha\sqrt{1+x^2}$ and $1 + x\beta = -\beta\sqrt{1+x^2}$.

By (1.5), $\alpha > 0$ and $\beta < 0$ for all real x. Furthermore, $\alpha > 1$ for $x > 0$ and $0 < \alpha < 1$ for $x < 0$. In addition, $|\beta| < 1$ for $x > 0$ and $|\beta| > 1$ for $x < 0$. From these considerations, (1.3), and (2.10), we have

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{(-1)^{r-1}}{P_{2r}(x)}\right) = \begin{cases}\tan^{-1}(1/\alpha), & \text{for } x > 0\\ \tan^{-1}(1/\beta), & \text{for } x < 0.\end{cases} \tag{2.11}$$

An argument similar to that used in deriving (2.10) shows that

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{(-1)^r}{P_{2r}(x)}\right) = \tan^{-1}\left(\frac{Q_1(x)}{Q_0(x)}\right) - \tan^{-1}\left(\frac{Q_{n+1}(x)}{Q_n(x)}\right) \tag{2.12}$$

$$= \tan^{-1}\left(\frac{xQ_n(x) - Q_{n+1}(x)}{Q_n(x) + xQ_{n+1}(x)}\right) = -\tan^{-1}\left(\frac{P_n(x)}{P_{n+1}(x)}\right).$$

Letting $n \to \infty$, we have another derivation of (2.11). Special manifestations of (2.9), (2.10), (2.11), and (2.12) are derived when $x = \frac{1}{2}$ and $x = 1$, yielding information about the Fibonacci and Lucas numbers, and the Pell and Pell-Lucas numbers, respectively.

For example, if $x = \frac{1}{2}$, then (2.11) with (1.5) yields the known result

$$\sum_{r=1}^{\infty} \tan^{-1}\left\{\frac{(-1)^{r-1}}{F_{2r}}\right\} = \frac{1}{2}\tan^{-1}2 = \tan^{-1}\left(\frac{\sqrt{5}-1}{2}\right),$$

which should be compared with the similar result for Lucas numbers preceding (2.9).

If $x = \frac{1}{2}$ in (2.9), then, with r replaced by $n + 1$, Theorem 3 (first part) of [1] results.

When $x = \frac{1}{2}$ in (2.12), we obtain Theorem 4 of [1].

Following the method used for (2.7), with appeal to (1.8) and (1.10), and then summing, we ascertain that

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{Q_{2r}(x)}{P_{2r}^2(x)}\right) = \frac{\pi}{4} + (-1)^{n-1}\tan^{-1}\left(\frac{1}{P_{2n+1}(x)}\right). \tag{2.13}$$

Summing to infinity gives

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{Q_{2r}(x)}{P_{2r}^2(x)}\right) = \frac{\pi}{4}, \text{ provided } P_{2n+1}(x) \to \infty.$$

When $x = \frac{1}{2}$, it follows that, for Fibonacci and Lucas numbers,

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{L_{2r}}{F_{2r}^2}\right) = \frac{\pi}{4}.$$

When $x = 1$, it follows that, for Pell and Pell-Lucas numbers,

$$\sum_{r=1}^{\infty} (-1)^{r-1}\tan^{-1}\left(\frac{Q_{2r}}{P_{2r}^2}\right) = \frac{\pi}{4}.$$

3. GENERALIZATIONS

Results (2.10) and (2.12) can be generalized as indicated below. Firstly, however, some extensions and generalizations of previous formulas must be established. Using the Binet forms (1.3) and (1.4), we may, with due diligence, demonstrate the validity of the following:

$$P_{(r+1)(2k-1)}(x)P_{(r-1)(2k-1)}(x) - P^2_{r(2k-1)}(x) = (-1)^{r(2k-1)}P^2_{2k-1}(x); \tag{3.1}$$

$$Q_{(r+1)(2k-1)}(x)Q_{(r-1)(2k-1)}(x) - Q^2_{r(2k-1)}(x)$$
$$= (-1)^{(r-1)(2k-1)}4(x^2+1)P^2_{2k-1}(x); \tag{3.2}$$

$$P_{r(2k-1)}(x)\{P_{(r+1)(2k-1)}(x) + P_{(r-1)(2k-1)}(x)\} = P_{2r(2k-1)}(x)P_{2k-1}(x); \tag{3.3}$$

$$Q_{r(2k-1)}(x)\{Q_{(r+1)(2k-1)}(x) + Q_{(r-1)(2k-1)}(x)\}$$
$$= 4(x^2+1)P_{2r(2k-1)}(x)P_{2k-1}(x). \tag{3.4}$$

The odd factor $2k-1$ is necessary to ensure the vanishing of certain terms that arise in the course of the algebraic manipulations. Of course, (3.1) and (3.2) are extensions of the Simson's formulas (1.8) and (1.9), respectively, when $k = 1$ $(P_1(x) = 1)$.

Now, consider

$$\tan^{-1}\left(\frac{Q_{(r-1)(2k-1)}(x)}{Q_{r(2k-1)}(x)}\right) - \tan^{-1}\left(\frac{Q_{r(2k-1)}(x)}{Q_{(r+1)(2k-1)}(x)}\right) \tag{3.5}$$

$$= \tan^{-1}\left(\frac{Q_{(r+1)(2k-1)}(x)Q_{(r-1)(2k-1)}(x) - Q^2_{r(2k-1)}(x)}{Q_{r(2k-1)}(x)\{Q_{(r+1)(2k-1)}(x) + Q_{(r-1)(2k-1)}(x)\}}\right)$$

$$= \tan^{-1}\left(\frac{(-1)^{(r-1)(2k-1)}\cdot 4(x^2+1)P^2_{2k-1}(x)}{4(x^2+1)P_{2r(2k-1)}(x)P_{2k-1}(x)}\right) \quad \text{by (3.2), (3.4)}$$

$$= \tan^{-1}\left(\frac{(-1)^{(r-1)(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right)$$

Put $k = 1$ in (3.5) and we obtain (2.9).

If we sum (3.5), as before, we have

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{(-1)^{(r-1)(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) \tag{3.6}$$

$$= \tan^{-1}\left(\frac{Q_0(x)}{Q_{2k-1}(x)}\right) - \tan^{-1}\left(\frac{Q_{n(2k-1)}(x)}{Q_{(n+1)(2k-1)}(x)}\right).$$

Recourse to (3.1) and (3.3) will likewise reveal that

$$\tan^{-1}\left(\frac{(-1)^{r(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) \tag{3.7}$$

$$= \tan^{-1}\left(\frac{P_{(r-1)(2k-1)}(x)}{P_{r(2k-1)}(x)}\right) - \tan^{-1}\left(\frac{P_{r(2k-1)}(x)}{P_{(r+1)(2k-1)}(x)}\right).$$

Therefore, since $P_0(x) = 0$,

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{(-1)^{r(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) = -\tan^{-1}\left(\frac{P_{n(2k-1)}(x)}{P_{(n+1)(2k-1)}(x)}\right). \tag{3.8}$$

Putting $k = 1$ in (3.6) and (3.8) leads us back to (2.10) and (2.12), respectively. Some advantage accrues in (3.7) if r is replaced by $r + 1$. Summation then leaves an additional (nonvanishing) term

$$\left[= \tan^{-1}\left(\frac{P_{2k-1}(x)}{P_{2(2k-1)}(x)}\right)\right]$$

on the right-hand side. If we put $k = 1$ in (3.7) and replace r by $r + 1$, then $x = 1$ gives us

$$\tan^{-1}\left(\frac{P_r}{P_{r+1}}\right) - \tan^{-1}\left(\frac{P_{r+1}}{P_{r+2}}\right) = \tan^{-1}\left(\frac{(-1)^{r+1}}{P_{2r+2}}\right), \tag{3.7a}$$

while $x = \frac{1}{2}$ gives Theorem 3 (second part) in [1].

In conclusion, we notice, using the results needed for (2.11), that

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{(-1)^{r(2k-1)}P_{2k-1}(x)}{P_{2r(2k-1)}(x)}\right) = -\begin{cases} \tan^{-1}(1/\alpha^{2k-1}), & \text{for } x > 0 \\ \tan^{-1}(1/\beta^{2k-1}), & \text{for } x < 0. \end{cases} \tag{3.9}$$

4. ACKNOWLEDGMENT

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A TWO-DIMENSIONAL GENERALIZATION OF GRUNDY'S GAME

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1. INTRODUCTION

A positional game with normal winning rule for two alternately moving persons may be defined by a pair $\Gamma = (P, S)$. P is the finite set of game positions and S is a mapping $S: P \to 2^P$ such that $y \in S(x)$ if and only if y is a position that can be brought about from position x by a legal move. We call $S(x)$ the set of successors of x. A play of Γ terminates when a position x with $S(x) = \emptyset$ has been reached. The normal winning rule states that the player loses who is first unable to move. It is assumed that there is an upper bound for the number of moves in any play. A disjunctive combination of a finite set of such games $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ may be defined thus: The players play alternately, each in turn making a move in one of the individual games. A player loses if unable to move.

P. M. Grundy [1] showed that for $\Gamma = (P, S)$ the function $G: P \to \mathbb{N}_0$ with

$$G(x) = \begin{cases} 0 & \text{if } S(x) = \emptyset \\ \min(\mathbb{N}_0 \backslash \{G(y) \mid y \in S(x)\}) & \text{if } S(x) \neq \emptyset \end{cases} \qquad (*)$$

has the properties

$P1$: A player who moves from a position x with $G(x) > 0$ can consistently move to a positive the G-value of which is 0 and so win the play.

$P2$: If Γ is a disjunctive combination of games $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, then for a combined position $x = (x_1, x_2, \ldots, x_k)$ the G-value is the nim-sum of the individual G-values, i.e.,

$G(x) = G(x_1) \overset{*}{+} G(x_2) \overset{*}{+} \cdots \overset{*}{+} G(x_k)$, where the nim-sum $a \overset{*}{+} b$ for $a, b \in \mathbb{N}_0$ is defined by

$$a \overset{*}{+} b = \begin{cases} 0 & \text{if } a = b = 0, \\ (a + b) \bmod 2 + 2(a \operatorname{div} 2 \overset{*}{+} b \operatorname{div} 2) & \text{otherwise,} \end{cases}$$

where $a \operatorname{div} 2$ is the integer division of a by 2.

So the G-value of $x = (x_1, x_2, \ldots, x_k)$ can easily be calculated if $G(x_i)$ is known for $i = 1, 2, \ldots, k$.

2. GRUNDY'S GAME

P. M. Grundy himself invented the following game: The starting position is a single heap of K matches. The first player, by his move, divides this heap into two heaps of unequal size. the second player selects one of the heaps and

divides it into two heaps of unequal size, and so forth. The play terminates when each individual heap contains only one or two matches. The second player makes the first move in a game that is the disjunctive combination of the two Grundy-games determined by the two heaps resulting from the first player's move. If the starting position of a Grundy-game is described by the number K of corresponding matches, we get

$$S(K) = \{(I,\ K - I) \mid 0 < I < K \text{ and } I \neq K - I\}.$$

The G-value of K according to the recursive definition (*) taking into consideration $P2$ is determined by

$$G(K) = \begin{cases} 0 & \text{if } K = 1 \text{ or } K = 2, \\ \min\left(\mathbb{N}_0 \backslash \{G(I) \overset{*}{+} G(K - I) \mid 0 < I < K,\ I \neq K - I\}\right) & \text{otherwise.} \end{cases}$$

For $K \leqslant 100$, the following G-values are obtained:

Table 1

K	$G(K)$			
1- 40	0010210210	2132132430	4304304123	1241241241
41- 80	5415415410	2102152132	1324324324	3243243245
81-100	2452437437	4374352352		

(See, also [4].)

3. THE GENERALIZED GRUNDY-GAME

A play in this game starts with an $(M \times N)$-rectangle. The first player breaks this rectangle into two rectangles of unequal size, to that the sides of the resulting rectangles are of integer length. (Imagine a bar of chocolate that can be broken along vertical and horizontal scores.) The second player breaks one of these rectangles into two new rectangles of unequal size, and so on. The play terminates when both sides of each individual rectangle are less than or equal to 2. According to the normal winning rule, it is lost by the person who is first unable to move.

$G(M, N)$ denotes the G-value corresponding to a single $(M \times N)$-rectangle, while $G(K)$ refers to a position in Grundy's original game. The following properties are obvious:

$Q1$: $G(M, N) = G(N, M)$, and

$Q2$: $G(1, N) = G(2, N) = G(N)$.

(The generalized game starting with a $(1 \times N)$-rectangle or a $(2 \times N)$-rectangle is obviously equivalent to the original Grundy-game starting with N matches.)

The set of positions succeeding to (M, N) is:

$$S(M, N) = \begin{cases} \emptyset & \text{if } M \leqslant 2 \text{ and } N \leqslant 2, \\ \{((m, N),\ (M - m, N)) \mid 0 < m < M \text{ and } m \neq M - m\} \\ \cup \{((M, n),\ (M, N - n)) \mid 0 < n < N \text{ and } n \neq N - n\} & \text{otherwise.} \end{cases}$$

So $G(M, N)$ can be calculated according to:

$$G(M, N) = \begin{cases} 0 & \text{if } M \leqslant 2 \text{ and } N \leqslant 2, \\ \min(\mathbb{N} \setminus (\{G(m, N) \overset{*}{+} G(M - m, N) \mid 0 < m < M,\ m \neq M - m\} \\ \cup \{G(M, n) \overset{*}{+} G(M, N - n) \mid 0 < n < N,\ n \neq N - n\})) & \text{otherwise,} \end{cases}$$

resulting in Table 2:

Table 2

$M \backslash N$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
1	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
2	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	2	0	
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
4	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
5	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
7	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
8	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
10	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
11	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
13	3	3	1	3	1	1	3	1	1	3	1	1	1	1	1	1	1	1	1	3	
14	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
16	3	3	1	3	1	1	3	1	1	3	1	1	1	1	1	1	1	1	1	3	
17	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
18	4	4	1	4	1	1	4	1	1	4	1	1	1	1	1	1	1	1	1	4	
19	3	3	1	3	1	1	3	1	1	3	1	1	1	1	1	1	1	1	1	3	
20	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
⋮																					

This table indicates that $G(M, N)$ is completely determined by the values $G(M)$ and $G(N)$, as the following theorem states.

Theorem:

$$G(M, N) = \begin{cases} G(N) & \text{if } G(M) = 0, \\ G(M) & \text{if } G(N) = 0, \\ 1 & \text{if } G(M) > 0 \text{ and } G(N) > 0, \end{cases}$$

for all $M, N \in \mathbb{N}$.

Proof (Induction on $M + N$):

$M + N = 2$: $G(1, 1) = 0 = G(1)$,

$M + N = 3$: $G(1, 2) = G(2, 1) = 0 = G(1) = G(2)$.

Now assume that there is an $(I \times K)$-rectangle such that $I + K = n + 1$, where $n \geqslant 2$, and that the theorem is true for all $(M \times N)$-rectangles with $M + N \leqslant n$.

Under this assumption, we must prove the following:

1. $G(I) = 0 \Rightarrow G(I, K) = G(K)$,
2. $G(K) = 0 \Rightarrow G(I, K) = G(I)$, and
3. $G(I) > 0$ and $G(K) > 0 \Rightarrow G(I, K) = 1$.

1. $G(I) = 0 \Rightarrow \min(\mathbb{N}_0 \backslash \{G(I, k) \overset{*}{+} G(I, K - k) | 0 < k < K, k \neq K - k\})$
$= \min(\mathbb{N}_0 \backslash \{G(k) \overset{*}{+} G(K - k) | 0 < k < K, k \neq K - k\}) = G(K)$.

It remains to prove that

$G(K) \notin \{G(i, K) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}$.

It is either $G(K) = 0$ or $G(K) > 0$. These two cases will be inspected separately.

(i) $G(K) = 0$:

$\min(\mathbb{N}_0 \backslash \{G(i, K) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}) =$

$\min(\mathbb{N}_0 \backslash \{G(i) \overset{*}{+} G(I - i) | 0 < i < I, i \neq I - i\}) = G(I) = 0$.

Since $G(K) = 0$, it is

$G(K) \notin \{G(i, K) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}$.

(ii) $G(K) > 0$:

$\{G(i, K) | 0 < i < I\} \subset \{1, G(K)\}$ as $i + K \leqslant n$.

Because of $1 \overset{*}{+} 1 = 0$,

$G(K) \overset{*}{+} G(K) = 0$, and

$$1 \overset{*}{+} G(K) = \begin{cases} G(K) + 1 & \text{if } G(K) \text{ is even,} \\ G(K) - 1 & \text{if } G(K) \text{ is odd,} \end{cases}$$

yields $G(i, K) \overset{*}{+} G(I - i, K) \neq G(K)$ for $0 < i < I$, and therefore

$G(K) \notin \{G(K, I) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}$.

2. $G(K) = 0 \Rightarrow G(K, I) = G(I)$, as was just proved, and $G(K, I) = G(I, K)$.

3. $G(I) > 0$ and $G(K) > 0$:

In this case, $I > 2$ and $K > 2$.

$G(I, K) = 1$ is equivalent to:

(i) $G(i, K) \overset{*}{+} G(I - i, K) \neq 1$ for all i such that $0 < i < I$ and $i \neq I - i$;

(ii) $G(I, k) \overset{*}{+} G(I, K - k) \neq 1$ for all k such that $0 < k < K$ and $k \neq K - k$;

(iii) $G(i, K) \overset{*}{+} G(I - i, K) = 0$ for some i such that $0 < i < I$ and $i \neq I - i$

or

$G(I, k) \overset{*}{+} G(I, K - k) = 0$ for some k such that $0 < k < K$ and $k \neq K - k$.

To prove (i), (ii), and (iii) of 3:

(i) By assumption, $G(i, K) \in \{1, G(K)\}$ for $0 < i < I$. So we have

$$G(i, K) \overset{*}{+} G(I - i, K) = \begin{cases} 1 \overset{*}{+} 1 = 0 & \text{or} \\ G(K) \overset{*}{+} G(K) = 0 & \text{or} \\ 1 \overset{*}{+} G(K) \neq 1 \end{cases}$$

for all i, $0 < i < I$.

(ii) $G(I, k) + G(I, K - k) = G(k, I) \overset{*}{+} G(K - k, I) \neq 1$ for all k, $0 < k < K$, because of (i).

(iii) Suppose $G(i, K) \overset{*}{+} G(I - i, K) \neq 0$ for a certain i, $0 < i < I$ and $i \neq I - i$. This is equivalent to $G(i, K) \neq G(I - i, K)$. Since, by assumption, $G(i, K)$ is determined by $G(i)$ and $G(K)$, it follows that $G(i) \neq G(I - i)$ for this i.

However, there must exist an i, $0 < i < I$ and $i \neq I - i$ such that $G(i) = G(I - i)$ because otherwise $G(i) \overset{*}{+} G(I - i) \neq 0$ for all i, $0 < i < I$ and $i \neq I - i$. That would mean $G(I) = 0$, in contradiction to the assumption $G(I) > 0$.

4. FURTHER GENERALIZATIONS

The theorem can be extended in different directions:

1. Consider the game in which a heap of K matches may be split into two nonempty heaps of size I and $K - I$ provided $|K - I| \geqslant d$, $d \in N_0$. (For $d = 1$, it is Grundy's original game.)

   For this game and its two-dimensional analogue, the above theorem remains valid. The proof requires only replacing the conditions $i \neq I - i$ and $k \neq K - k$ by $|I - i| \geqslant d$ and $|K - k| \geqslant d$.

2. The theorem can be generalized to more than two dimensions. For example:

$$G(L, M, N) = \begin{cases} G(M, N) & \text{if } G(L) = 0, \\ 1 & \text{if } G(L) > 0,\ G(M) > 0,\ G(N) > 0. \end{cases}$$

   The proof is completely analogous to the one given above.

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LUCAS SEQUENCES IN SUBGRAPH COUNTS OF SERIES-PARALLEL AND RELATED GRAPHS

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1. INTRODUCTION

We follow graph theoretic terminology as in [B&M]. Let $G = (V, E)$ denote a graph where V is a set of vertices and E is a set of nonoriented edges. Though we do not in general consider graphs with loops or multiple edges, we make reference to such graphs for the purpose of proofs. When an edge e appears m times, we say e has *multiplicity* m. A *subgraph* $G' = (V', E')$ of G is any graph such that $V' \subseteq V$ and $E' \subseteq E$, and a *spanning subgraph* of G contains every vertex of V. A sequence of vertices $n_1, n_2, n_3, \ldots, n_k$ is a *path* of G if $n_i \in V$, $\{n_i, n_{i+1}\} \in E$, for all i, and no vertices are repeated. A path is a *cycle* if $n_1 = n_k$. A *tree* of G is a subgraph with no cycles; a *spanning tree* contains every vertex of G. Let $r(G)$ denote the count of spanning trees of G.

Spanning tree counts of general graphs can be obtained in $O(n^3)$ time by computing the determinant of its in-degree matrix [7], where n is the number of vertices. This function grows quickly; as well, the practical interest of circuit theory in counting spanning trees motivates the study of classes of graphs for which spanning tree counts can be obtained in linear time.

Sedlacek [19] notes that W_{n+1}, the *wheel* on $n + 1$ vertices, is obtained from a cycle on n points we call the *rim* by joining each point in the cycle to another point we call the *hub*. Vertices and edges on the rim are *rim vertices* and *rim edges*; an edge joining a rim vertex and the hub is a *spoke*. Sedlacek considers F_{n+1}, the *auxiliary fan* of W_{n+1}, derived from W_{n+1} by removing a single rim edge and proves

$$r(F_{n+1}) = \frac{(3 + \sqrt{5})^{n+1} - (3 - \sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

and

$$r(W_{n+1}) = \left(\frac{3 + \sqrt{5}}{2}\right)^{n+1} + \left(\frac{3 - \sqrt{5}}{2}\right)^{n+1} - 2.$$

It is remarkable that $r(F_{n+1})$ generates every second number of the Fibonacci series.

Myers [14] and Bedrosian [2] derive similar formulas for wheels and multigraph wheels in a circuit theory setting. Hilton [10] presents formulas for $r(G)$ of fans and wheels in terms of Fibonacci and Lucas numbers, and Fielder [8] provides tree counts for *sector graphs*, fans with certain multiple edges. Slater [21] shows that all maximal outerplane graphs with exactly two vertices of degree two have the same spanning tree count as fans. (We coin the term *generalized fan* to refer to these graphs in [16] and [17].) Shannon [20] derives $r(W_{n+1})$ with a number theoretic approach. Bange, Barkauskas, and Slater

[1] show that generalized fans have more spanning trees than any other maximal outerplane graph. Most of these studies have been motivated by the remarkable involvement of Fibonacci numbers in spanning tree counts.

The study of *network reliability* demands counts of subgraphs other than spanning trees. Previously, formulas for subgraph counts apparently existed only for complete graphs [9]. A network is commonly modeled as a *probabilistic graph* where each edge e fails independently with probability p and vertices never fail. The probability that such a graph is connected is called *probabilistic connectedness*, and is a standard measure of network reliability. This can be generalized in two different ways. In some applications, a network may not be considered operational unless it has edge connectivity or *cohesion* of at least k; this we call k-cohesive connectedness. Alternately, a network may be considered operational if it has broken down into no more than k components; we call this k-component connectedness. In Section 2, we use Lucas recurrences to count various types of subgraphs of generalized fans and related graphs. Section 3 counts connected spanning subgraphs with cohesion of at least two (two-cohesive). Section 4 presents the rank polynomial as a technique for classifying subgraphs of generalized fans both by number of edges and by number of components. We conclude in Section 5 with some applications. By noting that probabilities can be encoded in the coefficients of some of these recurrences, we obtain reliability formulas as well as subgraph counting formulas. As in previous studies, we find that the required enumerations are given in two-term recurrence relations; hence, the desired subgraph counts are Lucas numbers.

2. COUNTING CONNECTED SPANNING SUBGRAPHS

We begin by counting connected spanning subgraphs of generalized fans that satisfy a Lucas recurrence. Generalized fans are a subset of *2-trees* [18], defined recursively as follows:

1) A single edge is a 2-tree.
2) If G is a 2-tree with edge $\{x, y\}$, adding a new vertex z, and the two edges $\{x, z\}$ and $\{y, z\}$ creates a new 2-tree. If G is not a single edge, $\{x, y\}$ becomes an *interior edge* of the new graph.

When parallel edges are not allowed, 2-trees are equivalent to *maximal series-parallel networks* as in [6], [16], [17]; other definitions of series-parallel networks do appear in the literature.

Any vertex of degree two is a *peripheral vertex*; an edge incident on a peripheral vertex is a *peripheral edge*. To illustrate the counting technique, we reproduce in part this lemma from [16] which counts connected spanning subgraphs of generalized fans.

Lemma 2.1: The number of connected spanning subgraphs of an n-vertex generalized fan, $sc(n)$, satisfies the recurrence:

$$sc(n) = 4sc(n - 1) - 2sc(n - 2).$$

Proof: Let **peripheral** vertex z be attached to edge $\{x, y\}$ of generalized fan G by edges $\{x, z\}$ and $\{y, z\}$. A connected spanning subgraph of G induces on $G - z$ either a connected spanning subgraph or a disconnected spanning subgraph which the addition of $\{x, y\}$ would connect. To handle this latter case, we define $dc(n)$ to be the number of spanning subgraphs of an n-vertex generalized fan which the addition of a specific peripheral edge would connect.

Any connected spanning subgraph of G must contain at least one of $\{x, z\}$ and $\{y, z\}$. If both are selected, the graph induced on $G - z$ must either be connected, or be one of the graphs counted by dc. In this case, there are $sc(n - 1) + dc(n - 1)$ induced subgraphs. Otherwise, a connected spanning subgraph contains either $\{x, z\}$ or $\{x, z\}$, but not both. But then the graph induced on $G - z$ must be connected; the number in this case is $2sc(n - 1)$. Therefore,

$$sc(n) = 3sc(n - 1) + dc(n - 1).$$

By a similar argument,

$$dc(n) = sc(n - 1) + dc(n - 1).$$

These two recurrences may be combined to yield

$$sc(n) = 4sc(n - 1) - 2sc(n - 2). \blacksquare$$

Since $sc(2) = 1$ and $sc(3) = 4$, the recurrence yields the closed formula

$$sc(n) = \frac{(2 + \sqrt{2})^{n-1} - (2 - \sqrt{2})^{n-1}}{2\sqrt{2}}.$$

From a reliability perspective, it is interesting that all generalized fans have the same number of connected spanning subgraphs; in addition, generalized fans have more connected spanning subgraphs than any other 2-tree [16]. We say F_i is a *subfan* of the fan F_n if h, the hub of F_n, is a vertex in F_i, F_i is a subgraph of F_n and F_i is a fan. From Lemma 2.1, we then show:

Lemma 2.2: For $n \geqslant 4$, the number of connected spanning subgraphs of a wheel on n vertices, $sc_W(n)$, is

$$sc_W(n) = 2\sum_{i=2}^{n} sc(i).$$

Proof: Consider the n-vertex wheel W_n with rim edge $\{a, b\}$. Denote by F_n the auxiliary fan of W_n created by removing $\{a, b\}$.

A connected spanning subgraph of W may or may not contain $\{a, b\}$. If not, there are $sc(n)$ connected spanning subgraphs of the auxiliary fan of W_n which are also connected spanning subgraphs of W_n. But we can also add the edge $\{a, b\}$ to any of the connected spanning subgraphs of F_n and get a connected spanning subgraph of W_n.

Lastly, the edge $\{a, b\}$ connects any *two-component* spanning subgraph of F_n, one containing a and the other containing b. Such a spanning subgraph of F_n must consist of a path on $n - i$ vertices and a connected spanning subgraph of the subfan of F_n on the remaining i vertices.

For each i, there are exactly two ways we can choose a path on $n - i$ vertices containing exactly one of a or b, and $sc(i)$ ways of obtaining a connected spanning subgraph on the remaining i vertices; hence, for each i we obtain $2sc(i)$ connected spanning subgraphs of W_n. We vary i from 2 to $n - 1$, and the result follows. $\blacksquare$

The above simplifies to:

$$sc_W(n) = (2 + \sqrt{2})^{n-1} + (2 - \sqrt{2})^{n-1} - 2.$$

This is analogous to Sedlacek's formula for spanning trees in a wheel.

3. COUNTING 2-COHESIVE SPANNING SUBGRAPHS

Sometimes a network must be at least k-cohesive, i.e., the order of the minimum edge cutset must be at least k, to be operational. This happens in an environoment where queuing delay is a problem [26]. The number of 2-cohesive spanning subgraphs of generalized fans satisfies a recurrence of the Lucas type [15]. We state the following without proof.

Lemma 3.1: For $n \geqslant 3$, $sc_2(n)$, the number of 2-cohesive spanning subgraphs of an n-vertex generalized fan is

$$sc_2(n) = dc_2(n) = 2sc_2(n-1) + sc_2(n-2)$$

$$= \frac{\sqrt{2}}{4}[(1+\sqrt{2})^{n-2} - (1-\sqrt{2})^{n-2}]. \blacksquare$$

As before, the count of two-connected spanning subgraphs is maximized by minimizing the number of peripheral vertices.

4. THE RANK POLYNOMIAL OF A GENERALIZED FAN

Subgraph counts have been studied in an algebraic setting by Tutte [22], [23], [24], and [25] and others [3] and [5]. In this section, we derive the *rank polynomial* of a generalized fan, by a similar technique.

Let $c(G)$ denote the number of components of a graph G. In addition, write

$$i(G) = |V| - c(G), \qquad j(G) = |E| - |V| + c(G).$$

If S is any subset of E, $G{:}S$ denotes the subgraph of G induced by S. Then denote by $RK(G;\ t,\ z)$ the rank polynomial of G where

$$RK(G;\ t,\ z) = \sum_{S \subseteq E} t^{i(G:S)} z^{j(G:S)}.$$

Note that $i(G{:}S) + j(G{:}S) = |S|$; thus, from the rank polynomial of a graph, we can quickly classify spanning subgraphs of G not only by number of edges but also by number of components. From [24], we can trivially derive the following three properties of the rank polynomial which completely characterize $RK(G;\ t,\ z)$:

1) If G consists of two vertex disjoint subgraphs H and K, then

$$RK(G;\ t,\ z) = RK(H;\ t,\ z)RK(K;\ t,\ z).$$

2) (Rank polynomial factoring theorem). If e is any edge in E,

$$RK(G;\ t,\ z) = RK(G = e;\ t,\ z) + tRK(G \bullet e;\ t,\ z),$$

where $G \bullet e$ is graph G less edge $e = \{x,\ y\}$ with endvertices x and y identified.

3) If G consists of a single vertex and k loops,

$$RK(G;\ t,\ z) = (1+z)^k.$$

Thus, the rank polynomial is a rich source of information about subgraph counts. We need some more identities:

Lemma 4.1: (a) If G is a single edge on two vertices, then

$$RK(G;\ t,\ z) = 1 + t.$$

(b) If G_x is the graph derived by adding a loop to any vertex x of G, then

$$RK(G_x;\ t,\ z) = (1 + z)RK(G;\ t,\ z).$$

(c) If $G = H \cup K$ and $H \cap K$ contains no edges and exactly one vertex,

$$RK(G;\ t,\ z) = RK(H;\ t,\ z)RK(K;\ t,\ z).$$

Proof: (a) Note that if H is any edgeless graph, then $RK(H;\ t,\ z) = 1$. A single application of the rank polynomial factoring theorem yields the result. ■

(b) Any spanning subgraph of G is a spanning subgraph of G_x; to each spanning subgraph of G, we can add the edge $\{x,\ x\}$ also yielding another spanning subgraph of G_x. This second set of spanning subgraphs can be represented by multiplying the rank polynomial of G by z, i.e., increasing the edge count of every term in the polynomial without disturbing any other information. ■

(c) Consider any subgraphs H' and K' of H and K, respectively. H' has n_H vertices, e_H edges, and c_H components. Similarly, K' has n_K vertices, e_K edges, and c_K components. The subgraph $H' \cup K'$ of G has $n_H + n_K - 1$ vertices, $e_H + e_K$ edges, and $c_H + c_K - 1$ components. Expressing the term of $RK(G;\ t,\ z)$ corresponding to $H' \cup K'$ in terms of the corresponding expressions for H' and K' in $RK(H;\ t,\ z)$ and $RK(K;\ t,\ z)$ yields the desired result. ■

We have seen that every generalized fan on n vertices, regardless of topology, has the same number of connected spanning subgraphs. Nevertheless, it is surprising that all n-vertex generalized fans have the same rank polynomial, again satisfying a two-term linear Lucas recurrence.

Lemma 4.2: The rank polynomial of any generalized fan on n vertices, $S(n)$, satisfies the recurrence

$$S(n) = (1 + 3t + tz)S(n - 1) - t(1 + t)(1 + z)S(n - 2)$$

which may be solved for the closed formula

$$S(n) = \frac{1 + 2t + 3t^2 + t^2z - tz + (1 + t)\alpha}{2\alpha}\left(\frac{1 + 3t + tz + \alpha}{2}\right)^{n-2} - \frac{1 + 2t + 3t^2 + t^2z - tz - (1 + t)\alpha}{2\alpha}\left(\frac{1 + 3t + tz - \alpha}{2}\right)^{n-2}$$

where $\alpha = \sqrt{(1 + 3t + tz)^2 + 4t(1 + t)(1 + z)}$.

Proof: As preliminaries, consider some special cases. Let $H_n = G_{n-1} \cup \{x,\ y\}$, be an n-vertex graph where G_{n-1} is an $n-1$-vertex generalized fan with peripheral vertex x and y is a new vertex not in G_{n-1}; then,

$$RK(H_n;\ t,\ z) = (1 + t)S(n - 1)$$

by Lemma 4.1(a) and (c). Let G be an n-vertex generalized fan and write $DD(n)$

for the rank polynomial of an n-vertex generalized fan with one peripheral edge of multiplicity 2. A single application of the rank polynomial factoring theorem to one of G's peripheral edges yields

$$S(n) = RK(H_n;\ t,\ z) + DD(n - 1) = (1 + t)S(n - 1) + DD(n - 1).$$

We obtain a recursive expression for $DD(n)$ by applying the rank polynomial factoring theorem to one of the edges of multiplicity 2. If G is an n-vertex generalized fan with one peripheral edge e of multiplicity 2, then $G - e$ is an n-vertex generalized fan and $G \bullet e$ is an $n - 1$-vertex generalized fan with a peripheral edge of multiplicity 2 and a loop at the peripheral vertex. Then

$$DD(n) = S(n) + t(1 + z)DD(n - 1)$$

by Lemma 4.1(b).

Combining these expressions provides the stated two-term Lucas recurrence, and solving gives the closed formula. ■

5. APPLICATIONS

Subgraph counts alone provide a measure of the connectedness of a graph. However, the recurrences in Section 2 can be generalized to compute probabilistic connectedness or, alternately, two-cohesive connectedness. If p is the probability that a single edge is up, then $R_p(n)$ is the probability that an n-vertex generalized fan is connected. Let $\rho_p(n)$ be the probability of obtaining a spanning subgraph on n vertices that *would become* connected if a *specific* peripheral edge were added. Since the context is clear, we omit the probability subscript. The following is a new proof of the main result in [17] using Lucas recurrences rather than generating functions.

Theorem 5.1: Let $x = q/p$. $R(n)$, the probability that an n-vertex generalized fan is connected is given by:

$$R(n) = p^2(3x + 1)R(n - 1) - p^4(x^2 + x)R(n - 2).$$

It is remarkable that $\rho(n)$ also obeys the same relation, that is:

$$\rho(n) = p^2(3x + 1)\rho(n - 1) - p^4(x^2 + x)\rho(n - 2).$$

Proof: Consider the n-vertex generalized fan G having peripheral vertex z and edge of attachment $\{x, y\}$. We measure $R(n)$ as a product of the states of edges $\{x, z\}$, $\{y, z\}$ and the subgraph induced by $G - z$. The probability that $G - z$ is connected is $R(n - 1)$; the probability that at least one of $\{x, z\}$ and $\{y, z\}$ is up is $2pq + p^2$. The probability of a connected spanning subgraph in this case is $R(n - 1)(2pq + p^2)$. Suppose, on the other hand, $G - z$ is disconnected but the addition of $\{x, y\}$ would connect it; if both $\{x, z\}$ and $\{y, z\}$ are up, the resultant subgraph of G is connected with probability $p^2\rho(n - 1)$. Then

$$R(n) = p^2\rho(n - 1) + (2pq + p^2)R(n - 1).$$

Similarly,

$$\rho(n) = pq\rho(n - 1) + q^2R(n - 1).$$

Combining these formulas yields the stated recurrences which can then be solved for a closed formula for probabilistic connectedness. ■

Such formulas are extremely useful, since it appears that no other exact measures of probabilistic connectedness exist except for complete graphs [9].

A formula for 2-cohesive connectedness can be derived similarly; in both instances, generalized fans are the most reliable maximal series-parallel network (see [15] and [17]).

The rank polynomial of a generalized fan yields a family of reliability measures. Let $t = z/r$ and $z = p/q$, and let $KC(n, k)$ be the probability of obtaining a subgraph of no more than k components. We multiply the rank polynomial by $r^{|V|}q^{2n-3}$ and collect terms by superscripts of z to yield

$$q^{2n-3}\sum_d c_d r^{|V|-i}\left(\frac{p}{q}\right)i^{i+j}.$$

From this, we can write

$$KC(n, k) = q^{2n-3}\sum_d c_d T(|V| - i \leqslant k)\left(\frac{p}{q}\right)^{i+j},$$

where $T(expression)$ returns 1 if its argument is true and 0 otherwise.

Lastly, these techniques apply to other classes of graphs. Generalizing Sedlacek [19], Mikola [13] describes $V_n^{(k)}$ as the path $v_0v_1v_2 \cdots v_{(n-1)(k-1)}$ and the edges wv_i for $i = 0, k + 1, 2(k + 1), \ldots, (n - 1)(k - 1)$, i.e., rim edges are replaced with paths of equal length. Then

$$r(V_n^{(k)}) = \frac{((k + 3 + K)^n - (k + 3 - K)^n)}{(2^nK)},$$

where $K = \sqrt{k^2 + 6k + 5}$. We generalize Mikola's result by replacing spokes with paths of equal length. Furthermore, a *generalized Mikola fan* is obtained from a generalized fan by replacing all the interior edges and any two nonadjacent peripheral edges by paths of length $j + 2$ and all the other edges by paths of length $k + 2$.

The connected spanning subgraph count of a generalized Mikola fan, $G(n)$, satisfies the recurrence:

$$G(n) = (k + 2j + 4)G(n - 1) - (j^2 + 3j + 2)G(n - 2),$$

where n is the index as in the definition. Solving this yields a formula for subgraph counts of yet another class of uniformly sparse graphs.

6. ACKNOWLEDGMENTS

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LONGEST SUCCESS RUNS AND FIBONACCI-TYPE POLYNOMIALS

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1. INTRODUCTION AND SUMMARY

Let L_n be the length of the longest run of successes in n ($\geqslant 1$) independent trials with constant success probability p $(0 < p < 1)$, and set $q = 1 - p$. In [3] McCarty assumed that $p = 1/2$ and found a formula for the tail probabilities $P(L_n \geqslant k)$ $(1 \leqslant k \leqslant n)$ in terms of the Fibonacci sequence of order k [see Remark 2.1 and Corollary 2.1(c)]. In this paper, we establish a complete generalization of McCarty's result by deriving a formula for $P(L_n \geqslant k)$ $(1 \leqslant k \leqslant n)$ for any $p \in (0, 1)$. Formulas are also given for $P(L_n \leqslant k)$ and $P(L_n = k)$ $(0 \leqslant k \leqslant n)$. Our formulas are given in terms of the multinomial coefficients and in terms of the Fibonacci-type polynomials of order k (see Lemma 2.1, Definition 2.1, and Theorem 2.1). As a corollary to Theorem 2.1, we find two enumeration theorems of Bollinger [2] involving, in his terminology, the number of binary numbers of length n that do not have (or do have) a string of k consecutive ones. We present these results in Section 2. In Section 3, we reconsider the waiting random variable N_k $(k \geqslant 1)$, which denotes the number of Bernoulli trials until the occurrence of the kth consecutive success, and we state and prove a recursive formula for $P(N_k = n)$ $(n \geqslant k)$ which is very simple and useful for computational purposes (see Theorem 3.1). We also note an interesting relationship between L_n and N_k. Finally, in Section 4, we show that $\sum_{k=0}^{n} P(L_n = k) = 1$ and derive the probability generating function and factorial moments of L_n. A table of means and variances of L_n when $p = 1/2$ is given for $1 \leqslant n \leqslant 50$.

We end this section by mentioning that the proofs of the present paper depend on the methodology of [4] and some results of [4] and [6]. Unless otherwise explicitly specified, in this paper k and n are positive integers and x and t are positive reals.

2. LONGEST SUCCESS RUNS AND FIBONACCI-TYPE POLYNOMIALS

We shall first derive a formula for $P(L_n \leqslant k)$ by means of the methodology of Theorem 3.1 of Philippou and Muwafi [4].

Lemma 2.1: Let L_n be the length of the longest success run in n ($\geqslant 1$) Bernoulli trials. Then

$$P(L_n \leqslant k) = p^n \sum_{i=0}^{k} \sum_{n_1, \ldots, n_{k+1}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_{k+1}}, \quad 0 \leqslant k \leqslant n,$$

where the inner summation is over all nonnegative integers $n_1, \ldots, n_{k+1}$, such that $n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n - i$.

Proof: A typical element of the event $(L_n \leqslant k)$ is an arrangement

$$x_1 x_2 \ldots x_{n_1 + \cdots + n_{k+1}} \underbrace{ss \ldots s}_{i},$$

such that n_1 of the x's are $e_1 = f$, n_2 of the x's are $e_2 = sf, \ldots, n_{k+1}$ of the x's are $e_{k+1} = \underbrace{ss \ldots s}_{k}f$, and $n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n - i$ $(0 \leqslant i \leqslant k)$. Fix $n_1, \ldots, n_{k+1}$ and i. Then the number of the above arrangements is

$$\binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}},$$

and each one of them has probability

$$\begin{aligned} &P(x_1x_2 \ldots x_{n_1+\cdots+n_{k+1}}\underbrace{ss \ldots s}_{i}) \\ &= [P\{e_1\}]^{n_1}[P\{e_2\}]^{n_2} \ldots [P\{e_{k+1}\}]^{n_{k+1}} P\{\underbrace{ss \ldots s}_{i}\} \\ &= p^n(q/p)^{n_1+\cdots+n_{k+1}}, \quad 0 \leqslant k \leqslant n, \end{aligned}$$

by the independence of the trials, the definition of e_j $(1 \leqslant j \leqslant k+1)$, and $P\{s\} = p$. Therefore,

$$\begin{aligned} &P(\text{all } x_1x_2 \ldots x_{n_1+\cdots+n_{k+1}}\underbrace{ss \ldots s}_{i};\ n_j\ (1 \leqslant j \leqslant k+1) \text{ and } i \text{ fixed}) \\ &= \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} p^n(q/p)^{n_1+\cdots+n_{k+1}}, \quad 0 \leqslant k \leqslant n. \end{aligned}$$

But n_j $(1 \leqslant j \leqslant k+1)$ are nonnegative integers which may vary, subject to the condition $n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n - i$. Furthermore, i may vary over the integers $0, 1, \ldots, k$. Consequently,

$$\begin{aligned} &P(L_n \leqslant k) \\ &= P\left(\text{all } x_1x_2 \ldots x_{n_1+\cdots+n_{k+1}}\underbrace{s \ldots s}_{i};\ n_j \geqslant 0 \ \ni \sum_{j=1}^{k+1} jn_j = n - i,\ 0 \leqslant i \leqslant n\right) \\ &= \sum_{i=0}^{k} \sum_{\substack{n_1, \ldots, n_{k+1} \ni \\ n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n-i}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} p^n\left(\frac{q}{p}\right)^{n_1+\cdots+n_{k+1}}, 0 \leqslant k \leqslant n, \end{aligned}$$

which establishes the lemma.

The formula for $P(L_n \leqslant k)$ derived in Lemma 2.1 can be simplified by means of the Fibonacci-type polynomials of order k [6]. These polynomials, as well as the Fibonacci numbers of order k [4], have been defined for $k \geqslant 2$, and the need arises presently for a proper extension of them to cover the cases $k = 0$ and $k = 1$. We shall keep the terminology of [6] and [4] despite the extension.

Definition 2.1: The sequence of polynomials $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ is said to be the sequence of Fibonacci-type polynomials of order k if $F_n^{(0)}(x) = 0$ $(n \geqslant 0)$, and for $k \geqslant 1$, $F_0^{(k)}(x) = 0$, $F_1^{(k)}(x) = 1$, and

$$F_n^{(k)}(x) = \begin{cases} x[F_{n-1}^{(k)}(x) + \cdots + F_1^{(k)}(x)] & \text{if } 2 \leqslant n \leqslant k+1 \\ x[F_{n-1}^{(k)}(x) + \cdots + F_{n-k}^{(k)}(x)] & \text{if } n \geqslant k+2. \end{cases}$$

Definition 2.2: The sequence $\{F_n^{(k)}\}_{n=0}^{\infty}$ is said to be the Fibonacci sequence of order k if $F_n^{(0)} = 0$ $(n \geqslant 0)$, and for $k \geqslant 1$, $F_0^{(k)} = 0$, $F_1^{(k)} = 1$, and

$$F_n^{(k)} = \begin{cases} F_{n-1}^{(k)} + \cdots + F_1^{(k)} & \text{if } 2 \leqslant n \leqslant k+1 \\ F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} & \text{if } n \geqslant k+2. \end{cases}$$

It follows from Definitions 2.1 and 2.2 that

$$F_n^{(k)}(1) = F_n^{(k)},\ n \geqslant 0. \tag{2.1}$$

The following lemma is useful in proving Theorem 2.1 below.

Lemma 2.2: Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order k $(k \geqslant 1)$. Then,

(a) $F_n^{(1)}(x) = x^{n-1}$, $n \geqslant 1$, and $F_n^{(k)}(x) = x(1+x)^{n-2}$, $2 \leqslant n \leqslant k+1$;

(b) $$F_{n+1}^{(k)}(x) = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} x^{n_1 + \cdots + n_k},\ n \geqslant 0.$$

Proof: Part (a) of the lemma follows easily from Definition 2.1. For $k = 1$, the right-hand side of (b) becomes x^n, which equals $F_{n+1}^{(1)}(x)$ $(n \geqslant 0)$ by (a), so that (b) is true. For $k \geqslant 2$, (b) is true because of Theorem 2.1(a) of [6].

Remark 2.1: Definition 2.2, Lemma 2.2(a), and (2.1) imply that the Fibonacci sequence of order k $(k \geqslant 1)$ coincides with the k-bonacci sequence (as it is given in McCarty [3]).

We can now state and prove Theorem 2.1, which provides another formula for $P(L_n \leqslant k)$. The new formula is a simplified version of the one given in Lemma 2.1, and it is stated in terms of the multinomial coefficients as well as in terms of the Fibonacci-type polynomials of order k. Formulas are also given for $P(L_n = k)$ $(0 \leqslant k \leqslant n)$ and $P(L_n \geqslant k)$ $(1 \leqslant k \leqslant n)$.

Theorem 2.1: Let $\{F_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci-type polynomials of order k, and denote by L_n the length of the longest run of successes in n $(\geqslant 1)$ Bernoulli trials. Then,

(a) $$P(L_n \leqslant k) = \frac{p^{n+1}}{q} \sum_{\substack{n_1, \ldots, n_{k+1} \ni \\ n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n+1}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_{k+1}}$$
$$= \frac{p^{n+1}}{q} F_{n+2}^{(k+1)}(q/p),\ 0 \leqslant k \leqslant n;$$

(b) $$P(L_n = k) = \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)],\ 0 \leqslant k \leqslant n;$$

(c) $$P(L_n \geqslant k) = 1 - \frac{p^{n+1}}{q} F_{n+2}^{(k)}(q/p),\ 1 \leqslant k \leqslant n.$$

Proof: (a) Lemma 2.1, Lemma 2.2(b) applied with $x = q/p$, and Definition 2.1 give

$$P(L_n \leqslant k) = p^n \sum_{i=0}^{k} F_{n+1-i}^{(k+1)}(q/p) = \frac{p^{n+1}}{q} F_{n+2}^{(k+1)}(q/p),\ 0 \leqslant k \leqslant n,$$
$$= \frac{p^{n+1}}{q} \sum_{\substack{n_1, \ldots, n_{k+1} \ni \\ n_1 + 2n_2 + \cdots + (k+1)n_{k+1} = n+1}} \binom{n_1 + \cdots + n_{k+1}}{n_1, \ldots, n_{k+1}} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_{k+1}},$$

which was to be shown. In order to show (b), we first observe that

$$P(L_n = k) = P(L_n \leqslant k) - P(L_n \leqslant k - 1)$$
$$= \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)], \quad 1 \leqslant k \leqslant n,$$

by (a). Next, we note that

$$P(L_n = 0) = P(L_n \leqslant 0) = \frac{p^{n+1}}{q} F_{n+2}^{(1)}(q/p) = \frac{p^{n+1}}{q}[F_{n+2}^{(1)}(q/p) - F_{n+2}^{(0)}(q/p)],$$

since $F_{n+2}^{(0)}(q/p) = 0$ by Definition 2.1. The last two relations show (b). Finally, (c) is also true, since

$$P(L_n \geqslant k) = 1 - P(L_n \leqslant k - 1) = 1 - \frac{p^{n+1}}{q} F_{n+2}^{(k)}(q/p), \quad 1 \leqslant k \leqslant n, \text{ by (a).}$$

We now have the following obvious corollary to the theorem.

Corollary 2.1: Let $\{F_n^{(k)}\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k and let L_n be as in Theorem 2.1. Assume $p = 1/2$. Then

(a) $P(L_n \leqslant k) = F_{n+2}^{(k+1)}/2^n, \quad 0 \leqslant k \leqslant n;$

(b) $P(L_n = k) = [F_{n+2}^{(k+1)} - F_{n+2}^{(k)}]/2^n, \quad 0 \leqslant k \leqslant n;$

(c) $P(L_n \geqslant k) = 1 - F_{n+2}^{(k)}/2^n, \quad 1 \leqslant k \leqslant n.$

Remark 2.2: McCarty [3] showed Corollary 2.1(c) by different methods.

We now proceed to offer the following alternative formulation and proof of Theorems 3.1 and 3.2 of Bollinger [2].

Corollary 2.2: For any finite set A, denote by $N(A)$ the number of elements in A, and let p, L_n, and $\{F_n^{(k)}\}_{n=0}^{\infty}$ be as in Corollary 2.1. Then

(a) $N(L_n < k) = F_{n+2}^{(k)}, \quad n \geqslant 1;$

(b) $N(L_n = k) = F_{n+2}^{(k+1)} - F_{n+2}^{(k)}, \quad n \geqslant 1.$

Proof: (a) Corollary 2.1(a) and the classical definition of probability give

$$\frac{N(L_n < k)}{2^n} = P(L_n < k) = P(L_n \leqslant k - 1) = \frac{F_{n+2}^{(k)}}{2^n}, \quad 1 \leqslant k \leqslant n + 1.$$

Furthermore, it is obvious that

$$N(L_n < 0) = 0 \quad \text{and} \quad N(L_n < k) = 2^n, \quad k \geqslant n + 2.$$

The last two relations and Lemma 2.2(a) establish (a). Part (b) follows from Corollary 2.1(b), by means of the classical definition of probability and Lemma 2.2(a), in an analogous manner.

3. WAITING TIMES AND LONGEST SUCCESS RUNS

Denote by N_k the number of Bernoulli trials until the occurrence of the first success run of length k ($k \geqslant 2$). Shane [7], Turner [8], Philippou and Muwafi [4], and Uppuluri and Patil [9] have all obtained alternative formulas for $P(N_k = n)$ ($n \geqslant k$). Presently, we derive another one, which is very simple and quite useful for computational purposes.

Theorem 3.1: Let N_k be a random variable denoting the number of Bernoulli trials until the occurrence of the first success run of length k ($k \geqslant 1$). Then

$$P(N_k = n) = \begin{cases} p^k, & n = k, \\ qp^k, & k + 1 \leqslant n \leqslant 2k, \\ P[N_k = n - 1] - qp^k P[N_k = n - 1 - k], & n \geqslant 2k + 1. \end{cases}$$

The proof will be based on the following lemma of [4] and [6]. (See also [5].)

Lemma 3.1: Let N_k be as in Theorem 3.1. Then

(a) $$P(N_k = n) = p^n \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n - k}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \quad n \geqslant k;$$

(b) $$P(N_k \leqslant n) = 1 - \frac{p^{n+1}}{q} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n + 1}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}, \quad n \geqslant k.$$

Proof of Theorem 3.1: By simple comparison, (a) and (b) of Lemma 3.1 give

$$P(N_k \leqslant n) = 1 - \frac{1}{qp^k} P(N_k = n + 1 + k), \quad n \geqslant k,$$

which implies

$$\begin{aligned} P(N_k = n) &= qp^k[1 - P(N_k \leqslant n - k - 1)] = qp^k\left[1 - \sum_{i=k}^{n-k-1} P(N_k = i)\right] \\ &= P[N_k = n - 1] - qp^k P[N_k = n - 1 - k], \quad n \geqslant 2k + 1. \end{aligned} \tag{3.1}$$

Next,

$$\begin{aligned} P(N_k = n) &= p^n F_{n-k+1}^{(k)}(q/p), \ n \geqslant k, \text{ by Lemma 3.1(a) and Lemma 2.2(b)}, \\ &= p^n\left(\frac{q}{p}\right)\left(1 + \frac{q}{p}\right)^{n-k-1}, \ k + 1 \leqslant n \leqslant 2k, \text{ by Lemma 2.2(a)}, \\ &= qp^k, \ k + 1 \leqslant n \leqslant 2k. \end{aligned} \tag{3.2}$$

Finally, we note that

$$P(N_k = k) = P\{\underbrace{ss \ldots s}_{k}\} = p^k. \tag{3.3}$$

Relations (3.1)-(3.3) establish the theorem.

Remark 3.1: An alternative proof of another version of Theorem 3.1, based on first principles, is given independently by Aki, Kuboki, and Hirano [1].

We end this section by noting the following relation between L_n and N_k.

Proposition 3.1: Let L_n be the length of the longest success run in n ($\geqslant 1$) Bernoulli trials, and denote by N_k the number of Bernoulli trials until the occurrence of the first success run of length k ($k \geqslant 1$). Then

$$P(L_n \geqslant k) = P(N_k \leqslant n).$$

Proof: It is an immediate corollary of Theorem 2.1 and Lemma 3.1(b).

4. GENERATING FUNCTION AND FACTORIAL MOMENTS OF L_n

In this section, we show that $\{P(L_n = k)\}_{k=0}^{n}$ is a probability distribution and derive the probability generating function and factorial moments of L_n. It should be noted that our present results are given in terms of finite sums of Fibonacci-type polynomials where the running index is the order of the polynomial. It is conceivable that they could be simplified, but we are not aware of any results concerning such sums, even for the Fibonacci sequence of order k. For the case $p = 1/2$, we give a table of the means and variances of L_n for $1 \leqslant n \leqslant 50$.

Proposition 4.1: Let L_n be the length of the longest success run in n ($\geqslant 1$) Bernoulli trials, and denote its generating function by $g_n(t)$. Also, set

$$x^{(0)} = 1 \quad \text{and} \quad x^{(r)} = x(x-1)\ \ldots\ (x-r+1), \quad r \geqslant 1.$$

Then

(a) $\displaystyle\sum_{k=0}^{n} P(L_n = k) = 1;$

(b) $\displaystyle g_n(t) = t^n - (t-1)\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}(q/p), \quad n \geqslant 1.$

(c) $\displaystyle E(L_n^{(r)}) = n^{(r)} - r\,\frac{p^{n+1}}{q}\sum_{k=r-1}^{n-1} k^{(r-1)} F_{n+2}^{(k+1)}(q/p), \quad 1 \leqslant r \leqslant n;$

(d) $\displaystyle E(L_n) = n - \frac{p^{n+1}}{q}\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}(q/p), \quad n \geqslant 1;$

(e) $$\sigma^2(L_n) = \frac{p^{n+1}}{q}\left[(2n-1)\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}(q/p) - 2\sum_{k=1}^{n-1} kF_{n+2}^{(k+1)}(q/p)\right] - \left[\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}(q/p)\right]^2, \quad n \geqslant 2.$$

Proof: (a) We observe that $F_{n+2}^{(0)}(q/p) = 0$, by Definition 2.1, and

$$F_{n+2}^{(n+1)}(q/p) = (q/p)[1 + (q/p)]^n = q/p^{n+1}, \text{ by Lemma 2.2(a).}$$

Then Theorem 2.1(b) gives

$$\sum_{k=0}^{n} P(L_n = k) = \sum_{k=0}^{n} \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)]$$

$$= \frac{p^{n+1}}{q}[F_{n+2}^{(n+1)}(q/p) - F_{n+2}^{(0)}(q/p)] = 1.$$

(b) By means of Theorem 2.1(b), Definition 2.1, and Lemma 2.2(a), we have

$$g_n(t) = E(t^{L_n}) = \sum_{k=0}^{n} t^k P(L_n = k)$$

$$= \sum_{k=0}^{n} t^k \frac{p^{n+1}}{q}[F_{n+2}^{(k+1)}(q/p) - F_{n+2}^{(k)}(q/p)]$$

$$= \frac{p^{n+1}}{q}\left[\sum_{k=0}^{n} t^k F_{n+2}^{(k+1)}(q/p) - \sum_{k=-1}^{n-1} t^{k+1} F_{n+2}^{(k+1)}(q/p)\right]$$

$$= \frac{p^{n+1}}{q}\left[t^n F_{n+2}^{(n+1)}(q/p) + \sum_{k=0}^{n-1}(t^k - t^{k+1})F_{n+2}^{(k+1)}(q/p)\right]$$

$$= t^n \frac{p^{n+1}}{q} F_{n+2}^{(n+1)}(q/p) + (1-t)\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}(q/p)$$

$$= t^n - (t-1)\frac{p^{n+1}}{q}\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}(q/p), \quad n \geqslant 1.$$

(c) It can be seen from (b), by induction on r, that the rth derivative of $g_n(t)$ is given by

$$\frac{\partial^r}{\partial t^r} g_n(t) = n^{(r)} t^{n-r} - r\frac{p^{n+1}}{q}\sum_{k=r-1}^{n-1} k^{(r-1)} t^{k-r+1} F_{n+2}^{(k+1)}(q/p)$$

$$- (t-1)\frac{p^{n+1}}{q}\sum_{k=r}^{n-1} k^{(r)} t^{k-r} F_{n+2}^{(k+1)}(q/p), \quad 1 \leqslant r \leqslant n.$$

The last relation and the formula

$$E(L_n^{(r)}) = \frac{\partial^r}{\partial t^r} g_n(t)\Big|_{t=1}$$

establish (c). Now (d) follows from (c) for $r = 1$. Finally, (e) follows from (c) by means of the relation

$$\sigma^2(L_n) = E(L_n^{(2)}) + E(L_n) - [E(L_n)]^2.$$

Corollary 4.1: Let L_n be as in Proposition 4.1 and assume $p = 1/2$. Then

(a) $g_n(t) = t^n - \dfrac{(t-1)}{2^n}\displaystyle\sum_{k=0}^{n-1} t^k F_{n+2}^{(k+1)}, \quad n \geqslant 1.$

(b) $E(L_n^{(r)}) = n^{(r)} - \dfrac{r}{2^n}\displaystyle\sum_{k=r-1}^{n-1} k^{(r-1)} F_{n+2}^{(k+1)}, \quad 1 \leqslant r \leqslant n;$

(c) $E(L_n) = n - \dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}, \quad n \geqslant 1.$

$$\text{(d)}\quad \sigma^2(L_n) = \frac{2n-1}{2^n}\sum_{k=0}^{n-1} F_{n+2}^{(k+1)} - \frac{1}{2^{n-1}}\sum_{k=1}^{n-1} kF_{n+2}^{(k+1)} - \frac{1}{2^{2n}}\left[\sum_{k=0}^{n-1} F_{n+2}^{(k+1)}\right]^2, \quad n \geqslant 2.$$

We conclude this paper by presenting a table of means and variances of L_n, when $p = 1/2$, for $1 \leqslant n \leqslant 50$.

n	$E(L_n)$	$\sigma^2(L_n)$	n	$E(L_n)$	$\sigma^2(L_n)$
1	.500000	.250000	26	4.090650	2.691060
2	1.000000	.500000	27	4.142980	2.713386
3	1.375000	.734375	28	4.193483	2.734376
4	1.687500	.964844	29	4.242285	2.754142
5	1.937500	1.183594	30	4.289496	2.772786
6	2.156250	1.381836	31	4.335215	2.790402
7	2.343750	1.553711	32	4.379535	2.807071
8	2.511719	1.702988	33	4.422539	2.822872
9	2.662109	1.829189	34	4.464300	2.837871
10	2.798828	1.938046	35	4.504889	2.852132
11	2.923828	2.031307	36	4.544370	2.865711
12	3.039063	2.112732	37	4.582799	2.878660
13	3.145752	2.184079	38	4.620233	2.891025
14	3.245117	2.247535	39	4.656719	2.902849
15	3.338043	2.304336	40	4.692306	2.914170
16	3.425308	2.355688	41	4.727035	2.925023
17	3.507553	2.402393	42	4.760948	2.935439
18	3.585327	2.445150	43	4.794080	2.945448
19	3.659092	2.484463	44	4.826468	2.955075
20	3.729246	2.520765	45	4.858143	2.964345
21	3.796131	2.554392	46	4.889137	2.973278
22	3.860043	2.585633	47	4.919477	2.981895
23	3.921239	2.614727	48	4.949192	2.990214
24	3.979944	2.641880	49	4.978305	2.998250
25	4.036356	2.667271	50	5.006842	3.006021

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SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

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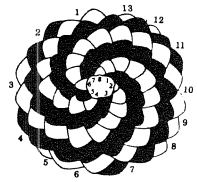

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CALL FOR PAPERS

The SECOND INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at San Jose State University, San Jose, CA, Aug. 13-16, 1986. This conference is sponsored jointly by The Fibonacci Association and San Jose State University.

Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Abstracts are requested by February 15, 1986. Manuscripts are requested by April 1, 1986. Abstracts and manuscripts should be sent to the chairman of the local committee. Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by July 15, 1986. All talks should be limited to one hour.

For further information concerning the conference, please contact either of the following:

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ON BERNSTEIN'S COMBINATORIAL IDENTITIES

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(Submitted February 1984)

*To the memory of Professor Dr. Leon Bernstein**

0. INTRODUCTION

Using elementary properties of algebraic numbers of certain finite extensions of $\mathbb{Q}$, L. Bernstein obtained in [1], [2], [3], [4], and [5] some combinatorial identities. In this paper, we want to give a clear and quick matrix treatment of Bernstein's technique, from which it will be seen that his combinatorial identities are in fact determinants.

In Section 1, writing the powers of an algebraic number ω of degree m over $\mathbb{Q}$ as

$$\omega^n = r_{1n} + r_{2n}\omega + \cdots + r_{mn}\omega^{m-1},$$

we give, in (1.4) and (1.5), the mth-order linear recurrences satisfied by the numbers

$$r_{jn}, \ n \in \mathbb{Z}, \ j = 1, 2, \ldots, m.$$

Let us note that L. Bernstein is always considering the case where $j = 1$ and ω is a unit of $\mathbb{Q}(\omega)$: see [3]; as far as [1], [2], [4], and [5] are concerned, L. Bernstein deals with the case $m = 3$.

In Section 2, Euler's generating functions are calculated in two ways: one with the help of the sums p_t of all symmetric functions of weight t; the other using the multinomial theorem. The second method is used by L. Bernstein, but the concluding remark of the last paragraph still applies.

A very general procedure combining the properties of the norm of an algebraic integer and Cramer's rule is described in Section 3, which leads to what can be called *combinatorial identities*.

In Section 4, we conclude this paper by giving a formula for r_{jn} involving the determinant of a Vandermonde matrix and the determinant of a matrix that is "almost" of the Vandermonde type.

1. RECURRENCE RELATIONS

Let ω be a root of the polynomial

$$f(X) = X^m + k_1X^{m-1} + \cdots + k_{m-1}X + k_m = (X - \alpha_1)(X - \alpha_2) \ldots (X - \alpha_m)$$

irreducible over $\mathbb{Q}$ with m distinct (nonzero) roots $\alpha_1 = \omega, \alpha_2, \ldots, \alpha_m$, whence the field $\mathbb{Q}(\omega)$ is of degree m over $\mathbb{Q}$. Let us consider the (positive, negative, zero) powers of ω.

**Professor L. Bernstein died on March 12, 1984, of a cerebral hemorrhage.*

For $n \in \mathbb{Z}$, let

$$\omega^n = r_{1n} + r_{2n}\omega + \cdots + r_{mn}\omega^{m-1}$$

with coefficients in $\mathbb{Q}$. Since

$$\omega^m = -k_1\omega^{m-1} - \cdots - k_{m-1}\omega - k_m,$$

we obtain the equality

$$\omega^{n+1} = -k_m r_{mn} + (r_{1n} - k_{m-1}r_{mn})\omega + (r_{2n} - k_{m-2}r_{mn})\omega^2 + \cdots + (r_{m-1,\,n} - k_1 r_{mn})\omega^{m-1},$$

which leads to the system

$$\begin{cases} r_{1,\,n+1} = 0r_{1n} + 0r_{2n} + \cdots + 0r_{m-1,\,n} - k_m r_{mn}, \\ r_{2,\,n+1} = 1r_{1n} + 0r_{2n} + \cdots + 0r_{m-1,\,n} - k_{m-1} r_{mn}, \\ r_{3,\,n+1} = 0r_{1n} + 1r_{2n} + \cdots + 0r_{m-1,\,n} - k_{m-2} r_{mn}, \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ r_{m,\,n+1} = 0r_{1n} + 0r_{2n} + \cdots + 1r_{m-1,\,n} - k_1 r_{mn}. \end{cases} \tag{1.1}$$

Define the matrices

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -k_m \\ 1 & 0 & \cdots & 0 & -k_{m-1} \\ 0 & 1 & \cdots & 0 & -k_{m-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -k_1 \end{bmatrix}, \qquad R_n = \begin{bmatrix} r_{1n} \\ r_{2n} \\ r_{3n} \\ \vdots \\ r_{mn} \end{bmatrix},$$

$$\Omega = [1 \quad \omega \quad \omega^2 \quad \cdots \quad \omega^{m-1}], \qquad I_m = \text{Identity matrix},$$

of dimension $m \times m$, $m \times 1$, $1 \times m$, $m \times m$, respectively.

Hence, we have $\omega^n = \Omega R_n$ and

$$R_n = I_m R_n, \quad R_{n+1} = CR_n, \quad \ldots, \quad R_{n+t} = C^t R_n, \quad \ldots,$$

from which we conclude

$$R_n = C^n R_0 \quad \text{with } R_0 = [1 \quad 0 \quad \cdots \quad 0]^t,$$

and

$$r_{jn} = (j,\ 1) \text{ element of } C^n. \tag{1.2}$$

It is worth noting that system (1.1) leads to the following matrix, which means that it suffices to have a formula for r_{1t} in order to know all the coefficients of ω^n:

$$R_n = \begin{bmatrix} r_{1n} \\ r_{1,\,n-1} + \dfrac{k_{m-1}}{k_m} r_{1n} \\ r_{1,\,n-2} + \dfrac{k_{m-1}}{k_m} r_{1,\,n-1} + \dfrac{k_{m-2}}{k_m} r_{1n} \\ \vdots \\ r_{1,\,n+1-m} + \dfrac{k_{m-1}}{k_m} r_{1,\,n+2-m} + \cdots + \dfrac{k_1}{k_m} r_{1n} \end{bmatrix}.$$

It is well known that the characteristic equation of C is

$$\lambda^m + k_1\lambda^{m-1} + \cdots + k_{m-1}\lambda + k_m,$$

whereupon the eigenvalues of C are $\alpha_1, \alpha_2, \ldots, \alpha_m$. Since

$$C^m = -k_1C^{m-1} - \cdots - k_{m-1}C - k_mI_m,$$

we deduce

$$R_{n+m} = C^mR_n = -k_1R_{n+m-1} - \cdots - k_{m-1}R_{n+1} - k_mR_n,$$

from which we conclude (with $1 \leqslant j \leqslant m$):

$$r_{jn} = -k_1r_{j,\,n-1} - \cdots - k_{m-1}r_{j,\,n-m+1} - k_mr_{j,\,n-m}. \tag{1.3}$$

In particular we obtain, for the coefficients of ω^t and ω^{-t} with $t \geqslant m$, the two following m^{th}-order linear recurrences (with $1 \leqslant j \leqslant m$):

$$r_{jt} = -k_1r_{j,\,t-1} - \cdots - k_{m-1}r_{j,\,t-m+1} - k_mr_{j,\,t-m}, \tag{1.4}$$

$$r_{j,\,-t} = -\frac{k_{m-1}}{k_m} r_{j,\,-t+1} - \cdots - \frac{k_1}{k_m} r_{j,\,-t+m-1} - \frac{1}{k_m} r_{j,\,-t+m}, \tag{1.5}$$

with the initial conditions for $0 \leqslant i \leqslant m-1$ being

$$r_{ji} = (j,\ 1) \text{ element of } C^i = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{elsewhere}, \end{cases} \tag{1.6}$$

$$r_{j,\,-i} = (j,\ 1) \text{ element of } C^{-i}. \tag{1.7}$$

Note that for the rest of this article, as opposed to [1], [2], [3], [4], and [5], we do not restrict ourselves to the case $j = 1$.

2. GENERATING FUNCTIONS

Using the mth-order linear recurrence given in (1.4) and the known values of r_{ji} in (1.6) we obtain, for $j = 1, \ldots, m$,

$$\left(\sum_{n=0}^{\infty} r_{jn}X^n\right)(1 + k_1X + \cdots + k_{m-1}X^{m-1} + k_mX^m) = X^{j-1} + k_1X^j + \cdots + k_{m-j}X^{m-1}.$$

So

$$\begin{aligned}\sum_{n=0}^{\infty} r_{jn} X^n &= \frac{X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1}}{1 + k_1 X + \cdots + k_{m-1} X^{m-1} + k_m X^m} \\ &= \frac{X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1}}{(1-\alpha_1 X)(1-\alpha_2 X)\ \ldots\ (1-\alpha_m X)} \\ &= (X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1})(1 + p_1 X + p_2 X^2 + \cdots + p_t X^t + \cdots),\end{aligned}$$

where p_t stands for the sum of all symmetric functions of weight t in $\alpha_1, \ldots, \alpha_m$. Hence, we conclude

$$r_{jn} = \begin{cases} \sum_{t=0}^{m-j} k_t p_{n-j+1-t} & \text{if } n \geqslant j-1, \\ 0 & \text{if } 0 \leqslant n < j-1, \end{cases} \tag{2.1}$$

where, as stated in [6],

$$p_t = \sum_{i=1}^{m} \frac{\alpha_i^{m-1+t}}{f'(\alpha_i)}, \quad \text{with } p_0 = k_0 = 1,\ p_{-1} = p_{-2} = \cdots = p_{-m+1} = 0.$$

Similarly, using (1.5), (1.3), and then (1.6), we obtain

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} r_{j,-n} X^n\right)(k_m + k_{m-1} X + \cdots + k_1 X^{m-1} + X^m) \\ &= k_m r_{j0} + k_m r_{j,-1} X + k_m r_{j,-2} X^2 + \cdots + k_m r_{j,-m+2} X^{m-2} + k_m r_{j,-m+1} X^{m-1} \\ &\quad + k_{m-1} r_{j0} X + k_{m-1} r_{j,-1} X^2 + \cdots + k_{m-1} r_{j,-m+3} X^{m-2} + k_{m-1} r_{j,-m+2} X^{m-1} \\ &\qquad + k_{m-2} r_{j0} X^2 + \cdots + k_{m-2} r_{j,-m+4} X^{m-2} + k_{m-2} r_{j,-m+3} X^{m-1} \\ &\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &\qquad\qquad + k_2 r_{j0} X^{m-2} + k_2 r_{j,-1} X^{m-1} \\ &\qquad\qquad\qquad + k_1 r_{j0} X^{m-1} \\ &= k_m r_{j0} - r_{j,m-1} X - r_{j,m-2} X^2 - \cdots - r_{j2} X^{m-2} - r_{j1} X^{m-1} \\ &\qquad - k_1 r_{j,m-2} X - k_1 r_{j,m-3} X^2 - \cdots - k_1 r_{j1} X^{m-2} \\ &\qquad\quad \vdots \qquad\qquad \vdots \\ &\qquad - k_{m-3} r_{j2} X - k_{m-3} r_{j1} X^2 \\ &\qquad - k_{m-2} r_{j1} X \\ &= K_j(X),\end{aligned}$$

where the polynomial $K_j(X)$ is defined by:

$$K_j(X) = \begin{cases} k_m & \text{if } j = 1, \\ -k_{m-j}X - k_{m-j-1}X^2 - \cdots - k_0X^{m-j+1} & \text{if } 2 \leqslant j \leqslant m. \end{cases} \tag{2.2}$$

Thus,

$$\sum_{n=0}^{\infty} r_{j,-n} X^n = \frac{K_j(X)}{k_m(1 - \alpha_1^{-1}X)(1 - \alpha_2^{-1}X) \ldots (1 - \alpha_m^{-1}X)}.$$

For $i = 1, \ldots, m$, let k_i^* denote the ith elementary symmetric functions in $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$, so that

$$k_i^* = (-1)^i k_{m-i}/k_m,$$

as is well known. Now, let p_t^* stand for the sum of all symmetric functions of weight t in $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$; letting

$$\begin{aligned} F(Y) &= Y^m + \frac{k_{m-1}}{k_m} Y^{m-1} + \cdots + \frac{k_1}{k_m} Y + \frac{1}{k_m} \\ &= (Y - \alpha_1^{-1})(Y - \alpha_2^{-1}) \ldots (Y - \alpha_m^{-1}), \end{aligned}$$

we have

$$p_t^* = \sum_{i=1}^{m} \frac{(\alpha_i^{-1})^{m-1+t}}{F'(\alpha_i^{-1})},$$

and $p_{-1}^* = p_{-2}^* = \cdots = p_{-m+1}^* = 0$.

We conclude that

$$\sum_{n=0}^{\infty} r_{j,-n} X^n = k_m^{-1}K_j(X)(1 + p_1^*X + p_2^*X^2 + \cdots + p_t^*X^t + \cdots),$$

and this leads to

$$r_{j,-n} = \begin{cases} p_n^* & \text{if } j = 1, \\ -\dfrac{1}{k_m} \displaystyle\sum_{t=0}^{m-j} k_{m-j-t} p_{n-1-t}^* & \text{if } j = 2, \ldots, m. \end{cases} \tag{2.3}$$

Instead of using p_t (resp. p_t^*), one can also use the multinomial theorem from [7] to find r_{jn} (resp. $r_{j,-n}$). For example, as in [10], we have (within an irrelevant radius of convergence)

$$\begin{aligned} &\sum_{n=0}^{\infty} r_{jn} X^n \\ &= (X^{j-1} + k_1X^j + \cdots + k_{m-j}X^{m-1}) \left[\sum_{j=0}^{\infty} (-1)^j (k_1X + \cdots + k_{m-1}X^{m-1} + k_mX^m)^j \right] \\ &= (X^{j-1} + k_1X^j + \cdots + k_{m-j}X^{m-1}) \left(\sum_{i=0}^{\infty} A(i)X^i \right), \end{aligned}$$

where

$$A(i) = \sum (-1)^{t_1+t_2+\cdots+t_m} \frac{(t_1 + t_2 + \cdots + t_m)!}{t_1!t_2! \ldots t_m!} k_1^{t_1} k_2^{t_2} \ldots k_m^{t_m},$$

the last sum being taken over all m-tuples $(t_1, t_2, \ldots, t_m)$ of $\mathbb{N}^m$ such that

$$t_1 + 2t_2 + \cdots + mt_m = i;$$

therefore,

$$r_{jn} = \sum_{t=0}^{m-j} k_t A(n - j + 1 - t), \tag{2.4}$$

with the convention $k_0 = 1$, and $A(i) = 0$ for i a negative integer.

Similarly, for $j = 2, \ldots, m$, we have

$$\sum_{n=0}^{\infty} r_{j,-n} X^n = (-k_{m-j}X - k_{m-j-1}X^2 - \cdots - k_0 X^{m-j+1})\left(\sum_{i=0}^{\infty} B(i)X^i\right),$$

where

$$B(i) = \sum \frac{(-1)^{t_1+t_2+\cdots+t_m}(t_1 + t_2 + \cdots + t_m)!}{k_m^{t_1+t_2+\cdots+t_m+1} t_1!t_2! \ldots t_m!} k_{m-1}^{t_1} k_{m-2}^{t_2} \ldots k_1^{t_{m-1}},$$

the last sum being taken over all m-tuples $(t_1, t_2, \ldots, t_m)$ such that

$$t_1 + 2t_2 + \cdots + mt_m = i.$$

Defining $B(i)$ to be 0 for $i < 0$, we therefore obtain

$$r_{j,-n} = \begin{cases} k_m B(n) & \text{if } j = 1, \\ -\sum_{t=0}^{m-j} k_{m-j-t} B(n - 1 - t) & \text{if } j = 2, \ldots, m. \end{cases} \tag{2.5}$$

Although formulas (2.4) and (2.5) with $j = 1$ may look different from Bernstein's formulas (1.14) and (1.14a) in [3], they are in fact equivalent.

To conclude this section, let us remark that if one wants a formula for the powers α^n for $\alpha = a_1 + a_2\omega + \cdots + a_m\omega^{m-1}$, one can use the characteristic polynomial of α to write an equation of the form

$$\alpha^m + b_1\alpha^{m-1} + \cdots + b_m = 0,$$

and apply the above procedure to get the powers of α as functions of the b_i's.

3. A GENERAL RESULT

If $\alpha = \sum_{i=1}^{m} a_{i1}\omega^{i-1} \in \mathbb{Q}(\omega)$ and if, for $j = 1, \ldots, m$,

$$\alpha\omega^{j-1} = \sum_{i=1}^{m} a_{ij}\omega^{i-1},$$

then $N_{\mathbb{Q}(\omega)/\mathbb{Q}}(\alpha) = \det A$, where $A = [a_{ij}]$; see [11].

Let us consider the equality

$$\gamma = \alpha\beta$$

with $\beta \in \mathbb{Q}(\omega)$ where, for $j = 1, \ldots, m$,

$$\beta\omega^{j-1} = \sum_{i=1}^{m} b_{ij}\,\omega^{i-1}, \quad \gamma\omega^{j-1} = \sum_{i=1}^{m} g_{ij}\,\omega^{i-1};$$

taking $B = [b_{ij}]$, $G = [g_{ij}]$, $\Omega = [1 \quad \omega \quad \dots \quad \omega^{m-1}]$, we have

$$\alpha\Omega = \Omega A, \quad \beta\Omega = \Omega B, \quad \gamma\Omega = \Omega G \quad \text{and} \quad (\alpha\beta)\Omega = (\beta\alpha)\Omega = \Omega(BA) = \Omega(AB),$$

hence, the identity

$$G = AB = BA.$$

If, for a matrix M, we denote its jth column by $M_{(j)}$, we conclude

$$\alpha\beta = \gamma = \Omega G_{(1)} = \Omega AB_{(1)} = \Omega BA_{(1)}.$$

In particular, we obtain $AB_{(1)} = G_{(1)}$, i.e.,

$$\begin{cases} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} = g_{11} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} = g_{21} \\ \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mm}b_{m1} = g_{m1}. \end{cases}$$

Let $\alpha \neq 0$; then $N_{\mathbb{Q}(\omega)/\mathbb{Q}}(\alpha) = \det A \neq 0$, and the matrix A of the coefficients of the above system has $\det A \neq 0$. For $i = 1, \dots, m$, Cramer's rule gives

$$b_{i1} = \frac{1}{\det A} \sum_{t=1}^{m} g_{t1}\ \operatorname{cof}(a_{ti}), \tag{3.1}$$

where $\operatorname{cof}(a_{ti})$ is the cofactor of the (t, i) element of A.

Similarly, if $\beta \neq 0$, we also have, for $i = 1, \dots, m$,

$$a_{i1} = \frac{1}{\det B} \sum_{t=1}^{m} g_{t1}\ \operatorname{cof}(b_{ti}). \tag{3.2}$$

In [2] Bernstein took $m = 3$, $k_1 = 0$, $k_2 = g \geqslant 2$, $k_3 = -1$, $\alpha = \omega^s$, $\beta = \omega^{-s}$, $\gamma = 1$, obtained recurrence relations for the rational coefficients of α and β, calculated the generating function of these coefficients, and then obtained his combinatorial identities, which turn out to be special cases of our formulas (3.1) and (3.2); see formulas (4.2) and (4.3) of [2], see also [1], [4], and [5]. The same was done in a lengthy way for arbitrary m in [3]. It turns out that in [3], Bernstein is considering $\alpha = \omega^{m-n+1}$, $\beta = \alpha^{-1}$, $\gamma = 1$; nevertheless, he forgot to write a_0 in front of the determinant appearing in his formula (2.3b), so formulas (2.4)-(2.7) must be modified accordingly [e.g., the power of a_0 in (2.7) is m].

Let us observe that, from a linear algebra point of view, the equality $G = AB$ with $\det A \neq 0$ immediately implies that one can solve for the entries of B in terms of the entries of G and minors of A.

4. VANDERMONDE MATRIX TREATMENT

Consider the matrix C defined in Section 1, the Vandermonde matrix V, and the diagonal matrix D shown on the following page:

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{m-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{m-1} \end{bmatrix}, \qquad D = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \alpha_m \end{bmatrix}.$$

Since $VC = DV$, we have $C = V^{-1}DV$, and

$$\det V = |V| = \prod_{j>i} (\alpha_j - \alpha_i).$$

It is possible to give an explicit formula for C in terms of det V and in terms of the determinant of a certain matrix that is almost of the Vandermonde type. Let us do it.

For $t = 1, \dots, m-1$, denote by $k_t(j)$ the tth elementary symmetric function in $\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_m$, whence

$$k_t(j) = k_t + k_{t-1}\alpha_j + \cdots + k_1\alpha_j^{t-1} + \alpha_j^t. \tag{4.1}$$

With respect to V, define

$$V_i = \operatorname{cof}(\alpha_i^{m-1}).$$

Then it is well known that

$$V^{-1} = \frac{1}{|V|}\begin{bmatrix} k_{m-1}(1)V_1 & k_{m-1}(2)V_2 & \dots & k_{m-1}(m)V_m \\ k_{m-2}(1)V_1 & k_{m-2}(2)V_2 & \dots & k_{m-2}(m)V_m \\ \vdots & \vdots & & \vdots \\ k_1(1)V_1 & k_1(2)V_2 & \dots & k_1(m)V_m \\ V_1 & V_2 & \dots & V_m \end{bmatrix}$$

(see for instance [9]). For a proof, call W the matrix $|V|V^{-1}$, and show that $WV = |V|I_m$ by comparing the (i, j) elements: if $i = j$, you obtain $|V|$; if $j < i$, you get 0, using (4.1); if $j > i$, you obtain 0, using the fact that

$$\alpha_t^{j-1}k_{m-i}(t) = -k_{m-i+1}\alpha_t^{j-2} - k_{m-i+2}\alpha_t^{j-3} - \cdots - k_{m-1}\alpha_t^{j-i} - k_m\alpha^{j-i-1}.$$

By definition, for all $n \in \mathbb{Z}$, let H_n be given by

$$H_n = \det\begin{bmatrix} 1 & \alpha_1 & \dots & \alpha_1^{m-2} & \alpha_1^{n-1} \\ 1 & \alpha_2 & \dots & \alpha_2^{m-2} & \alpha_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_m & \dots & \alpha_m^{m-2} & \alpha_m^{n-1} \end{bmatrix};$$

as is easily verified, this determinant H_n satisfies the mth-order linear recurrence

$$H_n = -k_1H_{n-1} - k_2H_{n-2} - \cdots - k_mH_{n-m}.$$

Keeping in mind formula (4.1), we find for the (j, t) element of $C^n = V^{-1}D^nV$ (with $1 \leqslant j, t \leqslant m$):

$$(j, t) \text{ element of } C^n = \frac{1}{|V|}\sum_{i=0}^{m-j} k_{m-j-i}H_{n+t+i}.$$

So, for every $n \in \mathbb{Z}$,

$$r_{jn} = \frac{1}{|V|}\sum_{i=0}^{m-j} k_{m-j-i}H_{n+1+i}. \tag{4.2}$$

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MULTIPLE OCCURRENCES OF BINOMIAL COEFFICIENTS

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I. INTRODUCTION

How many times can the same number appear in Pascal's triangle? After eliminating occurrences due to symmetry, $\binom{n}{k} = \binom{n}{n-k}$, and the uninteresting occurrences of $1 = \binom{n}{0}$ and $n = \binom{n}{1}$, the answer to this question is not clear. More precisely, if $1 < k \leqslant n/2$, we say that $\binom{n}{k}$ is a *proper* binomial coefficient. Are there integers that can be expressed in different ways as proper binomial coefficients?

Enumeration by hand or with a computer program produces some cases, given in Table 1. The smallest is 120, which equals

$$\binom{10}{3} \quad \text{and} \quad \binom{16}{2}.$$

Table 1. Small Multiple Occurrences of Binomial Coefficients

INTEGER	BINOMIAL COEFFICIENTS
120	$\binom{10}{3}$, $\binom{16}{2}$
210	$\binom{10}{4}$, $\binom{21}{2}$
1540	$\binom{22}{3}$, $\binom{56}{2}$
3003	$\binom{14}{6}$, $\binom{15}{5}$, $\binom{78}{2}$
7140	$\binom{36}{3}$, $\binom{120}{2}$
11628	$\binom{19}{5}$, $\binom{153}{2}$
24310	$\binom{17}{8}$, $\binom{221}{2}$

There is even an instance of a number, 3003, which can be expressed in three different ways. No clear pattern emerges; the cases just seem to be sprinkled among the binomial coefficients. We conjecture that, for any t, there exist infinitely many integers that may be expressed in t different (proper) ways as binomial coefficients.

Here we prove the conjecture for the case $t = 2$. The proof is constructive and depends in an unexpected way on the Fibonacci sequence.

II. THE CONSTRUCTION

We seek solutions to

$$\binom{n}{k} = \binom{n-1}{k+1}, \tag{1}$$

an especially tractable situation because it leads to a second-order equation. In particular, if (1) holds, then $n(k+1) = (n-k)(n-k-1)$. Let

$$x = n - k - 1. \tag{2}$$

Then $x(x+1) = n(n-x)$ so $n^2 - xn - (x^2 + x) = 0$ and

$$n = \frac{x + \sqrt{5x^2 + 4x}}{2} \tag{3}$$

(since n is positive). Integer solutions to (3) therefore lead to integer solutions to (1).

Since $5x^2 + 4x$ is even if and only if x is even, this means we must find integers x such that $5x^2 + 4x$ is a perfect square. Now x and $5x + 4$ have no common factors except possibly 2 or 4, so a natural slightly stronger condition would be that both x and $5x + 4$ be perfect squares. In other words, we need to find integers z such that $5z^2 + 4$ is a perfect square. These are given by the following lemma.

Lemma 1: Let F_j denote the Fibonacci sequence. Then, for all j,

$$(F_{j-1} + F_{j+1})^2 - 5F_j^2 = 4(-1)^j.$$

Proof: A straightforward calculation (see, e.g., [2], pp. 148-149) shows

$$(F_{j+1} + F_{j-1})^2 - 5F_j^2 = 4(F_{j-1}^2 + F_jF_{j-1} - F_j^2) = -4(F_j^2 + F_{j+1}F_j - F_{j+1}^2),$$

which yields the result by induction.

The lemma tells us that for any j even, $z = F_j$ gives the perfect square

$$5z^2 + 4 = (F_{j-1} + F_{j+1})^2.$$

This completes the construction.

Theorem 1: Let F_j denote the Fibonacci sequence. Then, for any even j, there exists a solution to (1), where $x = F_j^2$ and n and k are given by (2) and (3).

Remark: Letting L_j denote the Lucas sequence as usual, we can write this solution as

$$n = \frac{F_jL_j + F_j^2}{2}, \qquad k = \frac{F_jL_j - F_j^2}{2} - 1.$$

Theorem 2: Theorem 1 gives all solutions to (1).

Proof: It follows from the preceding discussion that any solution to (1) corresponds via (2) to some integer x such that $5x^2 + 4x$ is a perfect square. Let a (resp. b) be the number of times 2 divides $5x + 4$ (resp. x). If $a > 2$, then

$b = 2$ and, conversely, $b > 2$ implies $a = 2$. Since $5x + 4$ and x have no common factors except (possibly) 2 or 4, $(5x + 4)/2^a$ is a perfect square, as is $x/2^b$. Therefore, $a + b$ is even, so a and b are both even or both odd. In the former case, x and $5x + 4$ are perfect squares. We claim this leads precisely to the class of solutions given by Theorem 1. In the latter case, it follows that $a = b = 1$. Thus, we seek integers z such that $5z^2 + 2$ is a perfect square. We further claim that no such integers exist. The two claims can be shown to follows from the general theory of the so-called Pell equation (see, for example, [1] for the first claim, and [3, pp. 350-358] for the second claim). For completeness, we give a simple proof that does not rely on the general theory.

Let $\{A_n\}$ denote any sequence of positive numbers satisfying the recurrence $A_n + A_{n+1} = A_{n+2}$. The argument from Lemma 1 shows that, for all n,

$$(A_{n-1} + A_{n+1})^2 - 5A_n^2 = 4(A_{n-1}^2 + A_{n-1}A_n - A_n^2) = -(A_n + A_{n+2})^2 + 5A_{n+1}^2.$$

Therefore, given any solution z, y to $5z^2 + k = y^2$, we can construct smaller solutions by setting

$$A_i = z, \quad A_{i-1} = \frac{y - z}{2}, \quad A_{i+1} = \frac{y + z}{2},$$

and extending the sequence $\{A_n\}$ backward according to the recurrence

$$A_n + A_{n+1} = A_{n+2}.$$

[The solutions will be $z = A_j$, $y = A_{j-1} + A_{j+1}$, where $j \equiv i \pmod 2$, with

$$|k| = 4|A_n + A_nA_{n+1}^2 - A_{n+1}^2|, \text{ for all } n.]$$

Now, let (z, y) be any integer solution to $5z^2 + 4 = y^2$. Set

$$A_i = z \quad \text{and} \quad A_{i+1} = \frac{y + z}{2} \quad \text{(an integer)}.$$

Then extend $\{A_n\}$ backward to get the solution corresponding to A_{i-2} and A_{i-1}. If $z \geqslant 3$, then

$$.61z \leqslant \frac{z\sqrt{5} - z}{2} \leqslant A_{i-1} = \frac{y - z}{2} \leqslant \frac{z\sqrt{5} + .5 - z}{2} \leqslant .72z,$$

whence $.28z \leqslant A_{i-2} \leqslant .39z$. Therefore, if $A_i \geqslant 3$, the solution corresponding to A_{i-2} is smaller. Repeatedly extend $\{A_n\}$ backward until $A_j < 3$. Since the only such integer solution is $(1, 3)$, z must have been a Fibonacci Number. This verifies the first claim. The second claim, that $5z^2 + 2 = y^2$ has no solutions, follows immediately from the fact that $y^2 \equiv 2 \pmod 5$ has none.

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LINEAR RECURRENCE RELATIONS WITH BINOMIAL COEFFICIENTS

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A linear recurrence relation of the nth order is defined as

$$T_{i+n} = \sum_{j=1}^{n} a_j T_{i+n-j}, \qquad i = 0, 1, 2, \ldots, \tag{1}$$

where $a_1, a_2, \ldots, a_n$ are given coefficients. When all the coefficients are set equal to 1, the relation generates t-Fibonacci sequences [1], the Fibonacci sequence for $n = 2$, the Tribonacci sequence for $n = 3$ [2], and so on.

Another case arises when the coefficients in relation (1) are set equal to binomial coefficients, i.e.,

$$T_{i+n} = \sum_{j=1}^{n} \binom{n-1}{j-1} T_{i+n-j}. \tag{2}$$

For $n = 2$, relation (2) is reduced to the Fibonacci sequence and the recurring sequences generated by the recurrence relations with binomial coefficients (2) can be considered as another generalization of the Fibonacci sequence. These "binomial" sequences interest the author because of their relation to the dynamic development of self-replicating biochemical systems [3].

Consider self-replication of the type shown in Figure 1, i.e.,

$$A_1 \xrightarrow{k_1} A_2 + A_1 \tag{R1}$$

$$A_j \xrightarrow{k_j} A_{j+1}, \qquad j = 2, \ldots, n-1, \tag{Rj}$$

$$A_n \xrightarrow{k_n} A_1 \tag{Rn}$$

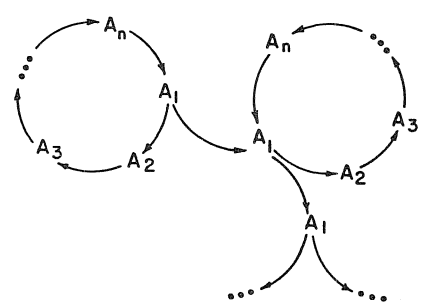

Figure 1. A Schematic Diagram of a Self-Replicating Process

Species A_1 forms species A_2 while reproducing itself in reaction (R1). Species A_2 undergoes $n-1$ transformations by reactions (R2)-(Rn) producing in the last step of this sequence the initial species A_1. Assume the first-order mass-action law for each of the reactions, that is, the rate of the jth reaction is proportional to the concentration of species A_j, and also assume that the rate coefficients are identical, i.e., $k_j = k$ for $j = 1, 2, \ldots, n$, the differential equations which describe the kinetics of the system take the form

$$\frac{d[A_1]}{dt} = k[A_n], \qquad \frac{d[A_j]}{dt} = k[A_{j-1}] - k[A_j], \qquad j = 2, \ldots, n,$$

with initial conditions

$$[A_1]_{t=0} = C_0,$$
$$[A_j]_{t=0} = 0, \qquad j = 2, 3, \ldots, n,$$

where:

$[A_j]$ is the concentration of species A_j;

C_0 is the initial concentration of species A_1;

t is time.

Dividing both sides of each differential equation by kC_0 and introducing dimensionless variables

$$a_j = [A]_j/C_0 \qquad \text{for } j = 1, 2, \ldots, n$$

and $\quad \tau = kt,$

these equations can be rewritten as

$$\frac{da_1}{d\tau} = a_n, \quad \frac{da_j}{d\tau} = a_{j-1} - a_j, \qquad j = 2, \ldots, n,$$

with initial conditions

$$a_j\Big|_{\tau=0} = \delta_{1j}.$$

The characteristic equation for this system of differential equations is

$$r(r+1)^{n-1} - 1 = 0. \tag{3}$$

Thus, the roots of (3) determine the kinetics of the reaction sequence.

Returning to the "binomial" sequence (2), the auxiliary polynomial for this sequence is

$$x^n - \sum_{j=1}^{n} \binom{n-1}{j-1} x^{n-j} = 0 \quad \text{or} \quad x^n - (x+1)^{n-1} = 0. \tag{4}$$

Defining $r = 1/x$, (4) becomes (3). Analysis of the "binomial" sequences and their relations can provide information necessary for understanding self-replication of the type considered here. It would be of interest to determine all possible relationships between the roots of equation (4) and their dependence on the order n.

For example, defining $z = r + 1$, equation (3) becomes

$$z^n - z^{n-1} - 1 = 0. \tag{5}$$

Equation (5) and its solution are discussed in a number of articles [4]-[7]. From the results of Ferguson [6] and Hoggatt & Alladi [7], the following conclusions can be made for roots of equation (3):

Property 1: For all n, there exists only one positive real root r_1—the dominant root of (3)—such that

$$r_1 = 1/\phi_n \tag{6}$$

where

$$\phi_n = \lim_{i \to \infty} \frac{T_{i+1}}{T_i} \tag{7}$$

is the limiting ration of the "binomial" sequence of the nth order.

Proof: It was proven in [6] and [7] that (5) has a single positive root with largest absolute value, λ_1. That is, λ_1 is the dominant root of (5). Since $r = z - 1$, $r_1 = \lambda_1 - 1$ is the dominant root of (3). Furthermore, since $x = 1/(z - 1)$, (4) has only one positive real root, $x_1 = 1/(\lambda_1 - 1)$. Root x_1 has the largest absolute value: It was proven in [6] that $\lambda_1 - 1 \leqslant |z - 1|$; therefore

$$\frac{1}{\lambda_1 - 1} \geqslant \frac{1}{|z - 1|} \qquad \text{or} \qquad x_1 \geqslant |x|.$$

Thus, there exists a single root of largest absolute value for (4); this satisfies the condition of the lemma in [7], proving the existence of limit (7) and that $x_1 = \phi_n$. Equation (6) follows from $x_1 = \phi_n$ and $r_1 = 1/x_1$.

Property 2: For n even, there is also one negative real root.

Proof: This follows from applying Descartes' Rule of Signs to equation (5) and using the relationship $r = z - 1$.

Property 3: $\lim_{n \to \infty} r_1 = \lim_{n \to \infty}(1/\phi_n) = 0$.

Proof: This follows from $r_1 = \lambda_1 - 1$ and the result of Theorem B in [6] that

$$\lim_{n \to \infty} \lambda_1 = 1.$$

Property 4: All the roots are distinct and lie in the intersection of the two annuli

$$\lambda_0 \leqslant |r_j + 1| \leqslant r_1 + 1 \quad \text{and} \quad r_1 \leqslant |r_j| \leqslant 1 + \lambda_0,$$

where r_j, $j = 2, 3, \ldots, n$, are the (complex) roots of equation (3) and λ_0 is the largest real solution of $u^n + u^{n-1} - 1 = 0$ $(0 < \lambda_0 < 1 < r_1 + 1 < 2)$.

Proof: These results follow from Theorem A in [6] and $r = z - 1$.

Species concentrations a_j are determined by linear combinations of n exponential terms $e^{r_\ell \tau}$, where r_ℓ $(\ell = 1, 2, \ldots, n)$ are the roots of (3). Based on properties (1)-(4) above, the dynamic behavior of reaction system (R1)-(Rn) is dominated by the term $e^{r_1 \tau}$ $(= e^{\tau/\phi_n})$. At $n \geqslant 14$ there are complex roots r_ℓ with positive real parts (e.g., $0.00617 \pm 0.38302i$), thus indicating the appearance

of nondecaying, oscillatory components in the concentration profiles. The exponential term for a complex root takes the form $e^{\alpha\tau}e^{\beta\tau i}$, where $r = \alpha + \beta i$. The term $e^{\beta\tau i}$ indicates oscillatory behavior of species concentrations in time. If α is negative, oscillations are decaying with increase in τ. For $\alpha > 0$, the oscillatory behavior is nondecaying. More detailed general analysis of the reaction kinetics depends on whether the roots of (3) and their dependence on n can be isolated further. Thus, it would be of interest to determine the frequencies and amplitudes of oscillatory components in concentration profiles.

The following recurrence expression,

$$\frac{\log \phi_n}{\log \phi_{n-1}} \approx \frac{\log n}{\log(n-1)}, \tag{8}$$

seems to be an approximate relationship between the limiting ratios (or the dominant roots) of different orders (see Figure 2). Since the dominant root of (3) is specified by ϕ_n, namely $r_1 = 1/\phi_n$, and the dominant root determines the main dynamic behavior of the reaction system, relationship (8) can be used to approximate such behavior. A question is: Can relationship (8) be justified and can it be improved?

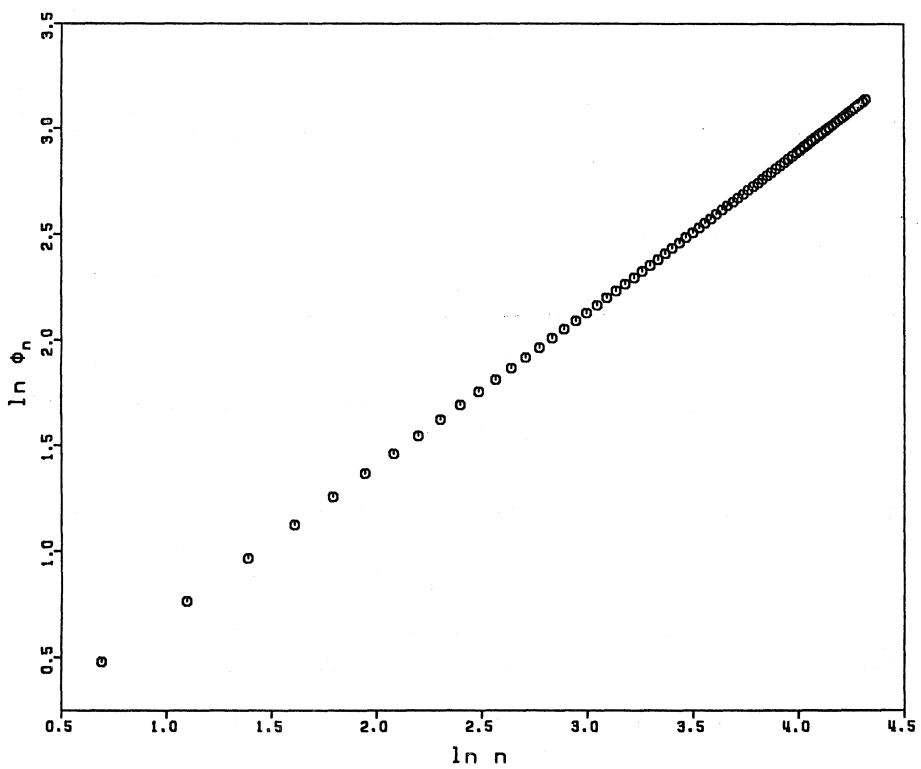

Figure 2. Logarithmic Dependence of the Limiting Ratio of the "Binomial" Sequence on the Order of the Sequence

The following proof that $\log \phi_n/\log n$ is bounded was suggested by the reviewer.

Conjecture: $\lim_{n\to\infty} \dfrac{\log \phi_n}{\log n}$ exists.

From $y = (1 + (1/y))^n$, where $y \equiv \phi_n$ and $1 < y < n$, we have

$$\log y < \log n \quad \text{and} \quad \log \phi_n < \log n \quad \text{or} \quad \frac{\log \phi_n}{\log n} < 1 \text{ is bounded.}$$

For large n, also,

$$y = \left(1 + \frac{n/y}{n}\right)^n \lesssim e^{n/y} \quad \text{or} \quad \log y + \log\log y < \log n$$

or

$$\frac{\log y}{\log n} + \frac{\log\log y}{\log n} \lesssim 1.$$

It may be that $\log \phi_n/\log n$ is eventually monotonically increasing. A short computer program shows, however, that it is not monotone at first.

ACKNOWLEDGMENT

The author would like to acknolwedge interesting and stimulating discussions with W. C. Gardiner, Jr., and H. R. P. Ferguson.

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THE SERIES OF PRIME SQUARE RECIPROCALS

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(Submitted May 1984)

The series

$$\sum_p 1/p^2 \quad \text{(the sum being extended over all primes } p\text{)}$$

converges very slowly. Fortunately, the convergence can be quickened:

Lemma: $\sum_p 1/p^2 = \sum_{k\geqslant 1} \frac{\mu(k)}{k} \log(\zeta(2k))$.

Proof: First,

$$\log(\zeta(2k)) = \log \prod_p (1 - (1/p^{2k}))^{-1}$$

by ([1], p. 246, Theorem 280).

$$\log \prod_p (1 - (1/p^{2k}))^{-1} = \sum_p - \log(1 - (1/p^{2k})) = \sum_p \sum_{s\geqslant 1} \frac{1}{sp^{2ks}}$$

$$= \sum_{s\geqslant 1} \frac{1}{s} \sum_p \frac{1}{p^{2ks}}.$$

We note the following for later use:

$$\log(\zeta(2k)) = \sum_{s\geqslant 1} \frac{1}{s} \sum_p \frac{1}{p^{2ks}}. \tag{*}$$

Now,

$$\sum_{k\geqslant 1} \frac{\mu(k)}{k} \log(\zeta(2k)) = \sum_p \sum_{k\geqslant 1} \sum_{s\geqslant 1} \frac{\mu(k)}{ks} \cdot \frac{1}{p^{2ks}} = \sum_p \sum_{n\geqslant 1} \sum_{k|n} \mu(k) \cdot \frac{1}{np^{2n}}$$

$$= \sum_p \sum_{n\geqslant 1} \frac{1}{np^{2n}} \sum_{k|n} \mu(k) = \sum_p \frac{1}{p^2}$$

the last equality by ([1], p. 235, Theorem 263).

To compute $\sum_p 1/p^2$ accurately (to 28 decimal places), we calculate the first seven terms in the Lemma using exact values for the relevant arguments of the zeta function ζ (computed from [2], p. 298, Table 54, and p. 40, 2.), and we approximate the next twenty-four terms using (*). Thus, we obtain

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AMS 1980 Mathematics Subject Classification: 10A40, 65B10.

$$\sum_p \frac{1}{p^2} \approx \log\left(\frac{\pi^2}{6}\right) - \frac{1}{2}\log\left(\frac{\pi^4}{90}\right) - \frac{1}{3}\log\left(\frac{\pi^6}{945}\right) - \frac{1}{5}\log\left(\frac{\pi^{10}}{93555}\right) + \frac{1}{6}\log\left(\frac{691\pi^{12}}{638512875}\right)$$

$$- \frac{1}{7}\log\left(\frac{2\pi^{14}}{18243225}\right) + \frac{1}{10}\log\left(\frac{174611\pi^{20}}{1531329465290625}\right)$$

$$- \frac{1}{11}\left\{(2^{-22} + 3^{-22} + 5^{-22} + \cdots + 23^{-22})\right.$$

$$\left. + \frac{1}{2}(2^{-44} + 3^{-44}) + \frac{1}{3}(2^{-66}) + \frac{1}{4}(2^{-88})\right\}\cdots$$

$$+ \frac{1}{46}(2^{-92}) - \frac{1}{47}(2^{-94}).$$

Our computer, when presented with this, answers:

$$\sum_p \frac{1}{p^2} \approx 0.45224742004106549850654336 49.$$

It is easy to see that the Lemma holds not only for the exponent 2, but for all exponents $t > 1$. Hence,

$$(\forall\, t > 1) \sum_{k \geqslant 1} \frac{\pi(k) - \pi(k-1)}{k^t} - \sum_{k \geqslant 1} \frac{\mu(k)}{k} \log(\zeta(tk)).$$

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COMBINATORIAL PROOF FOR A SORTING PROBLEM IDENTITY

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(Submitted June 1984)

1. In [2], L. Carlitz suggests that a combinatorial proof of the relation

$$H(m, n, p) - H(m-1, n, p) - H(m, n-1, p) - H(m, n, p-1) = \binom{m+n}{n}\binom{n+p}{p}\binom{p+m}{m} \tag{1.1}$$

might be interesting, where

$$H(m, n, p) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{j+k}{k} \cdot \binom{n-j+p-k}{p-k}\binom{p-k+i}{i}\binom{m-i+k}{k}. \tag{1.2}$$

We give such a proof.

By a lattice point is meant a point with integral coordinates. By a path is meant a minimal path via lattice points, taking unit horizontal and vertical steps. Unless stated otherwise, only nonnegative integers will be used.

2. To fix the idea, we first give the proof of Brock's original problem [1]; i.e., to show that

$$H(m, n) - H(m-1, n) - H(m, n-1) = \binom{m+n}{n}^2, \tag{2.1}$$

where

$$H(m, n) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{n-j+i}{i}\binom{m-i+j}{j}. \tag{2.2}$$

By Figure 1, the number of paths from $(0, 0)$ to (m, n) via (i, j) and then from (m, n) to $(m+n, n+m)$ via $(m+n-j, n+i)$ is

$$\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{m-i+j}{j}\binom{n-j+i}{i}. \tag{2.3}$$

Summed over $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$, (2.3) gives $H(m, n)$. $H(m, n)$ counts all the paths from $(0, 0)$ to $(m+n, n+m)$ via (m, n), but the paths are counted more than once.

For given i and j, each path from $(0, 0)$ to $(m+n, n+m)$ via (m, n) that passes over the segment joining (i, j) and $(i+1, j)$ is counted at (i, j) and again at $(i+1, j)$. The same is true for each path from (m, n) to $(m+n, n+m)$ that passes over the segment joining $(m+n-j, n+i)$ and $(m+n-j, n+i+1)$. The number of such paths is

$$\binom{i+j}{j}\binom{m-i-1+n-j}{n-j}\binom{m-i-1+j}{j}\binom{n-j+i}{i} \tag{2.4}$$

Summed over $i = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, n$, (2.4) gives $H(m-1, n)$.

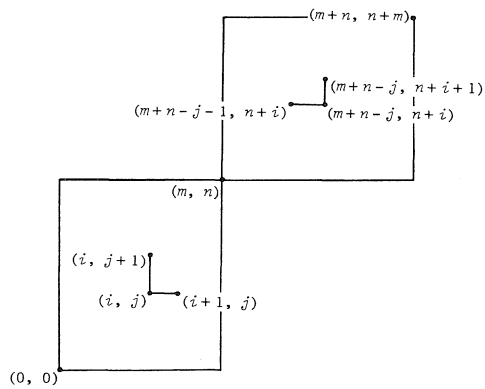

Figure 1

Similarly, $H(m, n - 1)$ is obtained by considering those paths that pass over the segment joining (i, j) and $(i, j + 1)$ and the segment joining $(m + n - j - 1, n + i)$ and $(m + n - j, n + i)$ for $j = 0, 1, \ldots, n - 1$. Alternatively, interchange m and n (and i and j) in the argument for $H(m - 1, n)$.

Thus the left member of (2.1) counts each path from $(0, 0)$ to $(m + n, n + m)$ via (m, n) exactly once. But the right member of (2.1) is just the number of such paths.

3. We now give the proof for (1.1).

By Figure 2, the number of paths from $(0, 0)$ to (m, n) via (i, j), from (m, n) to $(m + n, n + p)$ via $(m + j, n + k)$, and from $(m + n, n + p)$ to $(m + n + p, n + p + m)$ via $(m + n + p - k, n + p + i)$ is

$$\binom{i+j}{j}\binom{m-i+n-j}{n-j}\binom{j+k}{k}\binom{n-j+p-k}{p-k}\binom{p-k+i}{i}\binom{m-i+k}{k}. \tag{3.1}$$

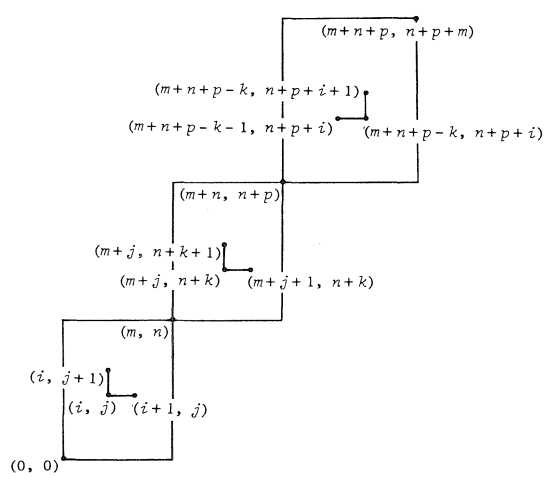

Figure 2

Sum (3.1) over $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$; and $k = 0, 1, \ldots, p$ to get $H(m, n, p)$. $H(m, n, p)$ counts all the paths from $(0, 0)$ to $(m + n + p, n + p + m)$ via (m, n) and $(m + n, n + p)$. Again the paths are counted more than once.

For given i, j, k, each path that passes over the segment joining (i, j) and $(i + 1, j)$ is counted at (i, j) and again at $(i + 1, j)$. The same is true along the segment joining $(m + n + p - k, n + p + i)$ and $(m + n + p - k, n + p + i + 1)$. The number of such paths is

$$\binom{i+j}{j}\binom{m-i-1+n-j}{n-j}\binom{j+k}{k}\binom{n-j+p-k}{p-k}\binom{p-k+i}{i}\binom{m-i-1+k}{k} \tag{3.2}$$

Summing (3.2) over permissible values of i, j, and k, we get $H(m - 1, n, p)$.

$H(m, n - 1, p)$ is obtained by counting the paths that pass over the segment joining (i, j) and $(i, j + 1)$ and that joining $(m + j, n + k)$ and $(m + j + 1, n + k)$.

$H(m, n, p - 1)$ is obtained by counting the paths that pass over the segment joining $(m + j, n + k)$ and $(m + j, n + k + 1)$ and that joining $(m + n + p - k, n + p + i)$ and $(m + n + p - k - 1, n + p + i)$.

Thus, the left member of (1.1) counts each path from $(0, 0)$ to $(m + n + p, n + p + m)$ via (m, n) and $(m + n, n + p)$. However, the right member of (1.1) is just that.

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A FIBONACCI AND LUCAS TANNENBAUM

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(Submitted October 1984)

The first 53 Fibonacci numbers expressed in base two are used for the Tannenbaum display. The right side of the display is a mirror image of the left side. The squares are 1's and the dots are 0's. Only the ending 11 digits of the Fibonacci numbers are given for the last six Fibonacci numbers in the Tannenbaum.

We hope that this whimsical display will prompt others to experiment with the artistic use of Fibonacci and Lucas numbers. The displays can also be done in various colors for striking and unusual effects.

FIGURE OF FIBONACCI TANNENBAUM

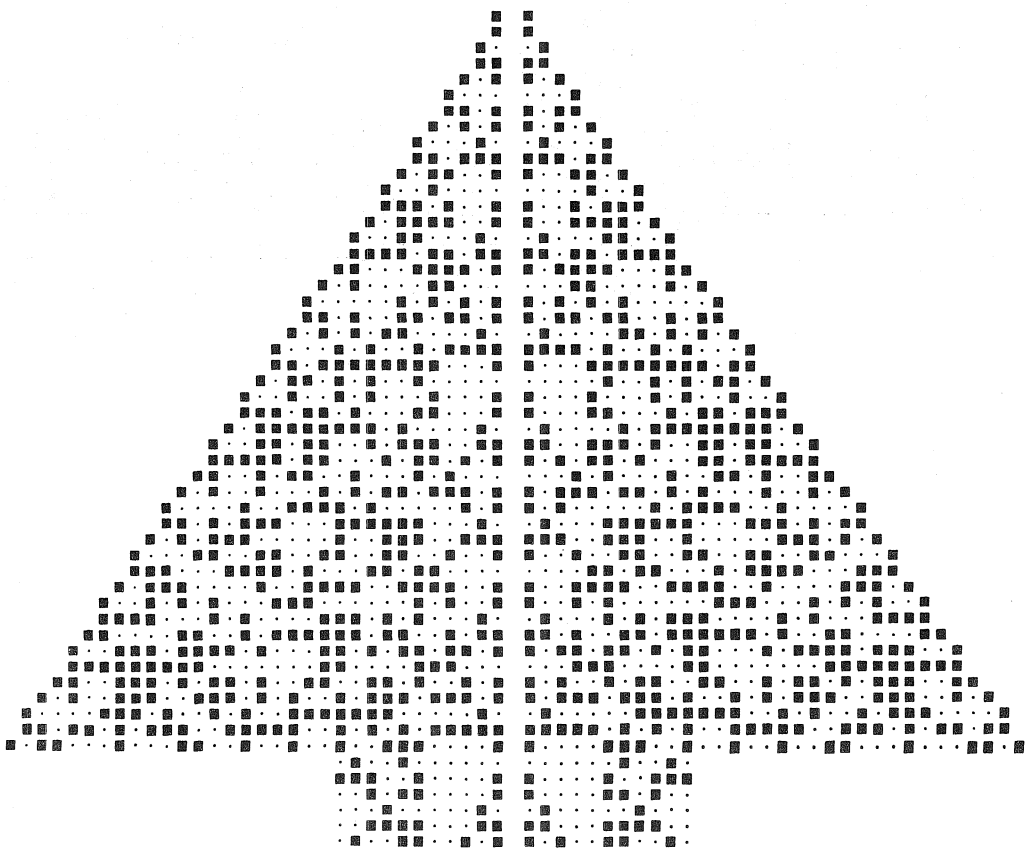
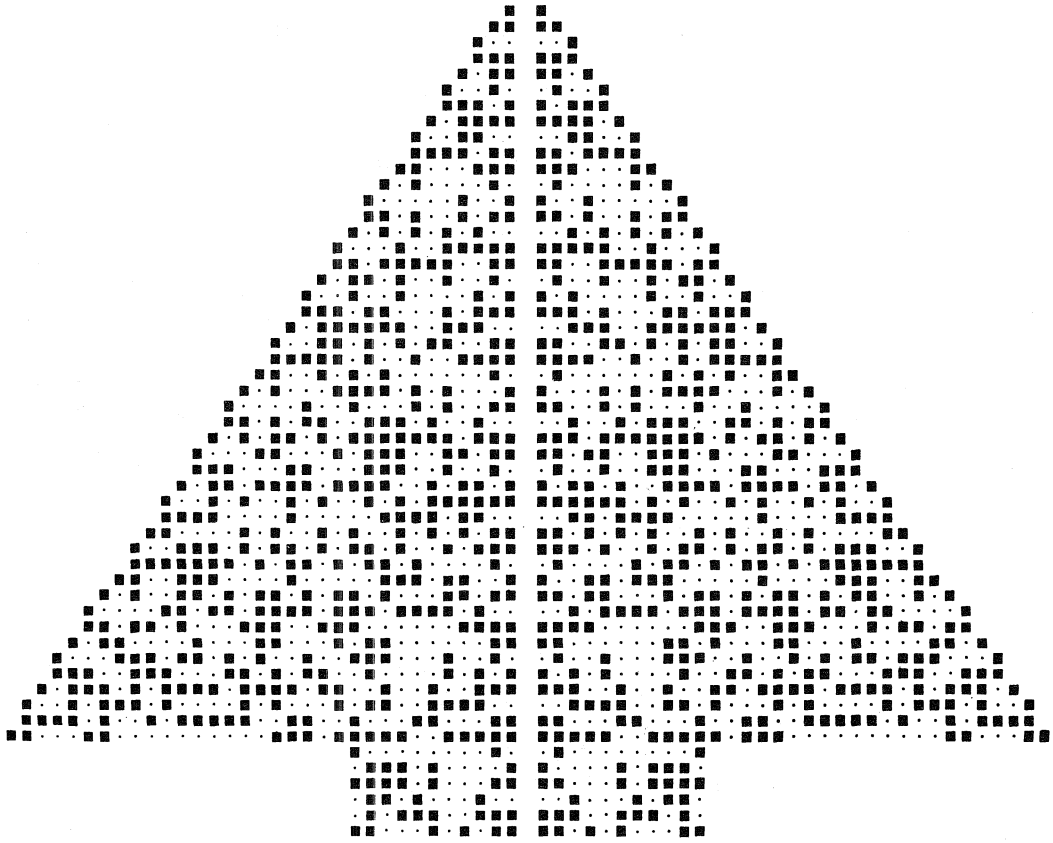


FIGURE OF LUCAS TANNENBAUM



ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN

Assistant Editors
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 Solano Dr., S.E.: Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, α and β designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-556 *Proposed by Valentina Bakinova, Rondout Valley, NY*

State and prove the general result illustrated by

$$4^2 = 16,\ 34^2 = 1156,\ 334^2 = 111556,\ 3334^2 = 11115556.$$

B-557 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there is no integer $n \geqslant 2$ such that

$$F_{3n-6}F_{3n-3}F_{3n+3}F_{3n+6} - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1985^8 + 1.$$

B-558 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there are no positive integers m and n such that

$$F_{4m}^2 - F_{3n} - 4 = 0.$$

B-559 *Proposed by László Cseh, Cluj, Romania*

Let $a = (1 + \sqrt{5})/2$. For positive integers n, prove that

$$[a + .5] + [a^2 + .5] + \cdots + [a^n + .5] = L_{n+2} - 2,$$

where $[x]$ denotes the greatest integer in x.

B-560 *Proposed by László Cseh, Cluj, Romania*

Let a and $[x]$ be as in B-559. Prove that

$$[aF_1 + .5] + [a^2F_2 + .5] + \cdots + [a^nF_n + .5]$$

is always a Fibonacci number.

B-561 *Proposed by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy*

(i) Let Q be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. For all integers n, show that

$$Q^n + (-1)^n Q^{-n} = L_n I, \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) Find a square root of Q, i.e., a matrix A with $A^2 = Q$.

SOLUTIONS

Double Product of 4 Consecutive Fibonacci Numbers

B-532 *Proposed by Herta T. Freitag, Roanoke, VA*

Find a, b, and c in terms of n so that

$$a^3(b - c) + b^3(c - a) + c^3(a - b) = 2F_nF_{n+1}F_{n+2}F_{n+3}.$$

Solution by Graham Lord, Princeton, NJ

The cyclic expression on the left-hand side factors into

$$-(a - b)(b - c)(c - a)(a + b + c).$$

The equality is quickly verified when $a = F_{n+1}$, $b = F_{n+2}$, and $c = F_{n+3}$.

Also solved by Wray Brady, Paul S. Bruckman, L. Cseh, L. A. G. Dresel, L. Kuipers, I. Merényi, Bob Prielipp, Sahib Singh, M. Wachtel, and the proposer.

Product of 5 Fibonacci Numbers

B-533 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$g(a, b, c) = a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2).$$

Determine an infinitude of choices for a, b, and c such that $g(a, b, c)$ is the product of five Fibonacci numbers.

Solution by Graham Lord, Princeton, NJ

The cyclic expression on the left-hand side factors into

$$-(b^2 - c^2)(c^2 - a^2)(a^2 - b^2).$$

With $a = F_n$, $b = F_{n+1}$, and $c = F_{n+2}$, this becomes $F_{n-1}F_nF_{n+2}F_{n+3}F_{2n}$.

Also solved by Wray Brady, Paul S. Bruckman, L. Cseh, L. A. G. Dresel, L. Kuipers, Bob Prielipp, Sahib Singh, M. Wachtel, and the proposer.

No Pythagorean Triangle with Square Area

B-534 *Proposed by A. B. Patel, India*

One obtains the lengths of the sides of a Pythagorean triangle by letting

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2,$$

where u and v are integers with $u > v > 0$. Prove that the area of such a triangle is not a perfect square when $u = F_{n+1}$, $v = F_n$, and $n \geqslant 2$.

I. Solution by L. A. G. Dresel, University of Reading, England

We have

$$a = (u - v)(u + v) = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1}F_{n+2},$$

and the area is given by $A = \frac{1}{2}ab = F_{n-1}F_nF_{n+1}F_{n+2}$. Also,

$$F_{n-1}F_{n+2} = F_nF_{n+1} + (-1)^n.$$

It follows that the area A is the product of two consecutive integers, and thus cannot be a perfect square if $F_{n-1} > 0$, i.e., $n \geqslant 2$. In fact,

$$A = a(a + 1) \quad \text{when } n \text{ is odd,}$$

and $$A = a(a - 1) \quad \text{when } n \text{ is even.}$$

II. Solution by L. Cseh (Cluj, Romania), J. M. Metzger (Grand Forks, ND), Bob Prielipp (Oshkosh, WI), Sahib Singh (Clarion, PA), and Lawrence Somer (Washington, D.C.), independently.

It is well known that the area of a Pythagorean triangle with integral sides is never a perfect square. For proof, see page 256 of *Elementary Number Theory* by David M. Burton (1980 edition). Thus, this result is true, in general, and therefore independent of involvement of Fibonacci numbers.

Also solved by Paul S. Bruckman, Piero Filipponi, Walther Janous, K. Klauser, L. Kuipers, I. Merényi, M. Wachtel, Tad White, and the proposer.

Impossible Sum

B-535 *Proposed by L. Cseh & I. Merényi, Cluj, Romania*

Prove that there is no positive integer n for which

$$F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$$

Solution by L. A. G. Dresel, University of Reading, England

We have the identity $F_1 + F_2 + F_3 + \cdots + F_{3n} = F_{3n+2} - 1$, and it remains to prove that there is no integer n for which $F_{3n+2} - 1 = 16!$ If there were such an integer n, then, since Wilson's theorem gives

$$16! \equiv -1 \pmod{17}$$

we would require

$$F_{3n+2} \equiv 0 \pmod{17}.$$

Now, the first Fibonacci number divisible by 17 is $F_9 = 34$, and therefore F_m

is divisible by 17 if and only if 9 divides m. Clearly, there is no integer n for which $3n + 2$ is divisible by 9, and the result follows.

Also solved by Paul S. Bruckman, Piero Filipponi, Walther Janous, Bob Prielipp, Sahib Singh, J. Suck, and the proposer.

Diophantine Equation

B-536 *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 2x^3 + 2x^2 + x + 1 = y^2.$$

Solution by Paul S. Bruckman, Fair Oaks, CA

If we make the substitution

$$w = 1 + 2x + 2x^2, \tag{1}$$

the given equation is transformed to the simpler one,

$$4y^2 - w = 3. \tag{2}$$

Thus, $(2y - w)(2y + w) = 3$, which has only the four solutions:

$$(w, y) = (1, 1), (-1, 1), (-1, -1), (1, -1).$$

Setting $w = 1$ in (1) yields $2x(x + 1) = 0$, which implies $x = 0$ or $x = -1$. This yields four solutions (x, y), given by

$$(x, y) = (0, 1), (0, -1), (-1, 1), (-1, -1). \tag{3}$$

On the other hand, setting $w = -1$ in (1) yields $x^2 + x + 1 = 0$, which is impossible for integral x (the solutions being the complex cube roots of unity). Thus, *all* the integer solutions of the original equation are given by (3).

Also solved by L. Cseh, L. A. G. Dresel, Walther Janous, L. Kuipers, J. M. Metzger, Bob Prielipp, Sahib Singh, J. Suck, M. Wachtel, and the proposer.

Another Diophantine Equation

B-537 *Proposed by L. Kuipers, Sierre, Switzerland*

Find all solutions in integers x and y of

$$x^4 + 3x^3 + 3x^2 + x + 1 = y^2.$$

Solution by John Oman & Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall show that the only solutions (x, y) in integers of the given equation are $(0, 1)$, $(0, -1)$, $(1, 3)$, $(1, -3)$, $(-1, 1)$, $(-1, -1)$, $(-3, 5)$, and $(-3, -5)$.

It is easily seen that (x, y) is a solution if and only if $(x, -y)$ is a solution. Hence, it suffices to find all solutions (x, y) in integers of the given equation where $y \geqslant 0$.

We begin with the following collection of equivalent equations:

$$x^4 + 3x^3 + 3x^2 + x + 1 = y^2$$

$$36x^4 + 108x^3 + 108x^2 + 36x + 36 = 36y^2$$

$$(6x^2 + 9x + 2)^2 + 3x^2 + 32 = (6y)^2 \tag{1}$$

$$(6x^2 + 9x + 3)^2 - 9(x + 3)(x - 1) = (6y)^2. \tag{2}$$

If $x > 1$, $6x^2 + 9x + 2 < 6y$ [by (1)] and $6y < 6x^2 + 9x + 3$ [by (2)]. Hence, there are no solutions when $x > 1$. If $x < -3$, $6x^2 + 9x + 2 < 6y$ [by (1)] and $6y < 6x^2 + 9x + 3$ [by (2)]. Hence, there are no solutions when $x < -3$. The problem is now easily completed.

Also solved by Paul S. Bruckman, L. Cseh, L. A. G. Dresel, Walther Janous, H. Klauser, J. M. Metzger, J. Suck, M. Wachtel, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-392 *Proposed by Piero Filipponi, Rome, Italy*

It is known [1], [2], [3], [4] that every positive integer n can be represented uniquely as a finite sum of *F-addends* (distinct nonconsecutive Fibonacci numbers). Denoting by $f(n)$ the number of F-addends the sum of which represents the integer n and denoting by $[x]$ the greatest integer not exceeding x, prove that:

(i) $f([F_k/2]) = [k/3], \quad (k = 3, 4, \ldots);$

(ii) $$f([F_k/3]) = \begin{cases} [k/4] + 1, \text{ for } [k/4] = 1 \pmod 2 \text{ and } k = 3 \pmod 4 \\ [k/4], \text{ otherwise.} \end{cases} \qquad (k = 4, 5, \ldots)$$

Find (if any) a closed expression for $f(F_k/p)$ with p a prime and k such that $F_k \equiv 0 \pmod p$.

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1. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly 2*, no. 4 (1964):163-168.
2. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly 3*, no. 1 (1965):1-8.
3. D. E. Daykin. "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers." *J. London Math. Soc. 35* (1960):143-160.
4. D. A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly 6*, no. 4 (1968):235-244.

H-393 *Proposed by M. Wachtel, Zürich, Switzerland*

Consider the triangle below:

A_{-n}		...		A_{-4}	A_{-3}	A_{-2}	A_{-1}	$A_0 = m^2$	A_1	A_2	A_3	A_4	...		A_n
							-1	1							
						-1	5	9	11						
					1	11	19	25	29	31					
				5	19	31	41	49	55	59	61				
			11	29	45	59	71	81	89	95	99	101			
		19	41	61	79	95	109	121	131	139	145	149	151		
	29	55	79	101	121	139	155	169	181	191	199	205	209	211	
41	71	99	125	149	171	191	209	225	239	251	261	269	275	279	281
			...					...				...			

This triangle shows two types of sequences:

a) with primes, or with composite terms with no divisors congruent to 3 or 7 modulo 10;

b) as described in a), and, in addition, terms with divisors congruent to (3 or 7 modulo 10)^{2k}.

In the above triangle, let:

$A_{0,m} = m^2$ (m odd)

$A_{-n,m}$ = the terms on the left of $A_{0,m}$

$A_{n,m}$ = the terms on the right of $A_{0,m}$

---

1. Establish general formulas for the sequences of every row, every column, and every diagonal.

2. Establish formulas:

   a) for the sequences showing terms that either are primes, or else composite integers with no divisors congruent to 3 or 7 modulo 10;

   b) for the sequences with terms as described in a) and also with composite terms showing periodically also divisors congruent to

   (3 or 7 modulo 10)^{2k}.

Remarks: Apart from the formulas

$$C - N^2 + rN$$

for the finite sequences, and

$$C + mN^2 + rN$$

for the infinite sequences, there exist other construction rules.

Some examples of relationships which can easily be established are:

Column A_{-2} Down	Diag. A_2 Down Left	Columns A_{-2} and A_2 Down
$-1 = -1 \cdot 2 + 1$	$31 = -2 \cdot 9 + 7$	Every term plus $5 = (m \mp 1)^2$
$11 = 1 \cdot 2 + 3$	$55 = -1 \cdot 9 + 8$	Columns A_{-4} and A_4 Down
$31 = 3 \cdot 2 + 5$	$81 = 0 \cdot 9 + 9$	Every term plus $5 \cdot 2^2 = (m \mp 2)^2$
$59 = 5 \cdot 2 + 7$	$109 = 1 \cdot 9 + 10$	
$\vdots$	$\vdots$	

According to what is stated in 2 above, the following rule holds true:

$A_{n,m} + 5(a^2 + a) + 1 = b^2 + b$, with infinitely many solutions whereby a and b are F/L numbers.

Example: $A_{2,11}$ and $A_{-2,13} = 139$

	a	b
I.	$5F_{2+6n} - \dfrac{F_{-1+6n} + 1}{2}$	$5L_{2+6n} - \dfrac{L_{-1+6n} + 1}{2}$
II.	$F_{6+6n} + \dfrac{L_{-1+6n} - 1}{2}$	$10F_{3+6n} - \dfrac{L_{-1+6n} + 1}{2}$
III.	$5F_{4+6n} - \dfrac{F_{1+6n} + 1}{2}$	$5L_{4+6n} - \dfrac{L_{1+6n} + 1}{2}$
IV.	$F_{8+6n} + \dfrac{L_{1+6n} - 1}{2}$	$10F_{5+6n} - \dfrac{L_{1+6n} + 1}{2}$

Special Properties: All sequences emerging out of this triangle show the following property:

$A_k \cdot A_{k+d} + B$ yield either a square or, alternately, a product of two consecutive integers. For brevity, example and formula are omitted.

Then there are combinations of different sequences, but it would take too much space to pursue the many things involved in this triangle.

SOLUTIONS

AB Surd

H-367 *Proposed by M. Wachtel, Zürich, Switzerland (Vol. 22, no. 1, February 1984)*

Problem A

Prove the identity:

$$\sqrt{(L_{2n} - L_{n-2}^2) \cdot (L_{2n+4} - L_n^2) + 30} = 5F_{2n} - 3(-1)^n$$

Problem B

Prove the identities:

$$\left.\begin{array}{l}\sqrt{(F_{n+1}^2 - F_{2n+3}) \cdot (F_{n+3}^2 - F_{2n+7})} \\ \sqrt{(F_{n+3}^2 - F_{2n+5}) \cdot (F_{n+5}^2 - F_{2n+9})} \\ \sqrt{(F_{n+4}^2 - F_{2n+6}) \cdot (F_{n+6}^2 - F_{2n+10})}\end{array}\right\} = F_{n+2}F_{n+4} \quad \text{or} \quad F_{n+3}^2 + (-1)^n$$

Solution by the proposer.

1. These are particular instances of the more general identities:

A) $$\sqrt{(L_{2n+m} - L_{n-2+m}^2) \cdot (L_{2n+4+m} - L_{n+m}^2) + \overbrace{5[L_{4-m} - (-1)^m]}^{A'}}$$
$$= L_{2n+2+m} - L_{n-2+m}L_{n+m}$$

B) $$\sqrt{(F_{2n+m} - F_{n-2+m}^2) \cdot (F_{2n+4+m} - F_{n+m}^2) - \overbrace{(-1)^m(F_{m-4} - 1)}^{B'}}$$
$$= F_{2n+2+m} - F_{n-2+m}F_{n+m}$$

$n, m = 0 \pm 1, 2, 3, \ldots$.

2. Squaring both sides of 1 and making use of the following identities on the left-hand side, we obtain (with these identities the congruence of both sides is established):

A) I. $L_{2n+m}L_{2n+4+m} \equiv L_{2n+2+m}^2 + 5(-1)^m$

(derived from $L_{2n}L_{2n+4} \equiv L_{2n+2}^2 + 5$)

II. $L_{2n+m}L_{n+m}^2 \equiv L_{2n+2+m}L_{n-2+m}L_{n+m} - 5(-1)^{n+m}L_{n+m}F_{n+2}$

[derived from $L_{2n}L_n \equiv L_{2n+2}L_{n-2} - 5(-1)^nF_{n+2}$
and $L_{2n+m}L_{n+m} \equiv L_{2n+2+m}L_{n-2+m} - 5(-1)^{n+m}F_{n+2}$]

III. $L_{2n+4+m}L_{n-2+m}^2 - 5L_{4-m} \equiv L_{2n+2+m}L_{n-2+m}L_{n+m} + 5(-1)^{n+m}L_{n+m}F_{n+2}$

(similarly derived as I and II)

B) I. $F_{2n+m}F_{2n+4+m} \equiv F_{2n+2+m}^2 - (-1)^m$

II. $F_{2n+m}F_{n+m}^2 \equiv F_{2n+2+m}F_{n-2+m}F_{n+m} + (-1)^{n+m}F_{n+m}F_{n+2}$

III. $F_{2n+4+m}F_{n-2+m}^2 + (-1)^mF_{m-4} \equiv F_{2n+2+m}F_{n-2+m}F_{n+m} - (-1)^{n+m}F_{n+m}F_{n+2}$

(similarly derived as A)

3. By establishing the values of

$A' = 5[L_{4-m} - (-1)^m]$ and $B' = -(-1)^m(F_{m-4} - 1)$

we obtain:

m	A'	m	B'
0	30	0	4
1	25	1	1
2	10	2	2
3	10	3	0
4	5	4	1
5	0	5	0
6	10	6	0

4. By application of the the formula 1 and the values 3, we find:

A) $\underline{m = 0}$ $\quad \sqrt{(L_{2n} - L_{n-2}^2) \cdot (L_{2n+4} - L_n^2) + 30} = L_{2n+2} - L_{n-2}L$

$$= 5F_{2n} - 3(-1)^n$$

B) $\underline{m = 3}$ $\quad \sqrt{(F_{2n+3} - F_{n+1}^2) \cdot (F_{2n+7} - F_{n+3}^2)} = F_{2n+5} - F_{n+1}F_{n+3}$

$\underline{m = 5}$ $\quad \sqrt{(F_{2n+5} - F_{n+3}^2) \cdot (F_{2n+9} - F_{n+5}^2)} = F_{2n+7} - F_{n+3}F_{n+5}$

$\underline{m = 6}$ $\quad \sqrt{(F_{2n+6} - F_{n+4}^2) \cdot (F_{2n+10} - F_{n+6}^2)} = F_{2n+8} - F_{n+4}F_{n+6}$ $\Big\}$ *

* = all three versions identical to $-(-1)^m F_{n+2}F_{n+4}$,

which had to be shown.

5. Some special properties of the sequences (each sequence shows its own distinct characteristic, depending on m):

	m	Sequence
A)	1	$5(F_{n-1}F_n + F_{n+1}^2)$, which implies: No integral divisor of any term is congruent to 3 or 7 modulo 10.
	2, 6	sequences identical, but with phase difference
	4	unique term for any n: $-3(-1)^n$
B)	1, 4, 7	$F_rF_{r+1} + F_{r+2}^2$, with phase differences. No integral divivor of any term is congruent to 3 or 7 modulo 10.
	3, 5, 6	sequences identical (see 4B)
	...	etc.

Also solved by P. Bruckman, L. Dresel, P. Filipponi, L. Kuipers, B. Prielipp, and H. Seiffert.

Sum Formula

H-368 (Corrected) *Proposed by Andreas N. Philippou, Patras, Greece (Vol. 22, no. 2, May 1984)*

For any fixed integers $k \geqslant 1$ and $r \geqslant 1$, set

$$f_{n+1,r}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k + r - 1}{n_1, \ldots, n_k, r - 1}, \quad n \geqslant 0$$

Show that

$$f_{n+1,r}^{(k)} = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)} f_{n+1-\ell,r-1}^{(k)}, \quad n \geqslant 0, \ r \geqslant 2. \tag{*}$$

The problem includes as special cases ($r = 2$) the following:

For any fixed integer $k \geqslant 2$,

$$\sum_{\substack{n_1,\ldots,n_k \\ n_1+2n_2+\cdots+kn_k=n}} \binom{n_1+\cdots+n_k+1}{n_1,\ \ldots,\ n_k,\ 1} = \sum_{\ell=0}^{n} f_{\ell+1}^{(k)} f_{n+1-\ell}^{(k)}, \quad n \geqslant 0, \tag{A}$$

where $f_n^{(k)}$ are the Fibonacci numbers of order k [1], [2].

In particular, for $k = 2$,

$$\sum_{\ell=0}^{[n/2]} (n+1-\ell)\binom{n-\ell}{\ell} = \sum_{\ell=0}^{n} F_{\ell+1}F_{n+1-\ell}, \quad n \geqslant 0. \tag{A.1}$$

The problem also includes as a special case ($k = 1$, $r \geqslant 2$) the following:

$$\binom{n+r-1}{r-1} = \sum_{\ell=0}^{n} \binom{n-\ell+r-2}{r-2}, \quad n \geqslant 0, \ r \geqslant 2. \tag{B}$$

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2. A. N. Philippou. "A Note on the Fibonacci Sequence of Order k and Multinomial Coefficients." *The Fibonacci Quarterly 21*, no. 2 (1983):82-86.

Solution by the proposer.

For any fixed $x \in (0, \infty)$ and k and r as in the problem, set

$$f_{n+1,r}^{(k)}(x) = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\cdots+n_k=n}} \binom{n_1+\cdots+n_k+r-1}{n_1,\ \ldots,\ n_k,\ r-1} x^{n_1+\cdots+n_k}, \quad n \geqslant 0. \tag{1}$$

We shall establish the more general result

$$f_{n+1,r}^{(k)}(x) = \sum_{\ell=0}^{n} f_{\ell+1,1}^{(k)}(x) f_{n+1-\ell,r-1}^{(k)}(x), \quad n \geqslant 0, \ r \geqslant 2. \tag{2}$$

To do so, we consider random variables $X_1, \ldots, X_r$ ($r \geqslant 2$) which are independent and identically distributed as $G_k(\cdot;\, p)$ ($0 < p < 1$) (see [3]). Then $X_1 + \cdots + X_r$ is distributed as $NB_k(\cdot;\, r, p)$ and $X_2 + \cdots + X_r$ is distributed as $NB_k(\cdot;\, r-1, p)$ [3]. Therefore,

$$P[X_1 = \ell + k] = p^{\ell+k} \sum_{\substack{\ell_1,\ldots,\ell_k \ni \\ \ell_1+2\ell_2+\cdots+k\ell_k=\ell}} \binom{\ell_1+\cdots+\ell_k}{\ell_1,\ \ldots,\ \ell_k} \left(\frac{1-p}{p}\right)^{\ell_1+\cdots+\ell_k}, \quad \ell \geqslant 0; \tag{3}$$

$$P[X_1 + \cdots + X_r = n + kr]$$

$$= p^{n+kr} \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\cdots+kn_k=n}} \binom{n_1+\cdots+n_k+r-1}{n_1,\ \ldots,\ n_k,\ r-1} \left(\frac{1-p}{p}\right)^{n_1+\cdots+n_k}, \quad n \geqslant 0; \tag{4}$$

and

$$P[X_2 + \cdots + X_r = n + k(r-1) - \ell]$$
$$= p^{n+k(r-1)-\ell} \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n-\ell}} \binom{n_1 + \cdots + n_k + r - 2}{n_1, \ldots, n_k, r-2} \left(\frac{1-p}{p}\right)^{n_1 + \cdots + n_k}, \quad n \geqslant \ell. \tag{5}$$

Next, for $n \geqslant 0$ and $n \geqslant 2$,

$$[X_1 + \cdots + X_r = n + kr] = \bigcup_{\ell=0}^{n} \{[X = \ell + k] \cap [X_2 + \cdots + X_r = n + k(r-1) - \ell]\}, \tag{6}$$

with

$$\{[X_1 = \ell + k] \cap [X_2 + \cdots + X_r = n + k(r-1) - \ell]\} \cap \{[X_1 = \ell' + k]$$
$$\cap [X_2 + \cdots + X_r = n + k(r-1) - \ell']\} = \emptyset \quad (0 < \ell' \leqslant n),$$

and

$$[X_1 = \ell + k],$$

$[X_2 + \cdots + X_r = n + k(r-1) - \ell]$ are independent events $(0 \leqslant \ell \leqslant n)$. Hence, for $n \geqslant 0$ and $r \geqslant 2$,

$$P[X_1 + \cdots + X_r = n + kr] = \sum_{\ell=0}^{n} P[X_1 = \ell + k]P[X_2 + \cdots + X_r = n + k(r-1) - \ell]. \tag{7}$$

Set $(1 - p)/p = x$, so that $x \in (0, \infty)$. Then relation (7) implies (2), by means of (1) and (3)-(5). Q.E.D.

For $x = 1$, relation (2) shows the proposed problem. In order to appreciate its generality, it is instructive to note the special cases (A), (A.1), and (B). (A) follows from (*) for $r = 2$, by means of the definition of $f_{n+1,r}^{(k)}$ and the formula of [1] and [2]:

$$f_{n+1}^{(k)} = \sum_{\substack{n_1, \ldots, n_k \ni \\ n_1 + 2n_2 + \cdots + kn_k = n}} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \quad n \geqslant 0.$$

(A.1) follows directly from (A), and (B) is a simple consequence of (*) and the definition of $f_{n+1,r}^{(k)}$.

References

1. See Reference 1 above (p. 381).
2. See Reference 2 above (p. 381).
3. A. N. Philippou, C. Georghiou, & G. N. Philippou. "A Generalized Geometric Distribution and Some of Its Properties." *Statistics and Probability Letters 1* (1983):171-175.

Also solved by P. Bruckman.

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Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972.

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