The Fibonacci Quarterly

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The Fibonacci Quarterly

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and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

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THE PARITY OF THE SUM-OF-DIGITS-FUNCTION OF GENERALIZED ZECKENDORF REPRESENTATIONS*

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1. INTRODUCTION

Let $G = (G_n)$ be a strictly increasing sequence of positive integers with $G_1 = 1$. Then every nonnegative integer n has a digital expansion

$$n = \sum_{i \geq 1} \varepsilon_i G_i$$

with respect to basis G, where the digits $\varepsilon_i = \varepsilon_i(n) \geq 0$ are integers. This digital expansion is unique, when one assumes that the digits ε_i are chosen in such a way that the digital sum $\sum_{i \geq 1} \varepsilon_i$ is as small as possible; in this case, we will call the digital expansion a *proper digital expansion*. It is easy to see that the following algorithm provides this expansion.

1. For $n = 0$, we have $\varepsilon_i(n) = 0$ for every $i \geq 1$.
2. If $G_j \leq n < G_{j+1}$ and $n' = n - G_j$ has the proper expansion $n' = \sum_{i \geq 1} \varepsilon_i' G_i$, then the expansion of $n = \sum_{i \geq 1} \varepsilon_i G_i$ is given by $\varepsilon_i = \varepsilon_i'$ for $i \neq j$ and by $\varepsilon_j = \varepsilon_j' + 1$.

The most prominent digital expansions are related to linear recurring sequences $G = (G_n)$, e.g., the binary (resp. the q-ary) expansion relies on $G_n = 2^{n-1}$ (resp. on $G_n = q^{n-1}$). If G_n are the Fibonacci numbers, i.e., $G_n = F_{n+1}$, then we obtain the Zeckendorf expansion.

For each digital expansion with respect to a basis G, we can define a partial order in a quite natural way. We will say $a \leq_G b$ if and only if $\varepsilon_i(a) \leq \varepsilon_i(b)$ for every $i \geq 1$. It is well known that for every partial order there is a Möbius function (see [10], [13]). Let $s_G(n)$ denote the sum of digits of n. Then it will turn out that the Möbius function μ_G of a digital expansion to a basis G is given by $\mu_G(n) = (-1)^{s_G(n)}$ if $\max_{i \geq 1} \varepsilon_i(n) \leq 1$ and by $\mu_G(n) = 0$ otherwise.

If G is a proper linear recurring sequence and if the initial conditions of G are properly chosen (see Section 3), then

$$M_G(N) := \sum_{n=0}^{N-1} \mu_G(n)$$

is either bounded or

$$M_G(N) = S_G(N) := \sum_{n=0}^{N-1} (-1)^{s_G(n)},$$

which we will see from calculating the Möbius function in Section 2. (We always define empty sums to be zero, i.e., $M_G(N) = S_G(N) := 0$ for $N \leq 0$.)

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In Section 3 we will formulate conditions for G, under which we will be able to derive formulas for $S_G(N)$. We will also obtain a recursive formula for the generating function of $S_G(G_n)$, which we will analyze in Section 4 in order to obtain asymptotic information about $S_G(N)$.

Our main interest lies in the distribution of the $S_G(N)$ (resp. $M_G(N)$) when $0 \le N < m$ for large m. This means that we count the number of times $S_G(N)$ takes a certain value k when $0 \le N < m$: let $d_m(k) := |\{0 \le N < m : S_G(N) = k\}|$ be this number and let X_m be a random variable with probability distribution $\mathbf{P}(X_m = k) = d_m(k)/m$. Then we are interested in the asymptotic distribution of X_m for $m \to \infty$. Depending on the nature of the recurrence relation for G, we will observe significantly different behavior of X_m. First, we distinguish two cases:

1. either $S_G(G_n)$ is bounded for all initial conditions of G (Section 4.1), or
2. there are initial conditions of G such that $S_G(G_n)$ is unbounded (Section 4.2).

Since we can establish a linear recurrence relation for the $S_G(G_n)$, the first case is equivalent to the assumption that the characteristic polynomial of this recursion is a product of some z^{r-v} $(r - v \ge 0)$ and certain different cyclotomic polynomials. In this case, we can derive asymptotic formulas for $\mathbf{E}X_m$ and $\mathbf{V}X_m$, provided that the sequence G satisfies a certain technical condition. Our main result (Theorem 2) says that, in the case of unbounded variance, X_m satisfies a central limit theorem. (Note that there are sequences G for which $\mathbf{V}X_m$ is bounded, e.g., $G_n = 2^{n-1}$.)

2. THE MÖBIUS FUNCTION OF A DIGITAL EXPANSION

Let $G = (G_n)$ be a strictly increasing sequence of integers with $G_1 = 1$. As mentioned above, every nonnegative integer n has a digital expansion $n = \sum_{i \ge 1} \varepsilon_i G_i$ with nonnegative integral digits ε_i. It is called *proper digital expansion for n* if the digital sum $\sum_{i \ge 1} \varepsilon_i$ is as small as possible.

Lemma 1: Let $n = \sum_{i \ge 1} \varepsilon_i G_i$ be a proper digital expansion for n. Then any sum of the form $\sum_{i \ge 1} \varepsilon_i' G_i$ with integral digits ε_i', $i \ge 1$, satisfying $0 \le \varepsilon_i' \le \varepsilon_i$ is a proper digital representation for some $n' \le n$.

Proof: First, note that it follows from the algorithm stated in the Introduction that any digital expansion of the form $n_j = \sum_{i=1}^{j} \varepsilon_i G_i \le n$ is a proper one.

Next, we will use induction on the digital sum $s' = \sum_{i \ge 1} \varepsilon_i'$, where $0 \le \varepsilon_i' \le \varepsilon_i$. Obviously, there is nothing to show if $s' = 0$.

Now suppose that $n' = \sum_{i \ge 1} \varepsilon_i' G_i$ has digital sum s'. There exists $j \ge 1$ such that $\varepsilon_j' > 0$ and $\varepsilon_i' = 0$ for $i > j$. Then $G_j \le n' \le n_j < G_{j+1}$. Therefore, $n'' = n' - G_j$ can be represented by $n'' = \sum_{i=1}^{j} \varepsilon_i'' G_i$ with $\varepsilon_j'' = \varepsilon_j' - 1$ and $\varepsilon_i'' = \varepsilon_i'$ for $i \ne j$. Since $0 \le \varepsilon_i'' \le \varepsilon_i$ and its digital sum satisfies $\sum_{i \ge 1} \varepsilon_i'' = s' - 1 < s'$, this expansion for n'' is proper. Consequently, $\sum_{i \ge 1} \varepsilon_i' G_i$ is a proper expansion for n'. □

Now we introduce the Möbius functions $\mu(x, y)$ of a locally finite partial order $\le$ on a set X, i.e., all intervals $[x, y] = \{u \in X : x \le u \le y\}$ are finite (see [10], [13]). Any function $f : X^2 \to \mathbf{C}$ that satisfies $f(x, y) = 0$ for $x \not\le y$ will be called an *arithmetical function*. The convolution $f * g$ of two arithmetical functions f, g is given by

$$(f * g)(x, y) = \sum_{x \le u \le y} f(x, u) g(u, y).$$

Obviously δ, defined by $\delta(x, y) = 1$ for $x = y$ and $\delta(x, y) = 0$ otherwise, is the unit element of $*$. Furthermore, if $f(x, x) \neq 0$ for every $x \in X$, then there always exists an inverse arithmetical function f^{-1} satisfying $f^{-1} * f = \delta$. The Möbius function μ is defined as the inverse function of ζ given by $\zeta(x, y) = 1$ if $x \leq y$ and by $\zeta(x, y) = 0$ otherwise. Especially, if $g = \zeta * f$, then f can be recovered by $f = \mu * g$. (We intend to use this Möbius function in future work for sieve methods in connection with specific problems of digital expansions.)

Theorem 1: Let $\leq_G$ be the partial order on the nonnegative integers induced by the digital expansion with respect to a strictly increasing sequence of integers $G = (G_n)$ and suppose $m = \sum_{i \geq 1} \varepsilon_i' G_i$ and $n = \sum_{i \geq 1} \varepsilon_i'' G_i$ are proper digital expansions of nonnegative integers m, n with $m \leq_G n$, i.e., $\varepsilon_i' \leq \varepsilon_i''$ for all i. Then

$$\mu(m, n) = \begin{cases} 0 & \text{if there is an } i \text{ with } \varepsilon_i'' - \varepsilon_i' > 1, \\ (-1)^{\sum_{i \geq 1} (\varepsilon_i'' - \varepsilon_i')} & \text{otherwise.} \end{cases}$$

Proof: Since there is a natural bijection between $[m, n] = \{d \in \mathbb{N}_0 | m \leq_G d \leq_G n\}$ and $[0, n-m]$, we have $\mu(m, n) = \mu(0, n-m)$ if $m \leq_G n$. (For $m \not\leq_G n$, we have $\mu(m, n) = 0$.)

Therefore, we will calculate only $\mu(0, n)$. From the definition of $\mu(x, y)$, it is clear that $\mu(0, 0) = 1$ and that

$$\sum_{0 \leq_G d \leq_G n} \mu(0, d) = 0 \quad \text{for } n > 0.$$

Assume for a moment that $\varepsilon_i'' \leq 1$ for all i. We show that $\mu(0, \sum_{j=0}^{k-1} G_{i_j}) = (-1)^k$ by induction on the digital sum $s = k$. Clearly, for $s = 0$, we have $\mu(0, 0) = 1 = (-1)^0$. Now assume that $s \geq 1$ and that $\mu(0, \sum_{j=0}^{k-1} G_{i_j}) = (-1)^k$ for all $k < s$. Then

$$\begin{aligned} 0 &= \sum_{0 \leq_G d \leq_G \sum_{j=0}^{s-1} G_{i_j}} \mu(0, d) \\ &= (\mu(0, 0)) + (\mu(0, G_{i_0}) + \mu(0, G_{i_1}) + \cdots + \mu(0, G_{i_{s-1}})) \\ &\quad + (\mu(0, G_{i_0} + G_{i_1}) + \mu(0, G_{i_0} + G_{i_2}) + \cdots + \mu(0, G_{i_{s-2}} + G_{i_{s-1}})) + \cdots + \left(\mu\left(0, \sum_{j=0}^{s-1} G_{i_j}\right)\right) \\ &= 1 + \binom{s}{1}(-1)^1 + \binom{s}{2}(-1)^2 + \cdots + \binom{s}{s-1}(-1)^{s-1} + \mu\left(0, \sum_{j=0}^{s-1} G_{i_j}\right). \end{aligned}$$

Because of $\sum_{j=0}^{s} \binom{s}{j}(-1)^j = 0$, it follows that $\mu(0, \sum_{j=0}^{s-1} G_{i_j}) = (-1)^s$, which proves the theorem in this special case.

Now suppose that kG_i with $i \geq 1$ and $k > 1$ is a proper digital expansion. Then $0 = \mu(0, 0) + \mu(0, G_i) + \cdots + \mu(0, kG_i)$. Notice that $\mu(0, 0) + \mu(0, G_i) = 0$. Hence, it follows that $\mu(0, 2G_i) = \mu(0, 3G_i) = \cdots = \mu(0, kG_i) = 0$.

Next, we show by induction on the digital sum $s(n) = \sum_{i \geq 1} \varepsilon_i''$ that $\mu(0, n) = 0$ whenever there is an i with $\varepsilon_i'' > 1$. We must start with $s(n) = 2$ because $\varepsilon_i'' > 1$ cannot be satisfied when $s(n) < 2$. Suppose that $s(n) = 2$ and that there is some i with $\varepsilon_i'' > 1$. Then $m = 2G_i$ and $\mu(0, m) = 0$. Now assume the assertion holds for all natural numbers l with $s(l) < s(n)$ and assume there is a j with $\varepsilon_j'' > 1$. Then

$$-\mu(0,n) = \sum_{0 \le_G d <_G n} \mu(0,d) = \sum_{0 \le_G d <_G n,\, \forall i:\varepsilon_i(d) \le 1} \mu(0,d) + \sum_{0 \le_G d <_G n,\, \exists i:\varepsilon_i(d) > 1} \mu(0,d)$$
$$= \sum_{0 \le_G d <_G n,\, \forall i:\varepsilon_i(d) \le 1} \mu(0,d).$$

Define $n_1 := \sum_{i \ge 1} \min(\varepsilon_i'', 1) G_i$. Because of the existence of j with $\varepsilon_j'' > 1$, we have $0 < n_1 < n$ and

$$\sum_{0 \le_G d <_G n,\, \forall i:\varepsilon_i(d) \le 1} \mu(0,d) = \sum_{0 \le_G d \le_G n_1} \mu(0,d).$$

The right-hand side is, of course, zero, due to (2), which completes our proof. □

Since $\mu_G(m,n) = \mu_G(0, n-m)$ (if $m \le_G n$), it is sufficient to consider the restricted Möbius function $\mu_G(n) = \mu_G(0,n)$. As mentioned above, the main topic of this paper is to discuss the partial sums

$$M_G(N) = \sum_{n=0}^{N-1} \mu_G(n).$$

Nevertheless, we will rather discuss the partial sums $S_G(N)$, see (1), which will be motivated by the following proposition.

Proposition 1: Suppose that $G_n \ge 2G_{n-1}$ for all $n > 1$. Then $M_G(N)$ is bounded by 1. On the other hand, if $G_n \le 2G_{n-1}$ for all $n > 1$, then

$$M_G(N) = S_G(N) := \sum_{n=0}^{N-1} (-1)^{s_G(n)},$$

where $s_G(n)$ denotes the digital sum $s_G(n) = \sum_{i \ge 1} \varepsilon_i$ of the proper digital representation

$$n = \sum_{i \ge 1} \varepsilon_i G_i.$$

Proof: Due to Theorem 1, only those n with expansion coefficients 0 or 1 enter the sum. If $G_n \ge 2G_{n-1}$ for all $n > 1$, then all the digital expansions $\sum_{i \ge 1} \varepsilon_i G_i$ with $\varepsilon_i \in \{0, 1\}$ are proper ones. Hence, $M_G(N)$ attains only the same values as in the binary case in which the corresponding sum is 0 or ± 1.

If $G_n \le 2G_{n-1}$ for all $n > 1$, then in all the proper digital expansions only the digits 0 and 1 can occur, and the assertion follows from Theorem 1 with $m = 0$. □

Remark 1: We will see later that for all G considered here, $(a_1 + 1)G_{n-1} \ge G_n \ge a_1 G_{n-1}$ holds for $n > r$; therefore, $G_n \le 2G_{n-1}$ for all $n > 1$ is equivalent to $a_1 = 2$ and $r = 1$ or $a_1 = 1$ when the initial conditions of G are properly chosen. But if $a_1 > 2$ or $a_1 = 2$ and $r > 1$, and if $G_n \ge 2G_{n-1}$ holds for the initial values, then Proposition 1 applies and $M_G(N)$ is bounded. Because of this, we will investigate the function $S_G(N)$ rather than $M_G(N)$, keeping in mind that, in most cases, when $M_G(N)$ is of interest, both are the same.

Remark 2: If $G_n = 2^{n-1}$, then $t_n = (-1)^{s_G(n)}$ is the Thue-Morse sequence [11]. Since $t_{2n} + t_{2n+1} = 0$, we have $S_G(2n+1) = t_{2n} = t_n$, and we also have $S_G(2n) = 0$. Thus, it is not really interesting to study $S_G(N)$ in this case.

3. DIGITAL EXPANSIONS AND GENERATING FUNCTIONS

From this point on we will consider only integral linear recurring sequences $G = (G_n)_{n\geq 1}$ that satisfy assumptions 1-5 below (in Section 4.1 we will also need assumption 6):

1. $G_1 = 1$ and $G_{n+1} > G_n$ for $n \geq 1$.
2. $G_n = \sum_{i=1}^{r} a_i G_{n-i}$ for $n > r$ with some integers $a_i \geq 0$.
3. $G_{n-j} \geq \sum_{i=j+1}^{r} a_i G_{n-i}$ for $n > r$ and $1 \leq j < r$.
4. G satisfies no linear recursion with constant integer coefficients with a smaller degree.
5. The characteristic polynomial $z^r - \sum_{i=1}^{r} a_i z^{r-1} = \prod_{i=1}^{r}(z - \alpha_i)$ (of the above recursion) has only one real, positive, and simple root α_1 of maximal modulus.
6. Let $b_i = (a_i \bmod 2)(-1)^{a_1+\cdots+a_{i-1}}$ ($a_i \bmod 2 = 0$ if a_i is even and $a_i \bmod 2 = 1$ otherwise). Then

$$z^r - \sum_{i=1}^{r} b_i z^{r-i} = z^{r-v} \prod_{h=1}^{k'} \Phi_{k_h}(z) \tag{1}$$

is a product of z^{r-v} $(r - v \geq 0)$ and different cyclotomic polynomials $\Phi_{k_h}(z)$ $(k_1 < k_2 < \cdots < k_{k'})$, all of them dividing $z^p - 1$ with some fixed $p > r$. Furthermore, none of the α_i and no quotient α_i / α_j $(i \neq j)$ is a pth root of unity.

Assumptions 1, 2, and 4 are natural. Therefore, only conditions 3, 5, and 6 need to be motivated.

Assumption 3 is necessary to show that $S(G_n)$ satisfies a linear recurrence; especially, it implies (6) in Proposition 2.

From assumption 5, we obtain $G_n = \beta\alpha_1^{n-1} + O((\alpha_1\gamma)^n)$ with some $\beta > 0$ and $0 \leq \gamma < 1$. Note that assumptions 2 and 3 imply $(a_1 + 1)G_{n-1} \geq G_n \geq a_1 G_{n-1}$ for $n > r$, which gives $a_1 \leq \alpha_1 \leq a_1 + 1$. Similarly, we get $a_1 \geq a_i$ for all i.

The first part of assumption 6 (concerning the cyclotomic factors) ensures that $S(G_n)$ is bounded. The assumption that α_i and α_i / α_j are not p^{th} roots of unity is frequently used in problems concerning digital expansions with respect to linear recurring sequences and avoids technical difficulties (see Lemma 2).

Usually, assumptions 3 and 5 are replaced by the stronger condition $a_1 \geq a_2 \geq \cdots \geq a_r$ and certain assumptions on the initial values of G (see, e.g., [8]; in this case, the second part of assumption 6 is also satisfied). However, there are other interesting examples, e.g., $a_1 = a_r = 1$, $a_2 = \cdots = a_{r-1} = 0$, that satisfy the above assumptions and are not of the form $a_1 \geq a_2 \geq \cdots \geq a_r$.

From here on, let $G = (G_n)$ be a fixed linear recurring sequence with assumptions 1-5. For notational convenience, we will omit the index G in the sequel.

Proposition 2: Let $b_i = (a_i \bmod 2)(-1)^{a_1+\cdots+a_{i-1}}$ ($a_i \bmod 2 = 0$ if a_i is even and $a_i \bmod 2 = 1$ otherwise). Then $S(G_n) = S_G(G_n)$ satisfies the linear recurrence

$$S(G_n) = \sum_{i=1}^{r} b_i S(G_{n-i}) \quad \text{for } n > r. \tag{2}$$

Furthermore, if n has the proper digital expansion $n = \sum_{j=1}^{l} \varepsilon_j G_j$, then

$$S\left(\sum_{j=1}^{l} \varepsilon_j G_j\right) = \sum_{j=1}^{l} (\varepsilon_j \bmod 2)(-1)^{\varepsilon_{j+1}+\cdots+\varepsilon_l} S(G_j). \tag{3}$$

Proof: We will first establish a set identity that holds for all nonnegative integers ε_j, regardless of whether $\sum_{j=1}^{l}\varepsilon_j G_j$ is a proper digital expansion or not:

$$\begin{aligned}\left\{a \,\middle|\, 0 \le a < \sum_{j=1}^{l}\varepsilon_j G_j\right\} &= \bigcup_{j=1}^{l}\left\{a \,\middle|\, \sum_{h=j+1}^{l}\varepsilon_h G_h \le a < \sum_{h=j}^{l}\varepsilon_h G_h\right\} \\ &= \bigcup_{j=1}^{l}\left\{\sum_{h=j+1}^{l}\varepsilon_h G_h + a \,\middle|\, 0 \le a < \varepsilon_j G_j\right\} = \bigcup_{j=1}^{l}\bigcup_{i=0}^{\varepsilon_j-1}\left\{\sum_{h=j+1}^{l}\varepsilon_h G_h + a \,\middle|\, iG_j \le a < (i+1)G_j\right\} \\ &= \bigcup_{j=1}^{l}\bigcup_{i=0}^{\varepsilon_j-1}\left\{\left(\sum_{h=j+1}^{l}\varepsilon_h G_h\right) + iG_j + a \,\middle|\, 0 \le a < G_j\right\},\end{aligned} \tag{4}$$

where each union is disjoint. (Again, empty sums are set at zero.)

Now set $l = n-1$, $\varepsilon_j = a_{n-j}$ for $n-r \le j < n$ and $\varepsilon_j = 0$ otherwise. Then one obtains for $n > r$, after interchanging i and j and shifting $i \to n-i$, $h \to n-h$,

$$\left\{a \,\middle|\, 0 \le a < \sum_{i=n-r}^{n-1} a_{n-i}G_i\right\} = \bigcup_{i=1}^{r}\bigcup_{j=0}^{a_i-1}\left\{\left(\sum_{h=1}^{i-1} a_h G_{n-h}\right) + jG_{n-i} + a \,\middle|\, 0 \le a < G_{n-i}\right\}. \tag{5}$$

From this we see that, for $n > r$,

$$\begin{aligned}S(G_n) &= \sum_{a=0}^{G_n-1}(-1)^{s(a)} = \sum_{i=1}^{r}\sum_{j=0}^{a_i-1}\sum_{a=0}^{G_{n-i}-1}(-1)^{s(\sum_{h=1}^{i-1} a_h G_{n-h} + jG_{n-i} + a)} \\ &= \sum_{i=1}^{r}\sum_{j=0}^{a_i-1}\sum_{a=0}^{G_{n-i}-1}(-1)^{(\sum_{h=1}^{i-1} a_h + j + s(a))} = \sum_{i=1}^{r}(-1)^{(\sum_{h=1}^{i-1} a_h)} S(G_{n-i})\sum_{j=0}^{a_i-1}(-1)^j \\ &= \sum_{i=1}^{r}(a_i \bmod 2)(-1)^{(\sum_{h=1}^{i-1} a_h)} S(G_{n-i}) = \sum_{i=1}^{r} b_i S(G_{n-1})\end{aligned}$$

with $b_i := (a_i \bmod 2)(-1)^{a_1+\cdots+a_{i-1}}$. Note that assumption 3 from above ensures that

$$s\left(\sum_{h=1}^{i-1} a_h G_{n-h} + jG_{n-i} + a\right) = \sum_{h=1}^{i-1} a_h + j + s(a). \tag{6}$$

You only have to start with $m = \sum_{h=1}^{i-1} a_h G_{n-h} + jG_{n-i} + a$ and apply the algorithms stated in the Introduction to deduce that $\varepsilon_{n-h}(m) = a_h$, $1 \le h < i$ and $\varepsilon_{n-i}(m) = j$. (Of course, this procedure is standard in the study of such digital sequences (cf. [8], [9]). This proves equation (2).

The proof of (3) is quite similar. If we set $\sum_{j=1}^{l}\varepsilon_j G_j =: m + \varepsilon_l G_l$ in (4), we get

$$\{a \mid 0 \le a < m + \varepsilon_l G_l\} = \bigcup_{i=0}^{\varepsilon_l-1}\{iG_l + a \mid 0 \le a < G_l\} \cup \{\varepsilon_l G_l + a \mid 0 \le a < m\}.$$

Let $\varepsilon_l G_l + m = \sum_{j=1}^{l}\varepsilon_j G_j$ be a proper digital expansion. Then it follows that

$$\begin{aligned}S(\varepsilon_l G_l + m) &= \sum_{a=0}^{\varepsilon_l G_l + m - 1}(-1)^{s(a)} = \sum_{i=0}^{\varepsilon_l-1}\sum_{a=0}^{G_l-1}(-1)^{s(iG_l+a)} + \sum_{a=0}^{m-1}(-1)^{s(\varepsilon_l G_l + a)} \\ &= \sum_{i=0}^{\varepsilon_l-1}(-1)^i\sum_{a=0}^{G_l-1}(-1)^{s(a)} + (-1)^{\varepsilon_l}\sum_{a=0}^{m-1}(-1)^{s(a)} = (\varepsilon_l \bmod 2)S(G_l) + (-1)^{\varepsilon_l}S(m).\end{aligned} \tag{7}$$

Iterated use of equation (7) gives (3). □

Corollary: Let $d_m(k) := |\{0 \le a < m | S(a) = k\}|$ and $D_m(z)$ the corresponding generating function

$$D_m(z) = \sum_{k \in Z} d_m(k) z^k = \sum_{a=0}^{m-1} z^{S(a)}. \tag{8}$$

Then $D_{G_n}(z)$ (and $D_{G_n}(z^{-1})$) satisfy, for $n > r$, the relation

$$D_{G_n}(z) = \sum_{i=1}^{r} \sum_{j=0}^{a_i - 1} z^{\left(\sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1 + \cdots a_{i-1}} (j \bmod 2) S(G_{n-i})\right)} D_{G_{n-i}}\left(z^{(-1)^{a_1 + \cdots a_{i-1} + j}}\right). \tag{9}$$

Proof: Suppose $n > r$. An iterated use of (7) gives, for $1 \le i \le r$, $j < a_i$, and $m < G_{n-i}$,

$$\begin{aligned}
&S(a_1 G_{n-1} + \cdots + a_{i-1} G_{n-i+1} + j G_{n-i} + m) \\
&= (a_1 \bmod 2) S(G_{n-1}) + (-1)^{a_1} (a_2 \bmod 2) S(G_{n-2}) + \cdots \\
&\quad + (-1)^{a_1 + \cdots + a_{i-2}} (a_{i-1} \bmod 2) S(G_{n-i+1}) + (-1)^{a_1 + \cdots + a_{i-1}} (j \bmod 2) S(G_{n-i}) \\
&\quad + (-1)^{a_1 + \cdots + a_{i-1} + j} S(m) \\
&= \sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1 + \cdots + a_{i-1}} (j \bmod 2) S(G_{n-i}) + (-1)^{a_1 + \cdots + a_{i-1} + j} S(m).
\end{aligned}$$

Note that, for $i = 1$, we just obtain $S(jG_{n-1} + m) = (j \bmod 2) S(G_{n-1}) + (-1)^j S(m)$. Hence, by using (5) and (8), we get

$$\begin{aligned}
D_{G_n}(z) &= \sum_{m=0}^{G_n - 1} z^{S(m)} = \sum_{i=1}^{r} \sum_{j=0}^{a_i - 1} \sum_{m=0}^{G_{n-i} - 1} z^{S(a_1 G_{n-1} + \cdots + a_{i-1} G_{n-i+1} + j G_{n-i} + m)} \\
&= \sum_{i=1}^{r} \sum_{j=0}^{a_i - 1} z^{\left(\sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1 + \cdots + a_{i-1}} (j \bmod 2) S(G_{n-i})\right)} \sum_{m=0}^{G_{n-i} - 1} z^{(-1)^{a_1 + \cdots + a_{i-1} + j} S(m)} \\
&= \sum_{i=1}^{r} \sum_{j=0}^{a_i - 1} z^{\left(\sum_{h=1}^{i-1} b_h S(G_{n-h}) + (-1)^{a_1 + \cdots + a_{i-1}} (j \bmod 2) S(G_{n-i})\right)} D_{G_{n-i}}\left(z^{(-1)^{a_1 + \cdots + a_{i-1} + j}}\right). \quad \square
\end{aligned}$$

4. ASYMPTOTIC ANALYSIS

We distinguish two cases: either $S(G_n)$ is bounded for all suitable initial conditions of G or it is not. The first case will be of special interest. It turns out that in this case the distribution of the values of $S(N)$ approximates a normal distribution for all suitable initial conditions of G (see Theorem 2).

4.1 Bounded $S(G_n)$

Proposition 3: Suppose that $S(G_n)$ is bounded. Then $S(G_n)$ satisfies a linear recursion for $n > N$ with some N, whose characteristic polynomial is a product of different cyclotomic polynomials.

Remark: This motivates the first part of assumption 6 in Section 3.

Proof: We know that every $S(m)$ is an integer and, therefore, can only attain a finite number of distinct values. So we see from (2) that $S(G_n)$ must be periodic (in n) for $n > N$. Let $p > r$ be

some period of $S(G_n)$ and assume $n > N$. Then $S(G_{n+p}) - S(G_n) = 0$, which implies that $S(G_n)$ is a linear combination of powers of pth roots of unity. Let $m(z)$ be the product of all cyclotomic polynomials corresponding to those roots of unity which appear in the representation of $S(G_n)$. Then $S(G_n)$ satisfies the linear recurrence related to $m(z)$. □

Proposition 4: Suppose that $S(G_n)$ is bounded. Then $D_{G_n}(z)$ (defined in (8)) and $D_{G_n}(z^{-1})$ satisfy, for $n > N$, a homogeneous linear recurrence with (in n) constant coefficients $a_i(z)$ that are analytic around $z = 1$ and satisfy $a_i(z) = a_i(z^{-1})$.

Proof: Let $p > r$ be a period of $S(G_n)$. Then, by splitting (9) into four parts, we get

$$\begin{aligned} D_{G_{k+sp}}(z) = &\sum_{i=\max(0,k-r)}^{k-1} \gamma_{k,i}(z) D_{G_{i+sp}}(z) + \sum_{i=k+p-r}^{p-1} \zeta_{k,i}(z) D_{G_{i+(s-1)p}}(z) \\ &+ \sum_{i=\max(0,k-r)}^{k-1} \gamma_{k,p+i}(z) D_{G_{i+sp}}(z^{-1}) + \sum_{i=k+p-r}^{p-1} \zeta_{k,p+i}(z) D_{G_{i+(s-1)p}}(z^{-1}), \end{aligned} \tag{10}$$

with

$$\begin{aligned} \gamma_{k,i}(z) &= h_{k-i} z^{\left(\sum_{h=1}^{k-i-1} b_h m_{k-h} - (a_1 + \cdots + a_{k-i-1} \bmod 2) m_i\right)} \\ \gamma_{k,p+i}(z) &= \overline{h}_{k-i} z^{\left(\sum_{h=1}^{k-i-1} b_h m_{k-h} + (a_1 + \cdots + a_{k-i-1} \bmod 2) m_i\right)} \\ \zeta_{k,i}(z) &= h_{k+p-i} z^{\left(\sum_{h=1}^{k+p-i-1} b_h m_{k-h} - (a_1 + \cdots + a_{k+p-i-1} \bmod 2) m_i\right)} \\ \zeta_{k,p+i}(z) &= \overline{h}_{k+p-i} z^{\left(\sum_{h=1}^{k+p-i-1} b_h m_{k-h} - (a_1 + \cdots + a_{k+p-i-1} \bmod 2) m_i\right)}, \end{aligned}$$

where $m_i := S(G_i)$, $0 \le k < p$ and $0 \le i < p$ and

$$h_i = \begin{cases} |\{0 \le j < a_i \mid j \equiv a_1 + \cdots + a_{i-1}(2)\}| & \text{for } 1 \le i < r, \\ 0 & \text{otherwise,} \end{cases}$$

$$\overline{h}_i = \begin{cases} |\{0 \le j < a_i \mid j \equiv a_1 + \cdots + a_{i-1} + 1(2)\}| & \text{for } 1 \le i \le r, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $1 \le i \le r$, we can calculate

$$\begin{aligned} h_i &= \left\lfloor \frac{a_i + 1}{2} \right\rfloor - \begin{cases} 1 & \text{for } a_i \equiv a_1 + \cdots + a_{i-1}\ (2) \equiv 1\ (2), \\ 0 & \text{otherwise,} \end{cases} \\ h_i + \overline{h}_i &= a_i, \\ h_i - \overline{h}_i &= b_i. \end{aligned} \tag{11}$$

Furthermore, we define $\gamma_{p+k,p+i}(z^{-1}) = \gamma_{k,i}(z)$, $\gamma_{p+k,i}(z^{-1}) = \gamma_{k,p+i}(z)$, $\zeta_{p+k,p+i}(z^{-1}) = \zeta_{k,i}(z)$, $\zeta_{p+k,i}(z^{-1}) = \zeta_{k,p+i}(z)$, and

$$\mathbf{d}_s(z) = \begin{pmatrix} \mathbf{d}_{1,s}(z) \\ \mathbf{d}_{2,s}(z) \end{pmatrix} = \begin{pmatrix} (D_{G_{0+sp}}(z), D_{G_{1+sp}}(z), \ldots, D_{G_{p-1+sp}}(z))^T \\ (D_{G_{0+sp}}(z^{-1}), D_{G_{1+sp}}(z^{-1}), \ldots, D_{G_{p-1+sp}}(z^{-1}))^T \end{pmatrix},$$

$$\Gamma(z)=\begin{pmatrix}\Gamma_{1,1}(z) & \Gamma_{1,2}(z)\\ \Gamma_{2,1}(z) & \Gamma_{2,2}(z)\end{pmatrix}=\begin{pmatrix}(\gamma_{k,i}(z))_{0\le k,i<p} & (\gamma_{k,p+i}(z))_{0\le k,i<p}\\ (\gamma_{p+k,i}(z))_{0\le k,i<p} & (\gamma_{p+k,p+i}(z))_{0\le k,i<p}\end{pmatrix},$$

$$\mathbf{Z}(z)=\begin{pmatrix}\mathbf{Z}_{1,1}(z) & \mathbf{Z}_{1,2}(z)\\ \mathbf{Z}_{2,1}(z) & \mathbf{Z}_{2,2}(z)\end{pmatrix}=\begin{pmatrix}(\zeta_{k,i}(z))_{0\le k,i<p} & (\zeta_{k,p+i}(z))_{0\le k,i<p}\\ (\zeta_{p+k,i}(z))_{0\le k,i<p} & (\zeta_{p+k,p+i}(z))_{0\le k,i<p}\end{pmatrix}.$$

Then the identities $\mathbf{d}_{2,s}(z)=\mathbf{d}_{1,s}(z^{-1})$, $\Gamma_{2,2}(z)=\Gamma_{1,1}(z^{-1})$, $\Gamma_{2,1}(z)=\Gamma_{1,2}(z^{-1})$, $\mathbf{Z}_{2,2}(z)=\mathbf{Z}_{1,1}(z^{-1})$, and $\mathbf{Z}_{2,1}(z)=\mathbf{Z}_{1,2}(z^{-1})$ hold and (10) becomes

$$\mathbf{d}_s(z)=\Gamma(z)\,\mathbf{d}_s(z)+\mathbf{Z}(z)\,\mathbf{d}_{s-1}(z),$$

or, formally,

$$\mathbf{d}_s(z)=((\mathbf{I}-\Gamma(z))^{-1}\mathbf{Z}(z))\,\mathbf{d}_{s-1}(z).$$

Since the quadratic matrix $\Gamma(1)$ consists of four quadratic $p\times p$-blocks that are lower triangle matrices with zero diagonal, it is an easy exercise to show that $\mathbf{I}-\Gamma(1)$ is invertible. Hence, $(\mathbf{I}-\Gamma(z))$ is invertible in a neighborhood of $z=1$.

Call $\mathbf{P}_{(z)}(l):=\det(l\mathbf{I}-\Theta(z))$ the characteristic polynomial of the matrix

$$\Theta(z):=(\mathbf{I}-\Gamma(z))^{-1}\mathbf{Z}(z).$$

Then, by the theorem of Cayley-Hamilton, $\mathbf{P}_{(z)}(\Theta(z))=\mathbf{0}$. From this, we see that the sequence $(D_{G_{i+sp}}(z))_{s\ge 0}$ satisfies a linear homogeneous recursion.

Finally, it follows from the definition of Γ and $\mathbf{Z}$ that $\mathbf{P}_{(z)}(l)=\mathbf{P}_{(z^{-1})}(l)$, from which we see that $a_i(z)=a_i(z^{-1})$. □

Let $A_i(z)$, $1\le i\le 2p$, denote the roots of the polynomial $\mathbf{P}_{(z)}(l)$, where z varies in a sufficiently small neighborhood of $z=1$. Since $a_i(z^{-1})=a_i(z)$, they satisfy $A_i(z^{-1})=A_i(z)$. Furthermore, there exist functions $B_{k,i}(z,s)$ that are polynomials in s such that

$$D_{G_{k+sp}}(z)=\sum_i B_{k,i}(z,s)A_i(z)^s. \tag{12}$$

Since $D_{G_{k+sp}}(1)=G_{k+sp}\sim\beta_1\alpha_1^{k-1}(\alpha_1^p)^s$, it might be expected that (locally around $z=1$) there exists a unique root $A_1(z)$ (satisfying $A_1(1)=\alpha_1^p$) of maximal modulus which is simple. The following lemma shows that this is true if assumption 6 in Section 3 holds.

Lemma 2: Suppose that assumptions 1-6 in Section 3 hold and let $v:=\max\{1\le i\le r\,|\,b_i\ne 0\}$. Then, with the above notation, the $2p$ roots of $\mathbf{P}_{(1)}(l)$ are α_i^p, $1\le i\le r$, where α_i, $1\le i\le r$, denote the roots of $z^r-\sum_{j=1}^r a_j z^{r-j}$, 0 with multiplicity $2p-r-v$, and 1 with multiplicity v.

Proof: From $D_{G_{k+sp}}(1)=G_{k+sp}=\sum_i\beta_i(k+sp)\alpha_i^{k+sp-1}\sim\beta_1\alpha_1^{k-1}(\alpha_1^p)^s$, we see that α_i^p surely are roots of $\mathbf{P}_{(1)}(l)$.

Since $\mathbf{I}-\Gamma(1)$ is invertible, the multiplicity of 0 is $2p$ minus the rank of $\mathbf{Z}(1)$. $\mathbf{Z}(1)$ has a simple block structure. It is an easy exercise to show that its rank equals $r+v$. (Recall that $h_i+\overline{h}_i=a_i$ and $h_i-\overline{h}_i=b_i$.)

Similarly, the multiplicity of 1 is $2p$ minus the rank of

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_{1,1} & \mathbf{K}_{1,2} \\ \mathbf{K}_{1,2} & \mathbf{K}_{1,1} \end{pmatrix} = \mathbf{I} - \mathbf{\Gamma}(1) - \mathbf{Z}(1).$$

Observe that

$$\operatorname{rk}\begin{pmatrix} \mathbf{K}_{1,1} & \mathbf{K}_{1,2} \\ \mathbf{K}_{1,2} & \mathbf{K}_{1,1} \end{pmatrix} = \operatorname{rk}\begin{pmatrix} \mathbf{K}_{1,1}+\mathbf{K}_{1,2} & \mathbf{0} \\ \mathbf{K}_{1,2} & \mathbf{K}_{1,1}-\mathbf{K}_{1,2} \end{pmatrix}$$

and that $\mathbf{K}_{1,1}+\mathbf{K}_{1,2}$ (resp. $\mathbf{K}_{1,1}-\mathbf{K}_{1,2}$) are cyclic matrices with entries $1, -a_1, \ldots, -a_r, 0, \ldots, 0$ (resp. $1, -b_1, \ldots, -b_r, 0, \ldots, 0$). By [3, Lemma 3], the rank of $\mathbf{K}_{1,1}+\mathbf{K}_{1,2}$ is p (resp. the rank of $\mathbf{K}_{1,1}-\mathbf{K}_{1,2}$ is $p-v$), v being equal to the number of different p^{th} roots of unity that are roots of $z^r - \sum_{j=1}^{r} b_j z^{r-j}$. Thus, rk $\mathbf{K} = 2p - v$, which completes the proof of the lemma. □

Let us define discrete random variables X_m by

$$\mathbf{P}(X_m = k) = \frac{d_m(k)}{m}. \tag{13}$$

(Recall that $d_m(k) := |\{0 \le a < m | S(a) = k\}|$.) It is well known that one can calculate mean and variance using the generating function:

$$\mu_m = \mathbf{E}X_m = \frac{1}{m} D_m'(1),$$

$$\sigma_m^2 = \mathbf{V}X_m = \frac{1}{m}\left(D_m''(1) + D_m'(1) - \frac{1}{m} D_m'(1)^2\right).$$

From here on, we will assume 1-6 in Section 3.

Lemma 3: Let $A_1(z)$ be the unique root of maximal modulus of $\mathbf{P}_{(l)}(z)$. Then we have $A_1''(1) \ge 0$,

$$\mu_{G_{k+sp}} := \mathbf{E}X_{G_{k+sp}} = O(1) \quad \text{and} \quad \sigma_{G_{k+sp}}^2 := \mathbf{V}X_{G_{k+sp}} = s\frac{A_1''(1)}{A_1(1)} + O(1)$$

as $s \to \infty$. Furthermore, if $A_1''(1) \ne 0$, then

$$\mathbf{E}\exp\left(it\frac{X_{G_{k+sp}} - \mu_{G_{k+sp}}}{\sigma_{G_{k+sp}}}\right) = \exp\left(-\frac{t^2}{2}\right)\left(1 + O\left(\frac{1}{\sqrt{s}}\right)\right)$$

as $s \to \infty$. This means that X_{G_m} is asymptotically Gaussian with mean μ_{G_m} and variance $\sigma_{G_m}^2$.

Proof: Let $A(z) = A_1(z)$ and $B_k(z) = B_{k,1}(z, s)$ in (12) (where the s-degree of the polynomial $B_{k,1}(z, s)$ is zero). Since $A'(1) = 0$, we obtain from (12) by differentiation,

$$D_{G_{k+sp}}(1) = B_k(1)A(1)^s + O((A(1)\gamma)^s),$$

$$D_{G_{k+sp}}'(1) = B_k(1)A(1)^s \frac{B_k'(1)}{B_k(1)} + O((A(1)\gamma)^s),$$

$$D_{G_{k+sp}}''(1) = B_k(1)A(1)^s\left(s\frac{A''(1)}{A(1)} + \frac{B_k''(1)}{B_k(1)}\right) + O((A(1)\gamma)^s),$$

with some $0 \le \gamma < 1$ properly chosen. From $D_{G_{k+sp}}(1) = G_{k+sp}$, we get

$$D'_{G_{k+sp}}(1) = G_{k+sp}\frac{B'_k(1)}{B_k(1)}(1+O(\gamma^s)),$$

$$D''_{G_{k+sp}}(1) = G_{k+sp}\left(s\frac{A''(1)}{A(1)}+\frac{B''_k(1)}{B_k(1)}\right)(1+O(\gamma^s)).$$

Both $D'_{G_{k+sp}}(1)$ and $D''_{G_{k+sp}}(1)$ are real, and because of $B_k(1)=\beta_1\alpha_1^{k-1}\in\mathbf{R}^+$, $B'_k(1)$ is real. Furthermore, $A''(1)$ and $B''_k(1)$ are real, too. From this, we obtain that

$$\mathbf{E}X_{G_{k+sp}} = \frac{B'_k(1)}{B_k(1)}(1+O(\gamma^s)) = O(1),$$

$$\mathbf{V}X_{G_{k+sp}} = \left(s\frac{A''(1)}{A(1)}+\frac{B''_k(1)}{B_k(1)}-\left(\frac{B'_k(1)}{B_k(1)}\right)^2\right)(1+O(\gamma^s)) = s\frac{A''(1)}{A(1)}+O(1),$$

from which it is clear that $A''(1)\geq 0$. Using $A'(1)=A'''(1)=0$, we get

$$A(e^t)^s = A(1)^s\exp\left(\frac{st^2}{2}\frac{A''(1)}{A(1)}\right)(1+O(st^4)).$$

Now suppose $A''(1)>0$, then we have

$$D_{G_{k+sp}}\left(e^{it/\sigma_{G_{k+sp}}}\right) = G_{k+sp}\exp\left(-\frac{t^2}{2}\right)\left(1+O\left(\frac{t}{\sqrt{s}}\right)+O\left(\frac{t^4+1}{s}\right)\right),$$

where the O-constants are independent of k. For any fixed t, we get

$$\mathbf{E}\exp\left(it\frac{X_{G_{k+sp}}-\mu_{G_{k+sp}}}{\sigma_{G_{k+sp}}}\right) = \frac{D_{G_{k+sp}}\left(e^{it/\sigma_{G_{k+sp}}}\right)}{G_{k+sp}}\exp\left(-it\frac{\mu_{G_{k+sp}}}{\sigma_{G_{k+sp}}}\right)$$

$$= \exp\left(-\frac{t^2}{2}\right)\left(1+O\left(\frac{1}{\sqrt{s}}\right)\right).$$

Thus, by Levi's theorem (see [7]), the normalized random variables $(X_{G_m}-\mu_{G_m})/\sigma_{G_m}$ converge weakly to normal distribution.

Remark: The use of generating functions for the proof of asymptotic normality probably started with Bender's paper [2]. Further references can be found in [5].

Now we will turn our attention to X_m, where m need not be an element of the basis G.

Theorem 2: Suppose that $G=(G_n)$ satisfies a linear recursion with restrictions 1-6 of Section 3. Then, with the above notation, we have

$$\mathbf{E}X_m = O(1) \quad\text{and}\quad \mathbf{V}X_m = \frac{l}{p}\frac{A''(1)}{A(1)}+O(1),$$

X_m being defined as in (13) and l being the length of the digital expansion of m. If $A''(1)>0$, then X_m is asymptotically Gaussian with mean value $\mathbf{E}X_m$ and variance $\mathbf{V}X_m\sim c\log m$ for some constant $c>0$, i.e.,

$$\lim_{m\to\infty}\frac{1}{m}\left|\left\{N<m: S(N)\leq \mathbf{E}X_m+x\sqrt{\mathbf{V}X_m}\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt.$$

Remark: The special case of $G_n = F_{n+1}$ (which leads to the original Zeckendorf representation) was discussed in [4]. There are also recent contributions to similar questions, e.g., Dumont and Thomas [6] prove asymptotic normality for substitution sequences by a different method, and Barat and Grabner [1] show the existence of a limiting distribution of G-additive functions.

Proof: Let $m = \sum_{i=1}^{l} \varepsilon_i G_i$ be the digital expansion of m. Iterated use of equation (7) yields, for $1 \le j \le l$, $i < \varepsilon_j$, and $a < G_j$,

$$\begin{aligned}
S\left(\sum_{h=j+1}^{l} \varepsilon_h G_h + iG_j + a\right) &= (\varepsilon_l \bmod 2)S(G_l) + (-1)^{\varepsilon_l} S\left(\sum_{h=j+1}^{l-1} \varepsilon_h G_h + iG_j + a\right) \\
&= (\varepsilon_l \bmod 2)S(G_l) + (-1)^{\varepsilon_l}(\varepsilon_{l-1} \bmod 2)S(G_{l-1}) + \cdots + (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+2}}(\varepsilon_{j+1} \bmod 2)S(G_{j+1}) \\
&\quad + (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2)S(G_j) + (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i} S(a) \\
&= \sum_{p=j+1}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}}(\varepsilon_p \bmod 2)S(G_p) + (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2)S(G_j) + (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i} S(a),
\end{aligned}$$

and from (4) we see that

$$\begin{aligned}
d_m(k) &= \left|\left\{0 \le a < \sum_{i=1}^{l} \varepsilon_i G_k \,\middle|\, S(a) = k\right\}\right| = \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j - 1} \left|\left\{0 \le a < G_j \,\middle|\, S\left(\sum_{h=j+1}^{l} \varepsilon_h G_h + iG_j + a\right) = k\right\}\right| \\
&= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j - 1} \left|\left\{0 \le a < G_j \,\middle|\, S(a) = (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i}\right.\right. \\
&\qquad \left.\left. \times \left(k - \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}} S(G_p) - (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2)S(G_j)\right)\right\}\right| \\
&= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j - 1} d_{G_j}\left((-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i}\left(k - \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}} m_p - (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2)m_j\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
D_m(z) &= \sum_{k \in \mathbf{Z}} d_m(k) z^k \\
&= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j - 1} \sum_{k \in \mathbf{Z}} z^k d_{G_j}\left((-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i}\left(k - \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}} S(G_p) - (-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2)S(G_j)\right)\right) \\
&= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j - 1} \sum_{k \in \mathbf{Z}} d_{G_j}(k) z^{\left((-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i} k + \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}} m_p + (-1)^{\varepsilon_l + \cdots \varepsilon_{j+1}}(i \bmod 2)m_j\right)}
\end{aligned}$$

$$= \sum_{j=1}^{l} z^{\left(\sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l+\cdots+\varepsilon_{p+1}} m_p\right)} \sum_{i=0}^{\varepsilon_j-1} z^{\left((-1)^{\varepsilon_l+\cdots+\varepsilon_{j+1}}(i \bmod 2) m_j\right)} D_{G_j}\left(z^{\left((-1)^{\varepsilon_l+\cdots+\varepsilon_{j+1}+i}\right)}\right)$$

$$= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j-1} z^{b(j,i)} D_{G_j}(z^{c(j,i)}), \tag{14}$$

in which

$$b(j,i) = \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l+\cdots+\varepsilon_{p+1}} m_p + (-1)^{\varepsilon_l+\cdots+\varepsilon_{j+1}} (i \bmod 2) m_j,$$

$$c(j,i) = (-1)^{\varepsilon_l+\cdots+\varepsilon_{j+1}+i}.$$

Differentiation of (14) yields

$$zD'_m(z) = \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j-1} \left(b(j,i) z^{b(j,i)} D_{G_j}(z^{c(j,i)}) + z^{b(j,i)} D'_{G_j}(z^{c(j,i)}) c(j,i) z^{c(j,i)} \right),$$

$$\begin{aligned} z\frac{\partial}{\partial z}(zD'_m(z)) = \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j-1} \Big(& b(j,i)^2 z^{b(j,i)} D_{G_j}(z^{c(j,i)}) + 2b(j,i) z^{b(j,i)} D'_{G_j}(z^{c(j,i)}) c(j,i) z^{c(j,i)} \\ & + z^{b(j,i)} \left(z^{c(j,i)} D'_{G_j}(z^{c(j,i)}) + z^{2c(j,i)} D''_{G_j}(z^{c(j,i)}) \right) \Big). \end{aligned}$$

It is an easy exercise to show $\sum_{j=1}^{l} (l-j+1)^k G_j \le C_k G_l$. Because the m_j are bounded, we get $b(j,i) = O(l-j+1)$ (uniformly in i) and

$$\begin{aligned} D'_m(1) &= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j-1} \left(b(j,i) D_{G_j}(1) + c(j,i) D'_{G_j}(1) \right) \\ &= O\left(\sum_{j=1}^{l} (l-j+1) G_j \right) = O(G_l) = O(m) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial z}(zD'_m(z))\Big|_{z=1} &= \sum_{j=1}^{l} \sum_{i=0}^{\varepsilon_j-1} \left(b(j,i)^2 D_{G_j}(1) + 2b(j,i) c(j,i) D'_{G_j}(1) + D'_{G_j}(1) + D''_{G_j}(1) \right) \\ &= \sum_{j=1}^{l} \varepsilon_j D''_{G_j}(1) + O\left(\sum_{j=1}^{l} (l-j+1)^2 G_j \right) + O\left(\sum_{j=1}^{l} (l-j+1) G_j \right) + O\left(\sum_{j=1}^{l} G_j \right) \\ &= \sum_{j=1}^{l} \varepsilon_j G_j \frac{j}{p} \frac{A''(1)}{A(1)} \left(1 + O\left(\frac{1}{j}\right) \right) + O(m) \\ &= \frac{1}{p} \frac{A''(1)}{A(1)} \left(l \sum_{j=1}^{l} \varepsilon_j G_j - \sum_{j=1}^{l} \varepsilon_j G_j (l-j) \right) + O\left(\sum_{j=1}^{l} \varepsilon_j G_j \frac{1}{p} \frac{A''(1)}{A(1)} \right) + O(m) \\ &= m \frac{l}{p} \frac{A''(1)}{A(1)} + O(m). \end{aligned}$$

Thus, we have

$$\mathbf{E}X_m = O(1) \quad \text{and} \quad \mathbf{V}X_m = \frac{l}{p}\frac{A''(1)}{A(1)} + O(1). \tag{15}$$

Furthermore, by using (14), we obtain

$$D_m(e^{it/\sigma_m}) = \sum_{j=1}^{l} \exp\left(\frac{it}{\sigma_m} \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}} m_p\right)$$
$$\times \sum_{i=0}^{\varepsilon_j - 1} \exp\left(\frac{it}{\sigma_m}(-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2) m_j\right) D_{G_j}\left(\exp\left(\frac{it}{\sigma_m}(-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i}\right)\right),$$

and for any fixed t,

$$D_{G_j}\left(\exp\left(\frac{it}{\sigma_m}(-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i}\right)\right) = D_{G_j}\left(\exp\left(\frac{it\frac{\sigma_{G_j}}{\sigma_m}}{\sigma_{G_j}}(-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1} + i}\right)\right)$$
$$= G_j \exp\left(-\frac{t^2 \frac{j}{l}\left(1 + O\left(\frac{1}{j}\right)\right)}{2}\right)\left(1 + O\left(\frac{1}{\sqrt{j}}\right)\right) = G_j e^{-t^2/2} \exp\left(\frac{t^2}{2}\frac{l-j}{l} + O\left(\frac{1}{\sqrt{j}}\right)\right)$$

and

$$\sum_{i=0}^{\varepsilon_j - 1} \exp\left(\frac{it}{\sigma_m}(-1)^{\varepsilon_l + \cdots + \varepsilon_{j+1}}(i \bmod 2) m_j\right) = \sum_{i=0}^{\varepsilon_j - 1}\left(1 + O\left(\frac{1}{\sqrt{l}}\right)\right)$$
$$= \varepsilon_j\left(1 + O\left(\frac{1}{\sqrt{l}}\right)\right) = \varepsilon_j \exp\left(O\left(\frac{1}{\sqrt{l}}\right)\right),$$

where the O-constants do not depend on l or j. Thus we get, for $0 < \vartheta < \frac{1}{2}$,

$$D_m(e^{it/\sigma_m})e^{t^2/2} = \sum_{j=1}^{l} \varepsilon_j G_j \exp\left(\frac{it}{\sigma_m} \sum_{\substack{p=j+1 \\ \varepsilon_p \equiv 1(2)}}^{l} (-1)^{\varepsilon_l + \cdots + \varepsilon_{p+1}} m_p + O\left(\frac{l-j}{l}\right) + O\left(\frac{1}{\sqrt{j}}\right)\right)$$
$$= \sum_{l - l^{\vartheta} \le j \le l} \varepsilon_j G_j \exp\left(O\left(\frac{l-j}{\sqrt{l}}\right) + O\left(\frac{1}{\sqrt{j}}\right)\right) + \sum_{1 \le j < l - l^{\vartheta}} \varepsilon_j G_j O(1)$$
$$= \sum_{l - l^{\vartheta} \le j \le l} \varepsilon_j G_j \exp\left(O\left(l^{\vartheta - \frac{1}{2}}\right) + O\left(\frac{1}{\sqrt{1/2}}\right)\right) + O\left(G_{\lfloor l - l^{\vartheta} \rfloor}\right)$$
$$= \sum_{l - l^{\vartheta} \le j \le l} \varepsilon_j G_j \left(1 + O\left(l^{\vartheta - \frac{1}{2}}\right)\right) + O\left(\alpha_1^{l - l^{\vartheta}}\right) = \sum_{l - l^{\vartheta} \le j \le l} \varepsilon_j G_j + O\left(m l^{\vartheta - \frac{1}{2}}\right) + O\left(\alpha_l^{l - l^{\vartheta}}\right)$$
$$= m + O\left(m l^{\vartheta - \frac{1}{2}}\right) + O\left(\alpha_1^{l - l^{\vartheta}}\right) = m + O\left(m l^{\vartheta - \frac{1}{2}}\right)$$

and, finally (for any fixed t),

$$\begin{aligned}\mathbf{E}\exp\left(it\frac{X_m-\mu_m}{\sigma_m}\right) &= \frac{D_m(e^{it/\sigma_m})}{m}\exp\left(-it\frac{\mu_m}{\sigma_m}\right)\\ &= \exp\left(-\frac{t^2}{2}\right)\left(1+O\left(l^{9-\frac{1}{2}}\right)\right)\exp\left(O\left(\frac{1}{\sqrt{l}}\right)\right)\\ &= \exp\left(-\frac{t^2}{2}\right)\left(1+O\left(l^{9-\frac{1}{2}}\right)\right)\end{aligned}$$

and X_m is asymptotically Gaussian with mean μ_m and variance σ_m^2. □

The condition that $z^r-\sum_{i=1}^r b_i z^{r-i}$ (where $v=\max\{1\le i\le r\,|\,b_i\ne 0\}$) is a product of z^{r-v} and different cyclotomic polynomials is rather restrictive in the case in which $G_n\le 2G_{n-1}$ for $n>1$.

Proposition 5: Suppose that $G=(G_n)$ satisfies a linear recursion with restrictions 1-5 of Section 3 such that $G_n\le 2G_{n-1}$ for $n>1$. Then $z^r-\sum_{i=1}^r b_i z^{r-i}$ is a product of z^{r-v} and different cyclotomic polynomials, where $v=\max\{1\le i\le r\,|\,b_i\ne 0\}$, if and only if one of the following conditions holds:

1. $r=1$ and $a_1=2$: the binary system, or
2. $a_1=a_2=\cdots+a_r=1$: a generalized Zeckendorf representation.

Proof: First, let $B(z)=z^r-\sum_{i=1}^r b_i z^{r-i}$ be of the above type, then if $a_1>1$ we are in the first case. So let us assume $a_1=1$, then it follows that $a_i\in\{0,1\}$, $a_r=1$, and therefore $v=r$. From this, we see that $z^r-\sum_{i=1}^r b_i z^{r-i}$ must be a symmetric polynomial that yields $a_i=a_{r-i}$ for all $1\le i<r$. Now suppose $a_1=\cdots=a_{i-1}=1=a_r=\cdots=a_{r-i+1}$ and $a_i=0=a_{r-i}$ for some $1<i\le r-i$. Then, by assumption 3 in Section 3, we have that $G_{n-r+i}\ge\sum_{j=r-i+1}^r a_jG_{n-j}=\sum_{j=r-i+1}^r G_{n-j}$ for $n>r$ or, equivalently, that $G_n\ge\sum_{j=1}^i G_{n-j}$ for $n>i$. Because $G_n=\sum_{j=1}^r a_jG_{n-j}$ for $n>r$, it follows that $\sum_{j=i+1}^r a_jG_{n-j}\ge G_{n-i}$ for $n>r$. On the other hand we have, again by assumption 3, that $G_{n-i}\ge\sum_{j=i+1}^r a_jG_{n-j}$ for $n>r$, from which we see that $G_n=\sum_{j=1}^{r-i} a_{i+j}G_{n-j}$ for $n>r-i$, a contradiction to assumption 4.

Now let $r=1$ and $a_1=2$, then $v=0$ and $B(z)=z$. Finally, suppose $a_1=a_2=\cdots=a_r=1$. Then $b_i=(-1)^{i+1}$ and

$$B(z)=\sum_{i=0}^r(-1)^i z^{r-i}=\frac{z^{r+1}+(-1)^r}{z+1}$$

is of the desired type. □

4.2 Unbounded $S(G_n)$

Proposition 6: If $S(G_n)$ is unbounded, then there exists some α with $1<\alpha<\alpha_1$ (α_1 defined as in Section 3), $k\ge 1$, real numbers $\varphi_1,\ldots,\varphi_k$, and polynomials $\beta_1(n),\ldots,\beta_k(n)$, $\overline{\beta}_1(n),\ldots,\overline{\beta}_k(n)$ not all of them zero, such that

$$S(G_n)=\alpha^n\sum_{i=1}^k(\beta_i(n)\cos(n\varphi_i)+\overline{\beta}_i(n)\sin(n\varphi_i))+O((\gamma\alpha)^n)$$

for some $\gamma\in(0,1)$.

Proof: Since $S(G_n)$ satisfies the linear recurrence of Proposition 2, this representation follows immediately. □

Theorem 3: Suppose that $G=(G_n)$ satisfies a linear recurrence as above such that $S(G_n)$ is unbounded. Then

$$\limsup_{m\to\infty}\frac{\log(|S(m)|)}{\log m}=\frac{\log\alpha}{\log\alpha_1}.$$

Proof: First, it follows from Proposition 6 that

$$\limsup_{m\to\infty}\frac{\log(|S(m)|)}{\log m}\geq\limsup_{m\to\infty}\frac{\log(|S(G_n)|)}{\log G_n}=\frac{\log\alpha}{\log\alpha_1}.$$

The upper bound follows from the second part of Proposition 2 and again by an application of Proposition 6: Let $m=\sum_{j=1}^{l}\varepsilon_j G_j$ be the proper digital expansion of m and let $C, K>0$ be large enough so that $|\beta_i(n)+\overline{\beta}_i(n)|<Cn^D$ for all n, i. Then we have, for $l\to\infty$,

$$\frac{\log(|S(m)|)}{\log m}\leq\frac{\log\left(\sum_{j=1}^{l}|S(G_j)|\right)}{\log(\varepsilon_l G_l)}\leq\frac{\log(l\alpha^l(Cl^D+C'\gamma^l))}{\log\varepsilon_l+\log G_l}$$
$$\leq\frac{l\log\alpha+(D+1)\log l+C''}{l\log\alpha_1+C'''}\to\frac{\log\alpha}{\log\alpha_1},$$

which completes our proof. □

Remark: It is also possible to discuss the function $F(m)=S(m)m^{-(\log\alpha)/(\log\alpha_1)}$ in more detail. It turns out that $F(m)$ is an almost periodic function, i.e., $S(m)$ has an almost fractal structure. You just have to adapt the methods used in [8] and [9].

5. CONCLUSIONS

Our starting point was the Möbius function $\mu_G(n)$ of the partial order which is induced by proper digital expansions with respect to a basis $G=(G_n)$. It turned out that $\mu_G(n)\in\{-1,0,1\}$, so it is a natural question to determine the distribution of these three values $-1, 0, 1$. If $G_{n+1}\geq 2G_n$ for all $n>1$, then the answer is very easy (see Proposition 1). Therefore, we restricted ourselves to the case $G_{n+1}\leq 2G_n$ for all $n>1$. Here $\mu_G(n)=(-1)^{s_G(n)}$. Thus, $\mu_G(n)\neq 0$ for all $n\geq 0$ and $M_G(N)=S_G(N)$ is exactly the difference between the number of $n<N$ with $\mu_G(n)=1$ and the number of $n<N$ with $\mu_G(n)=-1$. In the case of linear recurring sequences $G=(G_n)$ (satisfying certain natural conditions), we proved that in any case $M_G(N)=o(N)$, i.e., $-1,+1$ are asymptotically equidistributed.

More precisely, we discussed the distribution of values of $S_G(N)$ (which can also be considered in the case $G_{n+1}\geq 2G_n$). It turns out that there are two essentially different cases, the case of bounded $S_G(G_n)$ and the case of unbounded $S_G(G_n)$. If $S_G(G_n)$ is unbounded, then $S_G(N)$ has an almost fractal structure (see Theorem 3 and the Remark following it). However, if $S_G(G_n)$ is bounded for all suitable initial conditions of G, then the values $S_G(N)$ admit a Gaussian limit law in the following sense: If X_n is a random variable defined by

$$\mathbf{P}(X_N=k)=\frac{1}{N}\left|\{n<N\,|\,S_G(n)=k\}\right|$$

then X_N is asymptotically Gaussian with bounded mean value and variance $\mathbf{V}X_N\sim c\log N$, provided that $c\neq 0$ (Theorem 2).

Since $S_G(G_n)$ satisfies the linear recurrence (2), it follows that $S_G(G_n)$ is periodic (for sufficiently large n) if it is bounded. This can only occur for all suitable initial conditions of G if and only if the roots of the characteristic polynomial $B(z) = z^r - \sum_{j=1}^{r} b_j z^{r-j}$ of (2) are 0 or roots of unity. Therefore, the assumption on $B(z)$ in Theorem 2, this is assumption 6 in Section 3, is quite natural.

Finally, we want to recall that the only recurring sequences $G = G(n)$ satisfying assumptions 1-5 such that $a_1 = 1$ (i.e., $G_{n+1} < 2G_n$) and that $B(z)$ is the product of z^{r-v} and cyclotomic polynomials are generalized Fibonacci numbers (Proposition 5). They satisfy a recursion of the form $G_n = G_{n-1} + \cdots + G_{n-r}$. Here Theorem 2 applies. Hence, the values of $M_G(N)$ with respect to generalized Zeckendorf representations satisfy a central limit law.

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THE PASCAL-DE MOIVRE TRIANGLES*

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1. INTRODUCTION

The coefficients of the Pascal triangle were generalized in 1756 by de Moivre [5]. Each row of a Pascal triangle contains a sequence of numbers that are the coefficients of the power series expansion for the binary expression $(1+x)^N$. The de Moivre formula [2], [4], [5], [6] derives the coefficients of the power series for the generalized expansion of $(1+x+x^2+\cdots+x^{(J-1)})^N$. Thus, for integers ($J \geq 2$ and $N \geq 1$) and for $0 \leq h \leq N(J-1)$, we define $C(N, J; h)$ to be the coefficients of (x^h) in the expansion of

$$(1+x+x^2+\cdots+x^{(J-1)})^N = \sum C(N, J; h)x^h. \tag{1}$$

A Pascal-de Moivre triangle can be created from the coefficients $C(N, J; h)$ for each positive integer value (J). For example, with ($J = 3$), the Pascal-de Moivre triangle of $C(N, J; h)$ terms for row numbers $1 \leq N \leq 4$ is:

$N \backslash h$	0	1	2	3	4	5	6	7	8
1	1	1	1						
2	1	2	3	2	1				
3	1	3	6	7	6	3	1		
4	1	4	10	16	19	16	10	4	1

(2)

In this paper, the sequence of $C(N, J; h)$ terms in each row (N) of the Pascal-de Moivre triangle is examined for the series properties, at various arrangements of terms. Each $C(N, J; h)$ term in the Nth row of the Pascal-de Moivre triangle is assigned a coefficient factor (F_h), such that

$$\text{Series with coefficients } (F) = \sum\{(F_h)\ C(N, J; h)\}. \tag{3}$$

Sections 3 and 4 define summations of all $C(N, J; h)$ terms in the Nth row of the Pascal-de Moivre triangle that are separated by some fixed interval spacing (Δh). Then, from the set of coefficients (F), the factors (F_h) equal one at each interval step and equal zero otherwise. Section 3 examines these summations of the $C(N, J; h)$ terms at intervals that are a function of the distribution variable (J); i.e., for ($\Delta h = f(J)$). The quadruplet cycle of Section 4 adds the $C(N, J; h)$ terms with interval spacing ($\Delta h = 4$).

In Sections 5 and 6, the coefficient factors (F_h) are related to the moments of the $C(N, J; h)$ distribution. A quick review of the theory of moments from [3] will illustrate which coefficient factors (F_h) from set (F) are involved, and what form the series in (3) will take.

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The moment $(m_{(R,x)})$ about a point (x) for a discrete distribution of (C_h) terms can be expressed by a summation over all term indices (h). The Rth moment of $(m_{(R,x)})$ is defined by the summation in (4) for the distribution density $f(h)$ evaluated at each index (h):

$$m_{(R,x)} = \Sigma\{(h-x)^R f(h)\}, \tag{4}$$

where $f(h) = (C_h)/(\Sigma C_h)$.

We choose the distribution terms (C_h) to be the terms in the Nth row of the Pascal-de Moivre triangle, given by $C(N, J; h)$. By rearranging equation (4), we define a moment summation equation (5) for the $C(N, J; h)$ distribution as:

$$\Sigma\{(h-x)^R C(N, J; h)\} = (m_{(R,x)})(\Sigma\{C(N, J; h)\}). \tag{5}$$

The left-hand side of equation (5) is the same as equation (3) with the coefficient factors $(F_h) = (h-x)^R$. Section 5 uses equation (5) to obtain Summations Based on Moments. Section 6 uses a similar equation for (3), but with $(F_h) = (-1)^h(h-x)^R$, to obtain Summations Based on Alternating Signed Terms. In Sections 5 and 6, the $C(N, J; h)$ moment summations and moments $(m_{(R,x)})$ are evaluated relative to points at ($x = 0$, the origin) and ($x = M$, the mean).

2. DERIVATION AND TERMINOLOGY

De Moivre derived the formula for each $C(N, J; h)$ term by writing the left-hand side of equation (1) in the form $(1-x^J)^N(1-x)^{-N}$, expanded both factors with the binomial theorem, and collected terms. The resulting formula (6) is a summation over all integers $\{0 \le a \le [h/J]\}$, where $[h/J]$ is the "least integer function" for the largest integer not exceeding the value of h/J:

$$C(N, J; h) = \Sigma C(N, J; h, a) = \Sigma(-1)^a \binom{h-aJ+N-1}{N-1}\binom{N}{a}. \tag{6}$$

In a reduced format of factorials, with the substitution $(N) = (N!)/((N-1)!)$, the de Moivre formula becomes the summation in (7) over all integers $\{0 \le a \le [h/J]\}$:

$$C(N, J; h) = \Sigma(-1)^a \frac{(h-aJ+N-1)!}{(h-aJ)!}\frac{(N)}{(N-a)!(a)!}. \tag{7}$$

A standard terminology will be used for the coefficient terms of the Pascal-de Moivre triangles. A consistent notation for the $C(N, J; h)$ and $C(N, J; h, a)$ terms is described here:

C	capital letter for the term itself (the coefficient of the basic expansion);
N, J	capital letters for the independent variables of the $C(N, J; h)$ series;
h, a	small letters for the summation indices in their respective sums.

The power of the *Mathematica* [8] program allowed computations that could accurately generate numbers in excess of 100 digits. Therefore, large $C(N, J; h)$ distributions were evaluated with precision, including those defined by (N, J) values of $(100, 2)$ and $(20, 20)$ and $(2, 300)$.

3. COLUMNAL SUMMATIONS

The full N^{th} row sequence of terms $C(N, J; h)$ in the Pascal-de Moivre triangle has a known series value of (J^N) per [4], [7], when summed over all integers $0 \le h \le N(J-1)$:

$$\sum C(N, J; h) = (J^N). \tag{8}$$

The $C(N, J; h)$ sequence can also be partitioned by taking every (Q^{th}) term to obtain an ordered summation $S(N, J; Q, r)$ of the $C(N, J; h)$ terms. Reference to such partitioning is given for the binomial $(J = 2)$ case in [9] and for the $C(N, J; h)$ sequence in [1]. Here and in the next section, the derivation of $S(N, J; Q, r)$ will use a variation of the methods described in those references. The main difference between Hoggatt's approach in [1] and the one employed in this section is that here the least integer function for $[J^N / Q]$ is used rather than the simple ratio of (J^N / Q).

For pictorial convenience, the partition of a $C(N, J; h)$ sequence can be displayed in tabular form with (Q) columns. As a guide to the tabular display of partitions, the $C(N, J; h)$ sequence at values $(N, J) = (3, 3)$ from (2) will be analyzed for various spacings (Q) to obtain the sums $S(N, J;\ Q, r)$.

Table of Columnal Sums $S(N, J; Q, r)$ for $(N, J) = (3, 3)$ (9)

	$Q = (J-1) = 2$		$Q = J = 3$			$Q = (J+1) = 4$			
A	$r =$ 0	1	0	1	2	0	1	2	3
0	1	3	1	3	6	1	3	6	7
1	6	7	7	6	3	6	3	1	
2	6	3	1						
3	1								
$S(N, J; Q, r)$	14	13	9	9	9	7	6	7	7
	$[J^N / Q] = [3^3 / 2] = 13$		$[J^N / Q] = [3^3 / 3] = 9$			$[J^N / Q] = [3^4 / 4] = 6$			

Each row will have a row number (A) with values from $0 \le A \le [N(J-1)/Q]$, where the brackets indicate the least integer function for the greatest integer not exceeding the enclosed expression. The columns in this table will have column numbers (r) in the range $0 \le r \le (Q-1)$. The values of the column series $S(N, J; Q, r)$ are analyzed for various interval spacings (Q) at or near the value (J).

For $Q = J$, the terms at $(h = AQ + r)$ will be summed over integers $0 \le A \le [N(J-1)/(J)]$. Each column (r) in the range $0 \le r \le (J-1)$ will have the sum for $S(N, J; Q, r)$ per equation (10). Table (9) above shows that the $C(N, J; h)$ sequence at $(N, J) = (3, 3)$ has each columnal sum $S(N, J; Q, r)$ equal to (9).

$$S(N, J; Q, r) = \sum C(N, J; h) = J^{(N-1)} \quad \text{for each } (r). \tag{10}$$

In the binomial case of $(J = 2)$ for Pascal's (classical) triangle, formula (10) generalizes the familiar fact that the sum of alternate terms $(Q = 2)$ in any row is half the sum of the entire row from (8), since $(J^{N-1}) = 2^{(N-1)} = (1/2)2^N = (1/2)(J^N)$ when $(J = 2)$. It does not mean, however, that the sum of alternate terms of the generalized Pascal-de Moivre triangle with $(J \ne 2)$ is half the sum of the row terms (a case that is dealt with in equations (19) and (20) of the next section).

For $Q = (J-1)$, the $C(N, J; h)$ terms at $(h = AQ + r)$ are summed over integers $0 \le A \le N$. So then, for column locations (r) in the range $0 \le r \le (J-2)$, the column sums $S(N, J; Q, r)$ will

be given by equation (11) for all $N \geq 1$. In the $(N, J) = (3, 3)$ example, the first column at $(r = 0)$ has a sum of (14), which is one more than the sum of (13) at the other column(s), as seen in table (9). Equation (8) covers the linear $(Q = 1)$ case at $(J = 2)$.

$$\begin{aligned} S(N, J; Q, r) = \Sigma C(N, J; h) &= [(J^N)/(J-1)] + 1 \quad \text{for } J > 2 \text{ and } r = 0 \\ &= [(J^N)/(J-1)] \qquad \text{for } J > 2 \text{ and } 1 \leq r \leq (J-2). \end{aligned} \tag{11}$$

For $Q = (J+1)$, the column terms at $(h = AQ + r)$ are summed over the integers $0 \leq A \leq [N(J-1)/(J+1)]$. So, for column locations (r) in the range $0 \leq r \leq J$, the sums $S(N, J; Q, r)$ satisfy equation (12):

$$S(N, J; Q, r) = \Sigma C(N, J; h) = [(J^N)/(J+1)] + K. \tag{12}$$

The value of (K) in equation (12) is either one or zero, as determined in table (13) below.

Table for K Values (13)

Column Type	Condition for Column Type	N = Odd	N = Even
Unique	$(N + r) = 0 \pmod{(J+1)}$	$K = 0$	$K = 1$
Common	$(N + r) \neq 0 \pmod{(J+1)}$	$K = 1$	$K = 0$

In these $Q = (J+1)$ cases, one column is always unique. All the other columns will have identical sums that differ from the unique column by one. In the $(N, J) = (3, 3)$ example, where (N) is odd, the table for K values (13) indicates that the *unique* column will have $(K = 0)$. Thus, from equation (12) and table (9), the unique column sum $S(N, J; Q, r)$ equals (6). At $N = 3$, the only column location (r) within $0 \leq r \leq 3$ that satisfies $(N + r) = 0 \pmod 4$ is at $(r = 1)$, per the Condition in table (13). All other columns have $(K = 1)$ and *common* column sums $S(N, J; Q, r)$ equaling (7).

So in general, for any interval spacing $Q = \{J+1, J, \text{ or } J-1\}$, each columnal series with terms at $h = (AQ + r)$ is summed over integers $0 \leq A \leq [N(J-1)/(Q)]$. For the partition of the $C(N, J; h)$ sequence with these spacings (Q), each column (r) in the range $0 \leq r \leq (Q-1)$ will yield a series result for $S(N, J; Q, r)$ given by equation (14):

$$S(N, J; Q, r) = [(J^N)/(Q)] + b, \text{ where } b = \{0 \text{ or } 1\}. \tag{14}$$

4. SUMMATIONS WITH QUADRUPLET CYCLES

The Method of Ramus described in [1] and [9] uses the roots of unity, with its real and imaginary parts, to partition the terms in the Nth row of the Pascal and the Pascal-de Moivre triangles. The terms of the $C(N, J; h)$ sequence are likewise segmented here by using the second and fourth roots of unity. This segmentation creates four equations of series (A through D) in table (15), whose coefficients (F_r) repeat for every fourth term of the $C(N, J; h)$ sequence.

These repeating coefficients (F_r) in table (15) are the same as the coefficients (F_h) of the $C(N, J; h)$ terms from equation (3). For any series in (16), each (F_h) value equals the (F_r) entry in table (15) when $\{h = r \pmod 4\}$. The sum equation of series (A), for example, is expanded in (17).

Coefficient Table for Series $\{A, B, C, D\}$ and Quadruplet Partitions (P_r) (15)

Coefficients (F_r) at $r =$	0	1	2	3
Series A	+1	0	+1	0
Series B	0	+1	0	+1
Series C	+1	0	−1	0
Series D	0	+1	0	−1
Quadruplet $(P_r) =$	P_0	P_1	P_2	P_3

$$\text{Series } (A, B, C, D) = \Sigma\{(F_h)C(N, J; h)\} \text{ for } 0 \le h \le N(J-1). \tag{16}$$

$$\text{Series } (A) = \Sigma\{C(N, J; h = 4t) + C(N, J; h = 4t+2)\} \text{ for } 0 \le t \le N(J-1)/4. \tag{17}$$

The creation of a quadruplet cycle from table (15) using the series $\{A, B, C, D\}$ equations requires the identification of the relationships between the series equations and the quadruplet $\{P_0, P_1, P_2, P_3\}$ equations. In the nomenclature of the $S(N, J; Q, r)$ partition sums from [1], the quadruplet $\{P_r\}$ equations will be defined as

$$P_r = S(N, J; 4, r) \text{ since } Q = 4 \text{ and for integers } (r) \text{ within } 0 \le r \le 3.$$

The $C(N, J; h)$ sequences thus created, whose sums are (P_r), will have spacings between the nonzero terms of $(\Delta h = 4)$, compared with the nonzero term spacing of $(\Delta h = 2)$ for those of the series $\{A, B, C, D\}$. The corresponding transformation of the equations $\{A, B, C, D\}$ into the equations $\{P_r\}$ is given for each quadruplet location (r) from table (15) by:

$$\begin{aligned} P_0 &= (A+C)/2, & P_1 &= (B+D)/2, \\ P_2 &= (A-C)/2, & P_3 &= (B-D)/2. \end{aligned} \tag{18}$$

Now, in order to state the quadruplet equations, the actual formulas for the segmentation equations $\{A, B, C, D\}$ must be obtained. The first two equations $\{A$ and $B\}$ are the equations for the sum of alternate terms for $C(N, J; h)$ in the Nth row of the Pascal-de Moivre triangle. Here we state, from empirical analysis, that the series (A) starting at $(h = 0)$ and the series (B) starting at $(h = 1)$ have series summation formulas given by equations (19) and (20), where $(N \ge 1)$ and $(b) = J \pmod 2$:

$$A = ((J^N) + b)/2 \tag{19}$$

and

$$B = ((J^N) - b)/2, \tag{20}$$

where $b = 0$ for $(J = \text{even})$, or $b = 1$ for $(J = \text{odd})$.

The two segmentation series $\{C$ and $D\}$ have equations that are obtained from the tables (22) and (23), respectively, with values of $(\pm 1, 0, \text{or} \pm S)$, where S is defined by equation (21):

$$S = ((-1)^{[N/4]})(2^{[N/2]}), \tag{21}$$

where the bracketed expressions in the exponents are least integer functions.

Table for Series (C) (22)

Table for Series (C)	$J = 0$ (mod 4)	$J = 1$ (mod 4)	$J = 2$ (mod 4)	$J = 3$ (mod 4)
All $N \geq 1$	0	1		
$N = 0$ (mod 4)			S	1
$N = 1$ (mod 4)			S	0
$N = 2$ (mod 4)			0	–1
$N = 3$ (mod 4)			$-S$	0

Table for Series (D) (23)

Table for Series (C)	$J = 0$ (mod 4)	$J = 1$ (mod 4)	$J = 2$ (mod 4)	$J = 3$ (mod 4)
All $N \geq 1$	0	0		
$N = 0$ (mod 4)			0	0
$N = 1$ (mod 4)			S	1
$N = 2$ (mod 4)			S	0
$N = 3$ (mod 4)			S	–1

As an example of quadruplet analysis, the $C(N, J; h)$ sequence from table (2) with $(N, J) = (4, 3)$ is listed below in the tabular form of Section 3 with $(Q = 4)$. The values $\{A, B, C, D\}$ are calculated from equations (19), (20), and (21) and tables (22) and (23). Then the $\{P_r\}$ values, obtained from equations (18), give confirmation of their equality with the corresponding column sums.

Column Sums of the $C(N, J; h)$ Sequence

$r =$	0	1	2	3
	1	4	10	16
	19	16	10	4
	1			
Sum =	21	20	20	20

Series Equations

$$A = (3^4 + 1)/2 = 41$$
$$B = (3^4 - 1)/2 = 40$$
$$C = 1$$
$$D = 0$$

Quadruplet $\{P_r\}$ Equations

$$P_0 = (A + C)/2 = (41 + 1)/2 = 21$$
$$P_1 = (B + D)/2 = (40 + 0)/2 = 20$$
$$P_2 = (A - C)/2 = (41 - 1)/2 = 20$$
$$P_3 = (B - D)/2 = (40 - 0)/2 = 20$$

So, as just illustrated, the four quadruplet series $\{P_r\}$ may be built from combinations of the series $\{A, B, C, D\}$, according to equations (18). Also any desired arrangement of the four $\{P_r\}$ equations can be combined further with any choice of coefficients (G_r) represented as num-bers, variables, or functions, as shown:

$$\text{Quadruplet Arrangement} = \Sigma\{(G_r)\,(P_r)\} \text{ for } (r) \text{ within } 0 \leq r \leq 3.$$

5. SUMMATIONS BASED ON MOMENTS

The definition of moments about a point for the $C(N, J; h)$ distribution was introduced by equation (4) in Section 1. The two moments considered here are the moment (ν) taken about the origin and the moment (μ) taken about the mean (M). The connection between R^{th} moment calculations and the summations of $C(N, J; h)$ terms is indicated by their respective evaluation formulas from equation (5), when summed over $0 \le h \le N(J-1)$.

$$\Sigma\{(h)^R C(N, J; h)\} = (\nu_R)(J^N), \tag{24}$$

$$\Sigma\{(h-M)^R C(N, J; h)\} = (\mu_R)(J^N). \tag{25}$$

The mean (M) is the midpoint of the range of (h) values and thus equals $N(J-1)/2$, which may include half integers when N and $(J-1)$ are both odd integers.

The sum of all $C(N, J; h)$ row terms in a Pascal-de Moivre triangle equals (J^N), which was substituted for $\Sigma C(N, J; h)$ in the typical definition of moment equations [3]. Multiplied by (J^N), the moment equations (ν_R) and (μ_R) give the $C(N, J; h)$ moment summations (24) and (25).

To find the moment equations for (ν_R) and (μ_R), we derive their exponential generating functions: (μ_{egf}) and (ν_{egf}). And then, by expansion of the exponential generating functions, the coefficient of the term $((t^R)/R!)$ is the equation for the Rth moments for (μ_R) and (ν_R), in the summation over integers $0 \le i < \infty$, as outlined in [3]:

$$\mu_{\text{egf}} = \Sigma(\mu_i)(t^i)/(i!),$$
$$\nu_{\text{egf}} = \Sigma(\nu_i)(t^i)/(i!).$$

The exponential generating function for (μ_{egf}) turns out to be the exponential power series shown in (26), which is summed over integers $2 \le r < \infty$. The notation $\text{Exp}\{x\}$ is defined as (e^x).

$$\mu_{\text{egf}} = \text{Exp}\{N\,\Sigma((-1)^r(S_r)(t^r)/(r!))\}, \tag{26}$$

where $S_r = ((J^r)-1)(B_r)/r$ with B_r = the rth Bernoulli number.

Each Bernoulli number (B_r) can be derived as the coefficient of $((t^r)/r!)$ in the exponential generating function (B_{egf}) from the summation in (27) for $0 \le i < \infty$. From reference [10], the sequence of Bernoulli numbers (B_r) for $(r \ge 1)$ is $\{(-1/2), (1/6), 0, (-1/30), \ldots\}$.

$$B_{\text{egf}} = \Sigma(B_i)(t^i)/(i!) = t/\{(e^t)-1\}. \tag{27}$$

The accuracy of the (μ_{egf}) formula in (26) has been confirmed empirically by comparing the distribution results on the left-hand side of equation (25) with the moment equation results of the right-hand side of equation (25). These two approaches gave identical results through $R \le 32$, which represented the limit at which the author's computer hardware capability could complete the calculations in a reasonable time.

A couple of cases will illustrate the creation of the initial (μ_R) equations, in terms of their distribution variables (N and J). Also, a specific example with $(N, J) = (2, 3)$ can demonstrate the numerical equality between the left-hand and right-hand sides of equation (25).

For the case in which $R = 2$, the coefficient of $(t^2)/(2!)$ in the expansion of the (μ_{egf}) formula in (26) gives the variance (μ_2) of the $C(N, J; h)$ distribution:

$$\begin{aligned}&\mu_2 = N(S_2) = N(J^2-1)B_2/2, \quad \text{where } B_2 = (1/6),\\ &\mu_2 = N(J^2-1)/12.\end{aligned} \tag{28}$$

When multiplied by (J^N), this second moment (μ_2) formula (28) yields the $C(N, J; h)$ moment summation formula from (25).

$$\Sigma\{(h-M)^2 C(N, J; h)\} = \mu_2(J^N) = N(J^2-1)(J^N)/12. \tag{29}$$

The result of (29) generalizes the fact that the second moment summation of terms in the Nth row of Pascal's (classical) triangle equals the value $N(2^{(N-2)})$, as shown in (30) for $(J=2)$:

$$\Sigma\{(h-M)^2 C(N, 2; h)\} = \mu_2(2^N) = N(2^2-1)(2^N)/12 = N(2^{(N-2)}). \tag{30}$$

For symmetrical distributions like $C(N, J; h)$, the moment (μ_R) about the mean (M) will be zero whenever R is an odd integer. Thus, the coefficients of $(t^{2i+1})/((2i+1)!)$ in the expansion of (μ_{egf}) must be zero, for all integers $(i \geq 1)$. All of these coefficients (μ_{2i+1}) contain a factor with a Bernoulli number of odd index, which is zero per [10]. So the moments (μ_R) are zero when R is an odd integer, because the corresponding Bernoulli numbers are zero.

For the case in which $R=4$, the coefficient of $(t^4)/(4!)$ in the expansion of formula (26) for (μ_{egf}) gives the 4th moment (μ_4):

$$\begin{aligned}&\mu_4 = \{3N^2(S_2)^2\} + \{N(S_4)\},\\ &\mu_4 = \{3N^2((J^2-1)B_2/2)^2\} + \{N((J^4-1)B_4/4)\}, \quad \text{where } B_4 = (-1/30),\\ &\mu_4 = \{N^2((J^2-1)^2)/48\} - \{N(J^4-1)/120\}.\end{aligned} \tag{31}$$

The specific example of (μ_4) with $(N, J) = (2, 3)$ will be left to the reader to confirm that the formulas in equation (25) satisfy $\Sigma\{(h-2)^4 C(2, 3; h)\} = (\mu_4)(J^N) = (4)(3^2) = 36$.

Next, the exponential generating function (ν_{egf}) for the moment about the origin is shown to have the same formula as (μ_{egf}) in (26), except that the summation index (r) begins here at $(r=1)$ instead of at $(r=2)$.

$$\nu_{\text{egf}} = \text{Exp}\{N\Sigma((-1)^r(S_r)(t^r)/(r!))\}, \tag{32}$$

where $S_r = (((J^r)-1)(B_r)/r)$, with $B_r =$ the rth Bernoulli number.

By expanding the exponential generating function (ν_{egf}), the moment equation for (ν_R) can be obtained as the coefficient of the $((t^R)/R!)$ term. Again formula (32) was confirmed empirically to $R=32$. Select cases for $R=(1, 2,$ and $4)$ will show the moment equations in terms of the distribution variables (N) and (J). A numerical example of (ν_4) at $(N, J) = (2, 3)$ can illustrate the equality between calculations of the left-hand and right-hand sides of equation (24).

The first moment (ν_1) gives the value of the mean (M) of the $C(N, J; h)$ distribution. The mean (M) is the midpoint of (h) over the range $0 \leq h \leq N(J-1)$, which must equal $N(J-1)/2$. And this value is identical to the moment derivation from (32) for the first moment (ν_1):

$$\begin{aligned}&\nu_1 = -N(S_1) = -N(J^1-1)B_1/1 \quad \text{where } B_1 = (-1/2),\\ &\nu_1 = N(J-1)/2 = M.\end{aligned} \tag{33}$$

The result for (ν_1) generalizes the known binomial formula in [10] for the summation of $\Sigma\{(h)^1 C(N, J; h)\} = N(2)^{(N-1)}$, since at $(J=2)$ this also equals $(\nu_1)(J^N) = (N/2)(2^N) = N(2)^{(N-1)}$.

Also, from the (ν_{egf}) expansion at $(R=2)$, the second moment formula for (ν_2) becomes

$$v_2 = N^2((J-1)^2/4) + N((J^2-1)/(12)). \tag{34}$$

The binomial case for $\Sigma\{(h)^2C(N, J; h)\}$ in (24) is given at $(J=2)$ by taking (v_2) times (2^N), where $(v_2) = (1/4)N^2 + (1/4)N$. So $\Sigma\{(h)^2C(N, 2; h)\} = (v_2)(2^N) = N(N+1)(2^{(N-2)})$.

The fourth moment (v_4) is obtained from the coefficient of $((t^4)/4!)$ in the expansion of (v_{egf}) in formula (32), with the Bernoulli numbers for $(r \geq 1)$ of $\{(-1/2), (1/6), 0, (-1/30), \ldots\}$:

$$v_4 = N^4((J-1)^4/16) + N^3((J-1)^2(J^2-1)/8) + N^2((J^2-1)^2/48) - N((J^4-1)/120). \tag{35}$$

Calculation of (v_4) for a $C(N, J; h)$ distribution with $(N, J) = (2, 3)$ will be left to the reader to confirm that the formulas in equation (24) satisfy $\Sigma\{(h)^4C(2, 3; h)\} = (v_4)(J^N) = (52)(3^2) = 468$.

The mean (v_1) was shown in (33) to be equal to $(N(J-1)/2)$, and the variance (μ_2) was given in (28) as $(N(J^2-1)/12)$. Now, the two exponential generating functions from (26) and (32) can be redefined in terms of the mean and variance. To make the adjustment, the summation factor (N) is multiplied by $((J^2-1)/12)$ to get the variance (μ_2). To balance this multiplication, the terms (S_r) are then divided by the same $((J^2-1)/12)$ factor, thus creating a new summation term (T_r).

Applying this transformation to formulas (26) and (32), alternative definitions of the exponential generating functions for (μ_{egf}) and (v_{egf}) become the exponential power series in (36) and (37), with sums of integer index (r) over $2 \leq r < \infty$ for (μ_{egf}) and over $1 \leq r < \infty$ for (v_{egf}):

$$\mu_{\text{egf}} = \text{Exp}\{(\mu_2)\Sigma((-1)^r(T_r)(t^r)/(r!))\}, \tag{36}$$

$$v_{\text{egf}} = \text{Exp}\{(\mu_2)\Sigma((-1)^r(T_r)(t^r)/(r!))\}, \tag{37}$$

where $T_r = (12)S_r/(J^2-1) = \{(12)B_r/r\}\{((J^r)-1)/(J^2-1)\}$ with B_r = the rth Bernoulli number.

The special characteristics of the (T_r) sequence depend on the Bernoulli numbers (B_r), by definition. Like the (S_r) sequence, the values of (T_{2i+1}) are zero since (B_{2i+1}) are zero for integers $(i \geq 1)$. Additionally, the value of (T_2) is always equal to one for all (J).

$$T_2 = \{(12)B_2/2\}\{((J^2)-1)/(J^2-1)\} = \{(12)B_2/2\}\{1\} = 1, \text{ since } B_2 = 1/6.$$

With equations (36) and (37), various moment equations such as $(\mu_4$ and $v_4)$ can be derived:

$$\mu_4 = 3(\mu_2)^2(T_2)^2 + (\mu_2)(T_4), \tag{38}$$

$$v_4 = (\mu_2)^4(-T_1)^4 + 6(\mu_2)^3(-T_1)^2(T_2) + 3(\mu_2)^2(T_2)^2 + (\mu_2)(T_4). \tag{39}$$

Since $(\mu_2) = (N(J^2-1)/12)$ from equation (28), specific distributions can be evaluated exactly by knowing the values of $(N$ and $J)$. In the fourth moment example for a $C(N, J; h)$ distribution with $(N, J) = (2, 3)$, the (T_r) sequence for $r \geq 1$ begins with $\{(-3/2), 1, 0, (-1), \ldots\}$ and $(\mu_2) = 4/3$. The reader is invited to confirm that the fourth moments are again $(\mu_4 = 4)$ and $(v_4 = 52)$. Therefore, either interpretation of exponential generating functions, with S_r or T_r, gives the correct moment value.

The general rule for the leading term in the moment equation (μ_R) becomes apparent by observation of equation (36). Because the odd-indexed Bernoulli numbers (B_r) are zero for $r > 1$, and because $(T_2) = 1$, the first term in the (μ_R) moment equation will be a double factorial $(R-1)!!$ times (μ_2) to the power $(R/2)$. In the sample equation (38) for (μ_4), this first term was $3(\mu_2)^2$.

Since the summation in the exponential generating function for (v_{egf}) of (37) begins at index $(r=1)$, the leading term in the moment equation (v_R) is always $(-T_1)^R$ times $(\mu_2)^R$. In example (39) for (v_4), this first term was $(-T_1)^4(\mu_2)^4$, or simply $(-T_1\mu_2)^4$. But, by equation (40), the product term $(-T_1\mu_2)$ is just equal to $\{N(J-1)/2\}$, which is the value of the mean (v_1), as seen in equation (33). Therefore, the leading term in the moment equation (v_R) is always $(v_1)^R$.

$$(-T_1\mu_2) = \{6/(J+1)\}\{\mu_2\} = \{6/(J+1)\}\{N(J^2-1)/12\} = \{N(J-1)/2\} = (v_1), \tag{40}$$

since $T_1 = \{(12)B_1/1\}\{((J^1)-1)/(J^2-2)\} = \{-6/(J+1)\}$ with $B_1 = (-1/2)$.

Also, the term $(-T_1\mu_2)$ is the coefficient of $((t^r)/(r!))$ at $r=1$ in the exponential generating function (v_{egf}) from (37). This first term of the summation can now be rewritten by the equality $(-T_1\mu_2)(t^1)/(1!) = (v_1 t)$. If this term is extracted from the summation in (37), the remaining non-zero terms in the summation have even indices in (r), since all of the odd indexed terms have a factor that is zero; i.e., the odd indexed Bernoulli numbers B_{2i+1} for $i \geq 1$. Using this information, both exponential generating functions (μ_{egf}) and (v_{egf}) from (36) and (37) can be rewritten with the zero-valued summation terms excluded. The exponential generating functions are now both summed for the redefined index (r) over all integers in the range $1 \geq r > \infty$:

$$\mu_{\text{egf}} = \text{Exp}\{(\mu_2)\Sigma((T_{2r})(t^{2r})/((2r)!))\}, \tag{41}$$

$$v_{\text{egf}} = \text{Exp}\{(v_1 t) + (\mu_2)\Sigma((T_{2r})(t^{2r})/((2r)!))\}, \tag{42}$$

where $T_{2r} = (12)S_{2r}/(J^2-1) = \{(12)B_{2r}/(2r)\}\{((J^{2r})-1)/(J^2-1)\}$ with B_{2r} = the $(2r)^{\text{th}}$ Bernoulli number.

A comparison between moment generating functions of the discrete $C(N, J; h)$ distribution and a normal distribution in the continuous case is useful in this format. The exponential generating functions for the continuous and normal distribution are given in [3] as:

$$\mu_{\text{egf}} \text{ (continuous \& normal)} = \text{Exp}\{\mu_2(t^2)/2\} \text{ and}$$

$$v_{\text{egf}} \text{ (continuous \& normal)} = \text{Exp}\{(v_1 t) + \mu_2(t^2)/2\}.$$

Since (T_2) was shown to be equal to one, the continuous & normal (C&N) distribution and the discrete $C(N, J; h)$ distribution have moment generating functions that have identical initial summation terms. The relationships between the moment generating functions of both of these distributions are summarized in equations (43) and (44), with the second factor $\text{Exp}\{\Sigma\}$ being an exponential summation for integers $2 \leq r < \infty$:

$$\mu_{\text{egf}(C(N,J;h))} = \mu_{\text{egf(C\&N)}}\text{Exp}\{\Sigma((\text{T}_{2\text{r}})(t^{2r})/((2r)!))\} \tag{43}$$

and

$$v_{\text{egf}(C(N,J;h))} = v_{\text{egf(C\&N)}}\text{Exp}\{\Sigma((\text{T}_{2\text{r}})(t^{2r})/((2r)!))\}, \tag{44}$$

where $T_{2r} = \{(12)B_{2r}/(2r)\}\{((J^{2r})-1)/(J^2-1)\}$.

Besides the methodology of creating moment equations from exponential generating functions, another technique was originally used that involved recursion equations. The main recursion equation developed related the Rth moment (μ_R) about the mean (v_1) to the previously generated moments $v_{(R-h)}$ about the origin. The summation was taken over all integers of (h) in the range $0 \leq h \leq R$:

$$\mu_R = \Sigma(-1)^h \binom{R}{h} (v_1)^h (v_{(R-h)}). \tag{45}$$

The recursion equation (45) originated from observation of moment equations (μ_R) from [3] for $R = 0$ through 4, including the oft quoted $(\mu_2) = (v_2) - (v_1)^2$. This latter formula is apparent from equation (34) in the form $(v_2) = (v_1)^2 + (\mu_2)$.

When R is an odd integer, the moment (μ_R) is zero. Therefore, the recursion equation can also relate successive moments (v_R) to previous moments $(v_{(R-h)})$. And now the summation is taken over all integers of (h) in the range $1 \le h \le R$. Thus, after defining $(v_0) = 1$, the recursion equation for (v_R) at $(R = \text{odd})$ becomes

$$v_R = \Sigma(-1)^{(h+1)} \binom{R}{h} (v_1)^h (v_{(R-h)}). \tag{46}$$

6. SUMMATIONS OF ALTERNATING SIGNED TERMS

The alternating signed version of the moment equations yields the following summation formulas for the $C(N, J; h)$ terms, when summed over integers $0 \le h \le N(J-1)$:

$$\Sigma\{(-1)^h (h)^R C(N, J; h)\} = v_{Ra}, \tag{47}$$

$$\Sigma\{(-1)^h (h - M)^R C(N, J; h)\} = \mu_{Ra}, \tag{48}$$

where $M = (v_1) = N(J-1)/2$.

Two separate presentations on the moments for the alternating signed $C(N, J; h)$ terms will be given for cases when (J) is an odd integer and when (J) is an even integer. Thus, two different notations will be assigned for each case:

1. $(v_{Ra}) = (v_{R,d})$ and $(\mu_{Ra}) = (\mu_{R,d})$, when $J = \text{odd}$;
2. $(v_{Ra}) = (v_{R,e})$ and $(\mu_{Ra}) = (\mu_{R,e})$, when $J = \text{even}$.

Compared to equations (24) and (25) of the previous moment section, the most apparent difference in the summation equations of (47) and (48) is that they do not have the common factor (J^N).

In the initial definition of the moment equations (5), the common factor to be multiplied by (v_{Ra}) and (μ_{Ra}) was the sum of all the distribution terms in the Nth row of the Pascal-de Moivre triangle. From the discussion on quadruplet cycles, the difference of the two alternate term equations (A minus B) from equations (19) and (20) will give the sum of the alternating signed $C(N, J; h)$ distribution. For all cases when (J) is odd, it has a value of one:

$$\Sigma\{(-1)^h C(N, J; h)\} = 1 \text{ for } J = \text{odd}. \tag{49}$$

When (J) is an odd integer, the moment equations for $(v_{R,d})$ and $(\mu_{R,d})$ are derived directly from their exponential generating functions: $(\mu_{\text{egf},d})$ and $(v_{\text{egf},d})$. The exponential generating functions are expanded to obtain $(\mu_{R,d})$ and $(v_{R,d})$ as coefficients of the term $((t^R)/R!)$ in the summation over integers $0 \le i < \infty$, as outlined in [3]:

$$\mu_{\text{egf},d} = \Sigma(\mu_{i,d})(t^i)/(i!);$$
$$v_{\text{egf},d} = \Sigma(v_{i,d})(t^i)/(i!).$$

In this ($J =$ odd) case, the exponential generating functions for $(\mu_{\text{egf},d})$ and $(\nu_{\text{egf},d})$ turn out to have identical formulas, as expressed by the exponential power series in (50) and (51). Just the ranges of their summation index (r) differ, with $2 \le r < \infty$ for $(\mu_{\text{egf},d})$ and $1 \le r < \infty$ for $(\nu_{\text{egf},d})$:

$$\mu_{\text{egf},d} = \text{Exp}\{N\Sigma((-1)^r(S_{r,d})(t^r)/(r!))\}, \tag{50}$$

$$\nu_{\text{egf},d} = \text{Exp}\{N\Sigma((-1)^r(S_{r,d})(t^r)/(r!))\}, \tag{51}$$

where $S_{r,d} = ((2^r)-1)S_r = ((2^r)-1)((J^r)-1)(B_r)/r$ with $B_r =$ the rth Bernoulli number.

These exponential generating functions $(\mu_{\text{egf},d})$ and $(\nu_{\text{egf},d})$ for these alternating signed distributions are closely related to the exponential generating functions (μ_{egf}) and (ν_{egf}) of (26) and (32) for the positively signed distributions. Only the summation term $(S_{r,d})$ in (50) and (51) differs from the summation term (S_r) in (26) and (32) by a factor of $((2^r)-1)$.

The same transformation methods described in (36), (37), (41), and (42) will change the summation variables (N) and $(S_{r,d})$ in equations (50) and (51) into the variables (μ_2) and $(T_{r,d})$ for equations (52) and (53). These new exponential generating functions are both summed over all integers in the range $1 \le r < \infty$. Note: $(\nu_{1,d}) = \nu_1$ and $(\mu_{2,d}) = 3\mu_2$.

$$\mu_{\text{egf},d} = \text{Exp}\{(\mu_2)\Sigma((T_{2r,d})(t^{2r})/((2r)!))\}, \tag{52}$$

$$\nu_{\text{egf},d} = \text{Exp}\{(\nu_1 t + (\mu_2)\Sigma((T_{2r,d})(t^{2r})/((2r)!))\}, \tag{53}$$

where $T_{2r,d} = (12)S_{2r,d}/(J^2-1) = \{(2^{2r})-1\}\{(12)B_{2r}/(2r)\}\{((J^{2r})-1)/(J^2-1)\}$ with $B_{2r} =$ the $(2r)^{\text{th}}$ Bernoulli number.

The fourth moments $(\mu_{4,d})$ and $(\nu_{4,d})$, for example, can be obtained from the coefficient of $((t^4)/4!)$ in the expansions of $(\mu_{\text{egf},d})$ and $(\nu_{\text{egf},d})$ in formulas (52) and (53). The format of these formulas is the same as for (μ_4) and (ν_4) in (38) and (39), where the definition of each summation term $(T_{r,d})$ has only changed from (T_r) by a factor $((2^r)-1)$.

$$\mu_{4,d} = 3(\mu_2)^2(T_{2,d})^2 + (\mu_2)(T_{4,d}), \tag{54}$$

$$\nu_{4,d} = (\mu_2)^4(-T_{1,d})^4 + 6(\mu_2)^3(-T_{1,d})^2(T_{2,d}) + 3(\mu_2)^2(T_{2,d})^2 + (\mu_2)(T_{4,d}). \tag{55}$$

A specific fourth moment example for $(\mu_{4,d})$ and $(\nu_{4,d})$ of a distribution with $(N, J) = (2, 3)$ can be calculated by inserting the proper values for (μ_2) and $(T_{r,d})$ in (54) and (55). Here, the $(T_{r,d})$ sequence for $(r \ge 1)$ begins at $\{(-3/2), 3, 0, (-15), \ldots\}$, and $(\mu_2) = (N(J^2-1)/12) = (4/3)$. Thus, for $(N, J) = (2, 3)$, the reader can find that equations (54) and (55) agree with the moment values in the formulas from equations (47) and (48) to give $(\mu_{4,d}) = 28$ and $(\nu_{4,d}) = 140$.

Now for the other type of distribution, with (J) as an even integer. In this case, the sum of the alternating signed $C(N, J; h)$ distribution is zero for $N \ge 1$, as in the binomial $(J = 2)$ case from [10]. The left-hand side of equations (47) and (48) does exist. The interpretation of the terms $(\mu_{R,e})$ and $(\nu_{R,e})$ on the right-hand side may not be clear, since these moments and the distribution density $f(h)$ from equation (4) have a denominator of zero; thus, they may be undefined. For convenience, these terms $(\mu_{R,e})$ and $(\nu_{R,e})$ will still be called moments, in the sense that they are being used to generate the summation formulas in (47) and (48).

The simplest results for the Rth moment equations $(\mu_{R,e})$ and $(\nu_{R,e})$ for the ($J =$ even) cases occur when (N) is greater than (R). Then we do get moment equations of zero:

$$\mu_{R,e} = 0 \quad \text{and} \quad \nu_{R,e} = 0 \quad \text{for } N > R. \tag{56}$$

However, for $(R \geq N)$, the moment values are predominantly nonzero, when (J) is an even integer. The moment equations for $(\nu_{R,e})$ and $(\mu_{R,e})$ can be derived if they are broken down into two parts: a common factor (CF) and an equation (m_k) from the corresponding exponential generating functions for (μ) and (ν). The moment equations are:

$$\mu_{R,e} = (\text{CF})(m_k(\mu)), \tag{57}$$

and

$$\nu_{R,e} = (\text{CF})(m_k(\nu)), \tag{58}$$

where $(\text{CF}) = \{(-1)^N (J/2)^N (N!)\}\binom{R}{N}$.

The equation for (CF) was obtained by empirical analysis. The equations (m_k) are the coefficients of the term $((t^k)/k!)$ in the expansion of the exponential generating functions, where the value of (k) is defined as $(R-N)$. The summations in the exponential generating functions for the (m_k) coefficients are taken over all integers $r \geq a$, with $a = 2$ for $(\mu_{R,e})$ and at $a = 1$ for $(\nu_{R,e})$.

$$\mu_{\text{egf},e} = \text{Exp}\{N\,\Sigma((-1)^r (S_{r,e})(t^r)/(r!))\} \quad \text{for } 2 \leq r < \infty, \tag{59}$$

$$\nu_{\text{egf},e} = \text{Exp}\{N\,\Sigma((-1)^r (S_{r,e})(t^r)/(r!))\} \quad \text{for } 1 \leq r < \infty, \tag{60}$$

where $S_{r,e} = \{(J^r) - (2^r) + 1\}\{(B_r)/r\}$ with $B_r =$ the rth Bernoulli number.

Again a useful transformation takes summation variables (N) and $(S_{r,e})$ in (59) and (60) into the variables (μ_2) and $(T_{r,e})$ for equations (61) and (62). Both of these new exponential generating functions have summations taken for all integers $1 \geq r > \infty$. Note: $(\nu_{1,e}) = \nu_1$.

$$\mu_{\text{egf},e} = \text{Exp}\{(\mu_2)\Sigma((T_{2r,e})(t^{2r})/((2r)!))\}, \tag{61}$$

$$\nu_{\text{egf},e} = \text{Exp}\{(\nu_1 t) + (\mu_2)\Sigma((T_{2r,e})(t^{2r})/((2r)!))\}, \tag{62}$$

where $T_{2r,e} = (12)S_{2r,e}/(J^2-1) = \{(J^{2r}) - (2^{2r}) + 1\}\{(12)/(J^2-1)\}\{(B_{2r})/(2r)\}$.

The reader is challenged to find the $(\mu_{R,e})$ moment at $(R = 4)$ for the alternating signed $C(N, J; h)$ distribution with $(N, J) = (2, 2)$. In this example, the moment equation (57) has an index of $(k = R - N = 2)$, with factors (CF) = 12 and $m_k(\mu) = 1/6$, which give a value for $(\mu_{R,e})$ equal to 2.

A Final Note: Conditions involving distribution symmetry and the patterns of Bernoulli numbers in the exponential generating functions favor the proposition that all of the moment equations and their generating functions will remain valid for all possible $C(N, J; h)$ moment summations with distribution variables (N, J) and moment numbers (R) over all positive integers. Extensive empirical evidence suggests optimism.

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THE BRAHMAGUPTA POLYNOMIALS IN TWO COMPLEX VARIABLES*

In Commemoration of Brahmagupta's Fourteenth Centenary

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1. INTRODUCTION

Some of the properties of the Brahmagupta matrix [see (1) below], and polynomials x_n and y_n in two real variables (x, y) (see § 3) have been studied in [6]; we know that the Brahmagupta polynomials contain the Fibonacci polynomials, the Pell and Pell-Lucas polynomials [2], [5], and the Morgan-Voyce polynomials [4], [7]. The convolution properties that hold for the Fibonacci polynomials and for the Pell and Pell-Lucas polynomials also hold for Brahmagupta polynomials.

In this paper we extend analytically the properties of the Brahmagupta matrix and polynomials derived in [6] from two real variables to two complex variables z and w, which belong to two distinct complex planes. We denote this space by $\mathbf{C}^2$. A typical member in $\mathbf{C}^2$ has the form $\zeta = (z, w)$. Since $\mathbf{C}$ is simply $\mathbf{R}^2$ with the additional algebraic structure, we realize that $\mathbf{C}^2$ is (topologically) $\mathbf{R}^4$ with some additional algebraic properties. We have a natural way to identify points in $\mathbf{C}^2$ with points in $\mathbf{R}^4$. This is described by the scheme:

$$\mathbf{C}^2 \ni (z, w) \leftrightarrow (x+iy, u+iv) \leftrightarrow (x, y, u, v) \in \mathbf{R}^4.$$

In particular, we measure the distance in $\mathbf{C}^2$ in the customary Euclidean fashion: if $\zeta_1 = (z_1, w_1)$ and $\zeta_2 = (z_2, w_2)$ are points in $\mathbf{C}^2$, then $|\zeta_1 - \zeta_2| = (|z_1 - z_2|^2 + |w_1 - w_2|^2)^{1/2}$.

Another interesting feature of the Brahmagupta polynomials z_n and w_n in $\mathbf{C}^2$ is that, when the polynomials are expressed in terms of real and imaginary parts with $z = x+iy$ and $w = u+iv$, the resulting polynomials x_n, y_n, u_n, v_n satisfy recurrence relations (11)-(18). The functions x_n, y_n, u_n, v_n are solutions of the second-order partial differential equations (19) and (20).

Since the calculations go through without change in the complex case, we just list some of the properties.

2. BRAHMAGUPTA MATRIX

Let B be a matrix (a Brahmagupta matrix) of the form

$$B = \begin{bmatrix} z & w \\ tw & z \end{bmatrix}, \tag{1}$$

where t is the fixed real number and z and w are complex variables; further, we shall assume that $\det B = \beta = z^2 - tw^2 \neq 0$. Using its eigenrelations, B has the following diagonal form:

$$\begin{bmatrix} z & w \\ tw & z \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{t}{2}} & -\sqrt{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} z+w\sqrt{t} & 0 \\ 0 & z-w\sqrt{t} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2t}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2t}} \end{bmatrix}.$$

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Define

$$B^n = \begin{bmatrix} z & w \\ tw & z \end{bmatrix}^n = \begin{bmatrix} z_n & w_n \\ tw_n & z_n \end{bmatrix} = B_n.$$

Then the above diagonalization enables us to compute

$$B^n = \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{t}{2}} & -\sqrt{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} z+w\sqrt{t} & 0 \\ 0 & z-w\sqrt{t} \end{bmatrix}^n \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2t}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2t}} \end{bmatrix}. \tag{2}$$

Since $B^{n+1} = B^n B$, we have the following recurrence relations:

$$z_{n+1} = zz_n + tww_n, \qquad w_{n+1} = zw_n + wz_n, \tag{3}$$

with $z_n = z$ and $w_n = w$. From (2) we derive the following Binet forms for z_n and w_n:

$$z_n = \frac{1}{2}[(z+w\sqrt{t})^n + (z-w\sqrt{t})^n], \tag{4}$$

$$w_n = \frac{1}{2\sqrt{t}}[(z+w\sqrt{t})^n - (z-w\sqrt{t})^n], \tag{5}$$

and $z_n \pm \sqrt{t}w_n = (z \pm \sqrt{t}w)^n$. Note that if we set $z = 1/2 = w$ and $t = 5$ then $\beta = -1$; then $2w_n = F_n$ is the Fibonacci sequence, while $2z_n = L_n$ is the Lucas sequence, where $n > 0$.

Let $\xi = z + w\sqrt{t}$, $\eta = z - w\sqrt{t}$, $\xi_n = z_n + w_n\sqrt{t}$, $\eta_n = z_n - w_n\sqrt{t}$ and $\beta_n = z_n^2 - tw_n^2$, with $\eta_n = \eta$, $\xi_n = \xi$, and $\beta_n = \beta$. Then we have $\xi_n = \xi^n$, $\eta_n = \eta^n$, and $\beta_n = \beta^n$. To prove the last equality, consider $\beta^n = (z^2 - tw^2)^n = \xi^n\eta^n = \xi_n\eta_n = (z_n^2 - tw_n^2) = \beta_n$.

We also have the following property:

$$e^B = \frac{1}{4}\begin{bmatrix} e^\xi + e^\eta & \frac{1}{\sqrt{t}}(e^\xi - e^\eta) \\ \sqrt{t}(e^\xi - e^\eta) & e^\xi + e^\eta \end{bmatrix}, \quad \det e^B = e^{2z}.$$

To prove these results, set $2z_k = \xi^k + \eta^k$, $2\sqrt{t}w_k = \xi^k - \eta^k$. Since

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!} \quad \text{and} \quad \frac{B^k}{k!} = \frac{1}{k!}\begin{bmatrix} z_k & w_k \\ tw_k & z_k \end{bmatrix},$$

we express z_k and w_k in terms of ξ and η to obtain the desired results.

Recurrence relations (3) also imply that z_n and w_n satisfy the difference equations:

$$z_{n+1} = 2zz_n - \beta z_{n-1}, \qquad w_{n+1} = 2zw_n - \beta w_{n-1}. \tag{6}$$

Conversely, if $z_0 = 1$, $z_1 = z$, and $w_0 = 0$, and $w_1 = w$, then the solutions of the difference equations (6) are given by the Binet forms (4) and (5).

The expressions z_n and w_n can be extended to negative integers by defining $z_{-n} = z_n\beta^{-n}$ and $w_{-n} = -w_n\beta^{-n}$. Then we have

$$B^{-n} = \begin{bmatrix} z & w \\ tw & z \end{bmatrix}^n = \begin{bmatrix} z_{-n} & w_{-n} \\ tw_{-n} & z_{-n} \end{bmatrix} = B_{-n},$$

where we have used the property

$$\left(\begin{bmatrix} z & w \\ tw & z \end{bmatrix}^{-1}\right)^n = \left(\frac{1}{\beta}\begin{bmatrix} z & -w \\ -tw & z \end{bmatrix}\right)^n = \frac{1}{\beta^n}\begin{bmatrix} z_n & -w_n \\ -tw_n & z_n \end{bmatrix}.$$

All of the recurrence relations extend to the negative integers also. Notice that $B^0 = I$, where I is the identity matrix. For negative integers, z_n and w_n are rational functions of z and w.

3. THE BRAHMAGUPTA POLYNOMIALS

Using the Binet forms (4) and (5), we deduce some results: Write z_n and w_n as polynomials in z and w using the binomial expansion:

$$z_n = z^n + t\binom{n}{2}z^{n-2}w^2 + t^2\binom{n}{4}z^{n-4}w^4 + \cdots;$$

$$w_n = nz^{n-1}w + t\binom{n}{3}z^{n-3}w^3 + t^2\binom{n}{5}z^{n-5}w^5 + \cdots.$$

The first few polynomials are $z_0 = 1$, $z_1 = z$, $z_2 = z^2 + tw^2$, $z_3 = z^3 + 3tzw^2$, $z_4 = z^4 + 6tz^2w^2 + t^2w^4$, ..., $w_0 = 0$, $w_1 = w$, $w_2 = 2zw$, $w_3 = 3z^2w + tw^3$, $w_4 = 4z^3w + 4tzw^3, \ldots$. Notice that z_n and w_n are homogeneous in z and w; therefore, they are analytic (in the classical one-variable sense) in each variable separately. Also, z_n and w_n satisfy the Cauchy-Riemann equations in each variable separately: If $z_n = x_n + iy_n$, then

$$\frac{\partial x_n}{\partial x} = \frac{\partial y_n}{\partial y}, \qquad \frac{\partial x_n}{\partial y} = -\frac{\partial y_n}{\partial x}$$

and

$$\frac{\partial x_n}{\partial u} = \frac{\partial y_n}{\partial v}, \qquad \frac{\partial x_n}{\partial v} = -\frac{\partial y_n}{\partial u}.$$

Similar relations are satisfied by the polynomials $w_n = u_n + iv_n$.

If $t > 0$, then z_n and w_n satisfy:

$$\lim_{n\to\infty}\frac{z_n}{w_n} = \begin{cases} +\sqrt{t} & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| < 1, \\ -\sqrt{t} & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| > 1; \end{cases}$$

$$\lim_{n\to\infty}\frac{z_n}{z_{n-1}} = \lim_{n\to\infty}\frac{w_n}{w_{n-1}} = \begin{cases} z+\sqrt{t}w & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| < 1, \\ z-\sqrt{t}w & \text{if } \left|\frac{z-\sqrt{t}w}{z+\sqrt{t}w}\right| > 1; \end{cases}$$

$$\frac{\partial z_n}{\partial z} = \frac{\partial w_n}{\partial w} = nz_{n-1},$$

$$\frac{\partial z_n}{\partial w} = t\frac{\partial w_n}{\partial z} = ntw_{n-1}.$$

From the above relations, we infer that z_n and w_n are the polynomial solutions of the "wave equation":

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{t}\frac{\partial^2}{\partial w^2}\right)U = 0. \tag{7}$$

Since the partial differential equation (7) is linear, by the principle of superposition its general solution is

$$U(z, w) = \sum_{0}^{\infty} (A_n z_n + B_n w_n),$$

where A_n and B_n are constants.

4. RECURRENCE RELATIONS

From the Binet forms (4) and (5), we record the following obvious recurrence relations:

$$\begin{array}{llll}
\text{(i)} & z_{m+n} = z_m z_n + t w_m w_n, & \text{(vi)} & w_{m+n} + \beta^n w_{m-n} = 2 z_n w_m, \\
\text{(ii)} & w_{m+n} = z_m w_n + w_m z_n, & \text{(vii)} & z_{m+n} + \beta^n z_{m-n} = 2 t w_m w_n, \\
\text{(iii)} & \beta^n z_{m-n} = z_m z_n - t w_m w_n, & \text{(viii)} & w_{m+n} + \beta^n w_{m-n} = 2 z_m w_n, \\
\text{(iv)} & \beta^n w_{m-n} = z_n w_m - z_m w_n, & \text{(ix)} & 2(z_m^2 - z_{m+n} z_{m-n}) = \beta^{(m-n)}(\beta^n - z_{2n}), \\
\text{(v)} & z_{m+n} + \beta^n z_{m-n} = 2 z_m z_n, & \text{(x)} & z_{2m} - 2 t w_{m+n} w_{m-n} = \beta^{(m-n)} z_{2n}.
\end{array} \tag{8}$$

Putting $m = n$ in (i) and (ii) above, we see that $z_{2n} = z_n^2 + t w_n^2$ and $w_{2n} = 2 z_n w_n$; these relations imply that: (a) z_{2n} is divisible by $z_n \pm i\sqrt{t} w_n$ if $t > 0$; (b) z_{2n} is divisible by $z_n \pm \sqrt{t} w_n$ if $t < 0$; (c) w_{2n} is divisible by z_n and w_n and, if r divides s, then z_{rn} and w_{rn} are divisors of w_{sn}.

Let $\sum_{k=1}^{n} = \sum$. Then, using the Binet forms, it is not difficult to see the following facts:

(i) $\sum z_k = \dfrac{\beta z_n - z_{n+1} + z - \beta}{\beta - 2z + 1}$,

(ii) $\sum w_k = \dfrac{\beta w_n - w_{n+1} + w}{\beta - 2z + 1}$,

(iii) $\sum z_k^2 = \dfrac{\beta z_{2n} - z_{2n+2} + z_2 - \beta}{2(\beta - 2z_2 + 1)} + \dfrac{\beta(\beta^n - 1)}{2(\beta - 1)}$,

(iv) $\sum w_k^2 = \dfrac{\beta^2 z_{2n} - z_{2n+2} + z_2 - \beta^2}{2t(\beta^2 - 2z_2 + 1)} - \dfrac{\beta(\beta^n - 1)}{2t(\beta - 1)}$,

(v) $2\sum z_k z_{n+1-k} = n z_{n+1} + \dfrac{\beta w_n}{w}$,

(vi) $2t \sum w_k w_{n+1-k} = n z_{n+1} - \dfrac{\beta w_n}{w}$

(vii) $2\sum z_k w_{n-k+1} = 2\sum w_k z_{n-k+1} = n w_{n+1}$.

Now we generalize a result satisfied by the generating functions of Fibonacci (F_n) and Lucas (L_n) sequences; namely,

$$F(t) = \sum_{1}^{\infty} \frac{F_n}{n} t^n, \qquad L(t) = \sum_{1}^{\infty} L_n t^n.$$

Then $L(t) = e^{2F(t)}$ [3]. A similar result holds between z_n and w_n. Let Z and W be generating functions of z_n and w_n, respectively; that is,

$$Z=\sum_{1}^{\infty}\frac{z_n}{n}s^n,\qquad W=\sum_{1}^{\infty}w_n s^n. \tag{9}$$

Then $W(s)=swe^{2Z(s)}$. Since the proof is similar to the real case (see [6]), we omit it here.

5. SERIES SUMMATION INVOLVING RECIPROCALS OF z_n AND w_n

All the properties of infinite series summation involving x_n and y_n can be extended to the complex variables case also. Since the arithmetic goes through without any changes, we shall just list them here. For details, see [6].

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{z_{k+1}}\left(\frac{2z}{z_{k-1}}-\frac{\beta+1}{z_k}\right)=\frac{1}{z}.$

2. $\displaystyle\sum_{k=r+1}^{\infty}\left(\frac{2z}{z_{k-1}z_{k+1}}-\frac{\beta+1}{z_{k+1}z_k}\right)=\frac{1}{z_r z_{r+1}},\qquad \sum_{k=r+1}^{\infty}\left(\frac{2z}{w_{k-1}w_{k+1}}-\frac{\beta+1}{w_{k+1}w_k}\right)=\frac{1}{w_r w_{r+1}}.$

3. $\displaystyle\sum_{k=r+1}^{\infty}\frac{2zz_k}{z_{k-1}z_{k+1}}=\sum_{k=r+1}^{\infty}\left(\frac{1}{z_{k-1}}+\frac{\beta}{z_{k+1}}\right),\qquad \sum_{k=r+1}^{\infty}\frac{2zw_k}{w_{k-1}w_{k+1}}=\sum_{k=r+1}^{\infty}\left(\frac{1}{w_{k-1}}+\frac{\beta+1}{w_{k+1}}\right).$

4. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{z_{(k+1)r}}\left(\frac{2z_r}{z_{(k-1)r}}-\frac{\beta^r+1}{z_{kr}}\right)=\frac{1}{z_r z_{2r}},\qquad \sum_{k=2}^{\infty}\frac{1}{w_{(k+1)r}}\left(\frac{2z_r}{w_{(k-1)r}}-\frac{\beta^r+1}{w_{kr}}\right)=\frac{1}{w_r w_{2r}}.$

5. $\displaystyle\sum_{k=2}^{\infty}\frac{\beta^{2^{k-1}-2}}{y_{2^k}}=\frac{1}{(x+y\sqrt{t})^2}.$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{\beta^{n-1}}{z_n z_{n+k}}=\frac{1}{tww_k}\left(\sum_{1}^{k}\frac{z_{n-1}}{z_n}-k(z\pm\sqrt{t}w)\right),$

where the plus sign should be taken if $|\xi/\eta|<1$ and the minus sign should be taken if $|\xi/\eta|>1$. To show item 6, we consider

$$\begin{aligned}z_{n-1}z_{n+k}-z_{n+k-1}z_n&=z_{n-1}(zz_{n+k-1}+tww_{n+k-1})-z_{n+k-1}(zz_{n-1}+tww_{n-1})\\&=tw(z_{n-1}w_{n+k-1}-z_{n+k-1}w_{n-1})=tw\beta^{n-1}w_k.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{N}\frac{\beta^n}{z_n z_{n+k}}&=\frac{1}{tww_k}\frac{z_{n-1}z_{n+k}-z_{n+k-1}z_n}{z_n z_{n+k}}\\&=\frac{1}{tww_k}\sum_{n=1}^{N}\left(\frac{z_{n-1}}{z_n}-\frac{z_{n+k-1}}{z_{n+k}}\right)=\frac{1}{tww_k}\left(\sum_{n=1}^{k}\frac{z_{n-1}}{z_n}-\sum_{n=N+1}^{N+k}\frac{z_{n-1}}{z_n}\right).\end{aligned}$$

Now fix $k\ge 1$ and let N tend to infinity. Using the property we derived in Section 3, we obtain the required result. Similarly, we show that

$$\beta^k\sum_{n=1}^{\infty}\frac{\beta^{(n-1)}}{w_n w_{n+k}}=\frac{1}{ww_k}\left(\sum_{1}^{k}\frac{w_{n-1}}{w_n}-k(z\pm\sqrt{t}w)\right),$$

where the plus sign should be taken if $|\xi/\eta|<1$ and the minus sign should be taken if $|\xi/\eta|>1$.

6. CONVOLUTIONS FOR z_n AND w_n

Suppose that $a_n(z, w)$ and $b_n(z, w)$ are two homogeneous polynomial sequences in two variables z and w, where n is an integer ≥ 1. Their *first convolution sequence* is defined by

$$(a_n * b_n)^{(1)} = \sum_{j=1}^{n} a_j b_{n+1-j} = \sum_{j=1}^{n} b_j a_{n+1-j}.$$

In the above definition, we have written $a_n = a_n(z, w)$ and $b_n = b_n(z, w)$. To determine the convolutions $z_n * z_n$, $z_n * w_n$, and $w_n * w_n$, we use the matrix properties of B, namely,

$$\begin{bmatrix} z & w \\ tw & z \end{bmatrix}^{n+1} = \begin{bmatrix} z_{n+1} & w_{n+1} \\ tw_{n+1} & z_{n+1} \end{bmatrix} = B^{n+1} = B^j B^{n+1-j} = \begin{bmatrix} z_j & w_j \\ tw_j & z_j \end{bmatrix} \begin{bmatrix} z_{n+1-j} & w_{n+1-j} \\ tw_{n+1-j} & z_{n+1-j} \end{bmatrix}.$$

Let

$$B_n^{(1)} = \sum_{j=1}^{n} B_j B_{n+1-j} = \sum_{j=1}^{n} B^{n+1} = \begin{bmatrix} z_n^{(1)} & w_n^{(1)} \\ tw_n^{(1)} & z_n^{(1)} \end{bmatrix}.$$

Note that $B^n = B_n$. We prefer using the subscript notation. Since $\sum_{j=1}^{n} B_{n+1} = nB_{n+1}$, the above result can be written as

$$nB_{n+1} = \begin{bmatrix} z_n * z_n + tw_n * w_n & 2z_n * w_n \\ 2tz_n * w_n & z_n * z_n + tw_n * w_n \end{bmatrix} = \begin{bmatrix} z_n^{(1)} & w_n^{(1)} \\ tw_n^{(1)} & z_n^{(1)} \end{bmatrix} = B_n^{(1)},$$

where we have written $\sum_{j=1}^{n} = \sum$. Therefore, we have $z_n^{(1)} = nz_{n+1}$ and $w_n^{(1)} = nw_{n+1}$, or

$$2z_n * z_n = nz_{n+1} + \frac{\beta w_n}{w} \quad \text{and} \quad 2tw_n * w_n = nz_{n+1} - \frac{\beta w_n}{w},$$

from (8) parts (v) and (vi). The above result can be extended to the kth convolution by defining

$$B_n^{(k)} = \sum_{j=1}^{n} B_j \left(B_{n+1-j}^{(k-1)} \right).$$

We can show that

$$B_n^{(k)} = \binom{n+k-1}{k} B_{n+k}.$$

We shall prove the result by induction on k. Since $B^{(1)} = nB_{n+1}$, the result is true for $k = 1$. Now consider

$$\begin{aligned} B_n^{(k+1)} &= \sum B_j B_{n+1-j}^{(k)} = \sum B_{n+1-j} (B_j^{(k)}) \\ &= \sum B_{n+1-j} \binom{j+k-1}{k} B_{j+k} = B_{n+k+1} \sum \binom{j+k-1}{k} = \binom{n+k}{k+1} B_{n+k+1}, \end{aligned}$$

which completes the induction. We have used the property $\sum \binom{j+k-1}{k} = \binom{n+k}{k+1}$, to derive the above result.

From the above results, we can write the following k^{th} convolutions, namely,

$$z_n^{(k)} = \binom{n+k-1}{k} z_{n+k} \quad \text{and} \quad w_n^{(k)} = \binom{n+k-1}{k} w_{n+k}. \tag{10}$$

Result (10) enables us to write the convolutions $z_n * z_n^{(k)}$, $w_n * w_n^{(k)}$, $z_n * w_n^{(k)}$, and $w_n * z_n^{(k)}$. First, we shall show that

$$2z_n * z_n^{(k)} = \binom{n+k}{k+1} z_{n+k+1} + \sum_{j=1}^{n} z_j^k z_{n-j+1} \beta^{j+k} z_{n+1-2j-k}.$$

We consider

$$\begin{aligned}
2z_n * z_n^{(k)} &= 2\sum z_j^k z_{n-j+1} \\
&= 2\sum \binom{j+k-1}{k} z_{j+k} z_{n-j+1} \\
&= 2\sum \binom{j+k-1}{k} (z_j z_k + t w_j w_k) z_{n-j+1} \\
&= 2z_k \sum \binom{j+k-1}{k} z_j z_{n-j+1} + 2t w_k \sum_{j=1}^{n} \binom{j+k-1}{k} w_j z_{n-j+1} \\
&= z_k \sum \left[\binom{j+k-1}{k} z_{n+1} + \beta^j z_{n-2j+1} \right] + w_k \sum \binom{j+k-1}{k} (w_{n+1} - \beta^j w_{n-2j+1}) \\
&= \sum \binom{j+k-1}{k} (z_k z_{n+1} + t w_k w_{n+1}) \sum \beta^j \binom{j+k-1}{k} (z_k z_{n-2j+1} - t w_k w_{n-2j+1}) \\
&= \binom{n+k}{k+1} z_{n+k+1} + \sum \binom{j+k-1}{k} \beta^{j+k} z_{n+1-2j-k}.
\end{aligned}$$

We have used (10) and (8) part (i) to derive the above result. Similarly, we can show that

$$2tw_n * w_n^{(k)} = \binom{n+k}{k+1} z_{n+k+1} - \sum \binom{j+k-1}{k} \beta^{j+k} z_{n+1-2j-k},$$

$$2z_n^{(k)} * w_n = \binom{n+k}{k+1} w_{n+k+1} - \sum \binom{j+k-1}{k} \beta^{j+k} w_{n+1-2j-k},$$

$$2z_n * w_n^{(k)} = \binom{n+k}{k+1} w_{n+k+1} - \sum \binom{j+k-1}{k} \beta^{j+k} w_{n+1-2j-k}.$$

7. THE IMPLICATIONS OF z_n AND w_n IN $\mathbf{R}^4$

Let $z = x + iy$ and $w = u + iv$. Then $z_n = x_n + iy_n$, $w_n = u_n + iv_n$, and $\beta = z^2 - tw^2 = \alpha + i\gamma$, where $\alpha = x^2 - y^2 - t(u^2 - v^2)$ and $\gamma = 2(xy - tuv)$. Note that $\det B \neq 0$ implies that either $\alpha \neq 0$ or $\gamma \neq 0$. Recurrence relations (3) now become:

$$x_{n+1} = 2xx_n - 2yy_n - \alpha x_{n-1} + \gamma y_{n-1}, \tag{11}$$

$$y_{n+1} = 2yx_n + 2xy_n - \gamma x_{n-1} - \alpha y_{n-1}, \tag{12}$$

$$u_{n+1} = 2xu_n - 2yv_n - \alpha u_{n-1} + \gamma v_{n-1}, \tag{13}$$

$$v_{n+1} = 2xv_n + 2yu_n - \gamma u_{n-1} - \alpha v_{n-1}, \tag{14}$$

with $x_0 = 1$, $y_0 = 0$, $u_0 = 0$, $v_0 = 0$ and $x_1 = x$, $y_1 = y$, $u_1 = u$, $v_1 = v$. By (11)-(14), the first few polynomials are given by

$$\begin{aligned} x_2 &= x^2 - y^2 + t(u^2 - v^2), \\ y_2 &= 2(xy + tuv), \\ u_2 &= 2(xu - yv), \\ v_2 &= 2(xv + yu). \end{aligned}$$

$$\begin{aligned} x_3 &= x^3 - 3xy^2 + 3txu^2 - 3txv^2 - 6tyuv, \\ y_3 &= 3x^2y - y^3 + 6txuv + 3tyu^2 - 3tyv^2, \\ u_3 &= 3x^2u - 3y^2u - 6xyv - 3tuv^2 + tu^3, \\ v_3 &= 6xyu + 3x^2v - 3y^2v + 3tu^2v - tv^3, \ldots. \end{aligned}$$

By expressing equations (8) parts (i) and (ii) in terms of the real and imaginary components, we find that the recurrence relations transform to

$$x_{m+n} = x_n x_n - y_m y_n + t(u_m v_n - u_n v_m), \tag{15}$$

$$y_{m+n} = x_m y_n + x_n y_m + t(u_m v_n - u_n v_m), \tag{16}$$

$$u_{m+n} = x_m u_n + x_n u_m - y_m v_n - y_n v_m, \tag{17}$$

$$v_{m+n} = x_m v_n + x_n v_m + y_m u_n + y_n u_m. \tag{18}$$

To transform the partial differential equation (7) in z and w to the one in variables x, y, u, and v, we use the partial differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then equation (7) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{1}{t}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial u^2}\right)\right] f_n = 0, \tag{19}$$

$$\left(\frac{\partial^2}{\partial x \partial y} - \frac{1}{t}\frac{\partial^2}{\partial u \partial v}\right) g_n = 0. \tag{20}$$

where $f_n = x_n$ or u_n and $g_n = y_n$ or v_n. By the principle of superposition, the solution of differential equations (19) and (20) are, respectively,

$$f(x, y, u, v) = \sum_0^\infty (a_n x_n + b_n u_n) \quad \text{and} \quad g(x, y, u, v) = \sum_0^\infty (c_n y_n + d_n v_n),$$

where a_n, b_n, c_n, and d_n are constants.

Now we express relation (9) in Section 4, i.e., $W(s) = swe^{2Z(s)}$, in terms of real and imaginary parts. Let $Z(s) = X(s) + iY(s)$ and $W(s) = U(s) + iV(s)$. Then (9) transforms to

$$U(s) = use^{X(s)}(u\cos Y(s) - v\sin Y(s))$$

and

$$V(s) = vse^{X(s)}(v\cos Y(s) + u\sin Y(s)).$$

Now, let us turn our attention to the convolutions in Section 6. Result (11), expressed in terms of real and imaginary components, becomes

$$x_n^{(k)} = \binom{n+k-1}{k} x_{n+k}, \qquad y_n^{(k)} = \binom{n+k-1}{k} y_{n+k},$$

$$u_n^{(k)} = \binom{n+k-1}{k} u_{n+k}, \qquad v_n^{(k)} = \binom{n+k-1}{k} v_{n+k}.$$

We have seen here some of the properties of the matrix B with complex entries; we are sure there are many more of them.

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DIVISIBILITY TESTS IN $\mathbb{N}$

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This article will develop a method to test divisibility of arbitrary natural numbers by certain fixed natural numbers. The well-known tests for divisibility by 3, 9, and 11 will be obtained as special cases of the theorem. Note that all the variables in the following theorem are integers.

Theorem: If $(s, 10) = 1$, $t \equiv 10^{-1} \pmod{s}$, $n = \sum_{k=0}^{r} 10^k a_k$, and $m = \sum_{k=0}^{r} t^{r-k} a_k$, then $s|n \Leftrightarrow s|m$.

Proof: We will expand n and use standard congruence properties:

$$\begin{aligned} n &= 10^r a_r + 10^{r-1} a_{r-1} + \cdots + 10a_1 + a_0, \\ n &\equiv 10^r a_r + 10^{r-1} a_{r-1} + \cdots + 10a_1 + a_0 \pmod{s}, \\ 10^{-r} n &\equiv a_r + 10^{-1} a_{r-1} + \cdots + 10^{1-r} a_1 + 10^{-r} a_0 \pmod{s}, \\ (10^{-1})^r n &\equiv a_r + 10^{-1} a_{r-1} + \cdots + (10^{-1})^{r-1} a_1 + (10^{-1})^r a_0 \pmod{s}, \\ t^r n &\equiv m \pmod{s}. \end{aligned}$$

Now $t \equiv 10^{-1} \pmod{s} \Rightarrow 10t \equiv 1 \pmod{s} \Rightarrow s|(10t-1) \Rightarrow zs = 10t - 1$ for some $z \in \mathbb{Z}$. Hence, $10t - zs = 1$, which implies $(s, t) = 1$.

The statement $t^r n \equiv m \pmod{s}$ allows us to conclude that $s|n \Rightarrow s|m$; with the additional fact that $(s, t) = 1$, we can conclude that $s|m \Rightarrow s|n$.

Remark: This theorem generates a divisibility test for any natural number s that is relatively prime to 10. The practicality of the test comes into play for s with an associated t value close to 0.

Divisibility Tests for Specific Natural Numbers

1. Let $s = 3$. Then $t \equiv 10^{-1}$ (mod 3) allows us to choose $t = 1$. Hence, $3|n \Leftrightarrow 3|m$, where $m = \sum_{k=0}^{r} a_k$
2. Let $s = 9$. Then $t \equiv 10^{-1}$ (mod 9) allows us to choose $t = 1$. Hence, $9|n \Leftrightarrow 9|m$, where $m = \sum_{k=0}^{r} a_k$.
3. Let $s = 11$. Then $t \equiv 10^{-1}$ (mod 11) allows us to choose $t = -1$. Hence, $11|n \Leftrightarrow 11|m$, where $m = \sum_{k=0}^{r} (-1)^{r-k} a_k$.
4. Let $s = 19$. Then $t \equiv 10^{-1}$ (mod 19) allows us to choose $t = 2$. Hence, $19|n \Leftrightarrow 19|m$, where $m = \sum_{k=0}^{r} 2^{r-k} a_k$.
5. Let $s = 7$. Then $t \equiv 10^{-1}$ (mod 7) allows us to choose $t = -2$. Hence, $7|n \Leftrightarrow 7|m$, where $m = \sum_{k=0}^{r} (-2)^{r-k} a_k$.
6. Let $s = 29$. Then $t \equiv 10^{-1}$ (mod 29) allows us to choose $t = 3$. Hence, $29|n \Leftrightarrow 29|m$, where $m = \sum_{k=0}^{r} 3^{r-k} a_k$.
7. Let $s = 31$. Then $t \equiv 10^{-1}$ (mod 31) allows us to choose $t = -3$. Hence, $31|n \Leftrightarrow 31|m$, where $m = \sum_{k=0}^{r} (-3)^{r-k} a_k$.

Specific Examples

Ex. 1: $n = 5232$ is divisible by $s = 3$ because we can take $t = 1$ and
$m = 5(1)^0 + 2(1)^1 + 3(1)^2 + 2(1)^3 = 5 + 2 + 3 + 2 = 12$ is divisible by 3.

Ex. 2: $n = 7119$ is divisible by $s = 9$ because we can take $t = 1$ and
$m = 7(1)^0 + 1(1)^1 + 1(1)^2 + 9(1)^3 = 7 + 1 + 1 + 9 = 18$ is divisible by 9.

Ex. 3: $n = 80916$ is divisible by $s = 11$ because we can take $t = -1$ and
$m = 8(-1)^0 + 0(-1)^1 + 9(-1)^2 + 1(-1)^3 + 6(-1)^4 = 8 - 0 + 9 - 1 + 6 = 22$ is divisible by 11.

Ex. 4: $n = 2242$ is divisible by $s = 19$ because we can take $t = 2$ and
$m = 2(2)^0 + 2(2)^1 + 4(2)^2 + 2(2)^3 = 2 + 4 + 16 + 16 = 38$ is divisible by 19.

Ex. 5: $n = 686$ is divisible by $s = 7$ because we can take $t = -2$ and
$m = 6(-2)^0 + 8(-2)^1 + 6(-2)^2 = 6 - 16 + 24 = 14$ is divisible by 7.

Ex. 6: $n = 4350$ is divisible by $s = 29$ because we can take $t = 3$ and
$m = 4(3)^0 + 3(3)^1 + 5(3)^2 + 0(3)^3 = 4 + 9 + 45 + 0 = 58$ is divisible by 29.

Ex. 7: $n = 527000$ is divisible by $s = 31$ because we can take $t = -3$ and
$m = 5(-3)^0 + 2(-3)^1 + 7(-3)^2 + 0(-3)^3 + 0(-3)^4 + 0(-3)^5 = 5 - 6 + 63 = 62$ is divisible by 31.

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ELLIPSES, CARDIOIDS, AND PENROSE TILES

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1. INTRODUCTION

Many macroscopic properties in nature represent the response of a system to an applied disturbance. Such properties as electrical or thermal conductivity, magnetic permeability, and dielectric permittivity fall into this category. They can all be described by the same model of an induced flux produced by an applied field or potential gradient. In this study we shall present the solution to a problem in plane geometry involving cardioids and ellipses which has arisen in the study of the interaction of electromagnetic waves with matter.

A major area of research in the field of condensed matter physics is the optical response of composite materials. Moreover, recent advances in nanostructure technologies have generated particular interest in the physical properties of composite thin films [4]. Such structures are made up of an otherwise uniform thin film of one material into which are embedded shafts or cylinders of a different material. The film constituents can be chosen so as to obtain desired bulk properties. In practice, the major constituent is a dielectric material into which metal columnar inclusions are deposited. The optical properties of the metal-dielectric thin films can be intermediate between those of the metal and of the dielectric. These films also exhibit significant angular and spectral selectivity. The former feature has practical importance in the production of window coatings which minimize solar heating and glare while the latter feature is of use in solar collectors. Composite thin films have recently been analyzed mathematically by means of a conformal mapping technique [10]. A schematic diagram of the film microstructure for obliquely deposited circular cylindrical columns is shown in Figure 1.

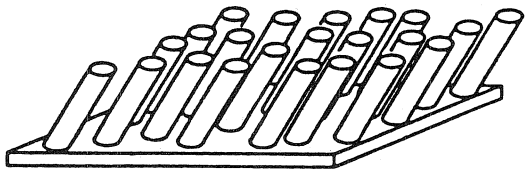

FIGURE 1. Film Microstructure

In general, the cylinder lengths are approximately equal to the film thickness. Therefore, by ignoring end effects such as fringing fields and restricting attention to a cross-section (normal to the cylinder axis), it becomes sufficient to model such a film as a plane figure. We therefore obtain a two-dimensional array of circles in the plane, each of which represents the cross-section of an individual cylindrical inclusion. During the production of the films it often happens that two

columns are deposited very close to each other and give the appearance of merging into one another. A mathematical model for this particular situation has recently appeared [9] which employs a symmetric pair of cardioids to describe the two-dimensional cross-section of the merging columns. During the analysis, the problem arose of determining the axis lengths of an ellipse of "best fit" enveloping the cardioid pair. The problem was solved and only the final numbers were presented. It was discovered that by choosing a suitable definition of best fit, the parameters of an optimal elliptical envelope for a cardioid pair could be determined exactly in terms of the *golden section*. We now present the full derivation of this interesting and unexpected result together with some concomitant findings that have been unearthed subsequently. An elliptical envelope for a pair of cardioids is shown in Figure 2.

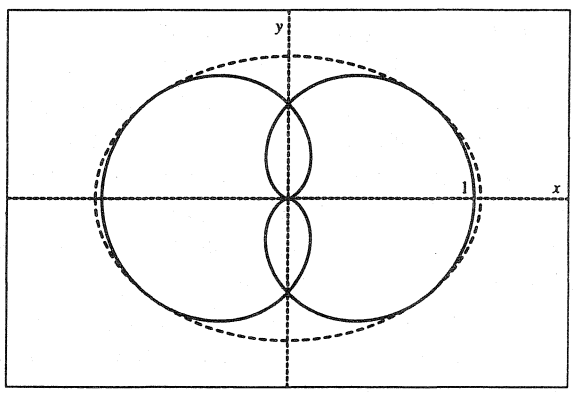

FIGURE 2. Cardioid Pair with Elliptical Envelope

2. GENERAL SCENARIO

The particular problem of interest can be considered as a special case of the following situation. We begin with the complex transformation

$$w_n = z^{-1/n} \tag{2.1}$$

in which $w = u + iv$ and $z = x + iy$. If we consider contours in the (Cartesian) z plane defined by $u = \mathrm{Re}(w) = \text{constant}$ and set $z = re^{i\theta}$, we obtain

$$r = \cos^n(\theta / n) \tag{2.2}$$

as the contour in the z plane which is mapped onto the straight line $u = 1$ in the w_n plane. When $n = 1$ we have a circle of radius $\frac{1}{2}$ centered at the Cartesian point $(\frac{1}{2}, 0)$. For $n = 2$ we obtain a cardioid symmetric about the x axis whose equation may be written as

$$r = \frac{1}{2}(1 + \cos\theta). \tag{2.3}$$

By superimposing the closed curves given by (2.2) with their respective reflections in the y axis, we obtain pairs of intersecting contours. The conformal mappings (2.1) corresponding to $n = 1$ and $n = 2$ have been used to study the polarization response of touching [7], [8] and intersecting [9] particles, respectively. As $n \to \infty$, the degree of merging of the particle pair increases until, in

the limit, the contour corresponding to $u = 1$ becomes the unit circle centered at the origin. In this paper we shall be considering elliptical envelopes for a pair of (left- and right-hand) cardioids (the $n = 2$ case).

The approach to be adopted here will be to eliminate θ and then ultimately express the ellipse area in terms of the radial coordinate of the point of tangency. Due to the symmetry of the cardioid pair with respect to both the x and the y axes, it will clearly be sufficient to work just within the first quadrant. Moreover, due to the shape of the cardioid pair, the horizontal axis of the desired optimal ellipse will be the major one. Hence, we can restrict attention to the right-hand cardioid in the first quadrant where $0 \le \theta \le \pi/2$ and search for unrotated ellipses centered at the origin with horizontal and vertical semi-axis lengths of a and b, respectively, where $a \ge b > 0$.

The first step in determining our optimal ellipse is, naturally, to find the points of intersection of the relevant curves. In the general case, we must therefore begin by finding the points of intersection of the n-cardioid (2.2) and the ellipse. The polar equation of an ellipse with horizontal and vertical semi-axis lengths of a and b, respectively, is given by

$$r = \frac{ab}{\sqrt{a^2 + (b^2 - a^2)\cos^2\theta}}, \quad a \ne 0 \ne b. \tag{2.4}$$

Eliminating θ between (2.2) and (2.4) leads to the following polynomial equation for the value of the radial coordinate of the point(s) of intersection (ρ, φ):

$$\rho^2\left(T_n^2(\sqrt[n]{\rho}) - \lambda - 1\right) + \mu = 0, \quad a \ne b, \tag{2.5}$$

where the functions $T_n(s)$ are the Chebyshev polynomials of the first kind [11] and

$$\lambda = \frac{b^2}{a^2 - b^2}, \quad \mu = \frac{a^2b^2}{a^2 - b^2}, \quad a \ne b,$$

which can be rearranged as

$$a = \sqrt{\frac{\mu}{\lambda}}, \quad b = \sqrt{\frac{\mu}{\lambda + 1}}. \tag{2.6}$$

In order that the solutions ρ represent points of tangency, we must also require that the slopes of the ellipse and the n-cardioid be the same at their point(s) of intersection (ρ, φ). We can specify the slope of a curve at a given point by considering the angle γ between the tangent and radial vectors at that point. If we denote these angles for the ellipse and the n-cardioid by γ_E and γ_C, respectively, then the tangency condition at (ρ, φ) can be written

$$\tan\gamma_E = \tan\gamma_C, \tag{2.7}$$

where

$$\tan\gamma = \frac{r\,d\theta}{dr} = \frac{r(\theta)}{r'(\theta)}. \tag{2.8}$$

Substituting (2.2) and (2.4) into (2.7)-(2.8) yields, for $a \ne b$,

$$\frac{\lambda + \sin^2\varphi}{\sin 2\varphi} = \frac{1}{2}\cot\left(\frac{\varphi}{n}\right), \quad \varphi \ne 0,\ \pi/2,$$

or, by using (2.2) at (ρ, φ):

$$\lambda + 1 = T_n^2(\rho^{1/n}) + \rho^{1/n} T_n(\rho^{1/n}) U_{n-1}(\rho^{1/n}), \quad \rho \neq 1, \tag{2.9}$$

where the $U_n(s)$ are the Chebyshev polynomials of the second kind [11].

In the next section, we shall consider the interesting case of enveloping a pair of standard $(n = 2)$ cardioids by an ellipse. In anticipation of the results to be obtained, we conclude this section by introducing the *golden section*, $\tau = (1+\sqrt{5})/2$, a number also familiar as the positive solution to

$$x^2 - x - 1 = 0, \tag{2.10}$$

which is the characteristic equation for the sequence of Fibonacci numbers $\{F_n\}$. This sequence has many connections with the Chebyshev polynomials mentioned above [6], [11]. In addition, the identity

$$\frac{1}{m+n\tau} = \frac{m+n-n\tau}{m^2+mn-n^2}, \tag{2.11}$$

which is well known from the field $\mathbf{Q}(\sqrt{5})$, will be found useful in later sections.

3. THE ELLIPSE

We shall define an optimal elliptical envelope (or ellipse of best fit) to be an ellipse of *minimal area* which is tangent to and completely contains the cardioid pair. This provides precisely the right number of conditions necessary to determine the three key parameters: the value of the radial coordinate of the point of tangency, $r = \rho$, and the semi-axis lengths a and b of the desired ellipse. We will solve the problem in terms of ρ and then substitute back to find the required ellipse dimensions.

The first condition is that the cardioid and the ellipse must intersect. For $a \neq b$ this is just (2.5) with $n = 2$, which leads to the following equation for ρ:

$$\mu = \lambda\rho^2 + 4\rho^3 - 4\rho^4. \tag{3.1}$$

The second condition is the tangency requirement which is (2.9) with $n = 2$. This yields

$$\lambda = 2\rho(4\rho - 3). \tag{3.2}$$

We now obtain an expression for μ in terms of ρ by substituting (3.2) into (3.1) to find that

$$\mu = 2\rho^3(2\rho - 1). \tag{3.3}$$

Substitution into (2.6) of the respective expressions (3.2) and (3.3) for λ and μ, together with some subsequent simplification, leads to:

$$a = \rho\sqrt{\frac{2\rho-1}{4\rho-3}}, \quad b = \rho\sqrt{\frac{2\rho}{4\rho-1}}, \quad \rho \neq 1/4,\ 3/4. \tag{3.4}$$

The expression for the area of an ellipse tangent to the cardioid pair in terms of ρ then follows directly from (3.4):

$$A(\rho)=\frac{\pi}{2}\rho^2\sqrt{\frac{\rho(\rho-1/2)}{(\rho-3/4)(\rho-1/4)}},\quad \rho\neq 1/4,\ 3/4. \tag{3.5}$$

Expression (3.5) for the ellipse area is defined only when $\rho=0$, $1/4<\rho\leq 1/2$ or $3/4<\rho\leq 1$. In other words, an ellipse intersecting the cardioid at a point (ρ,φ) at which both curves have the same slope is only possible for ρ values in the above intervals. The graph of the ellipse area $A(\rho)$ for $0\leq\rho\leq 1$ is plotted in Figure 3.

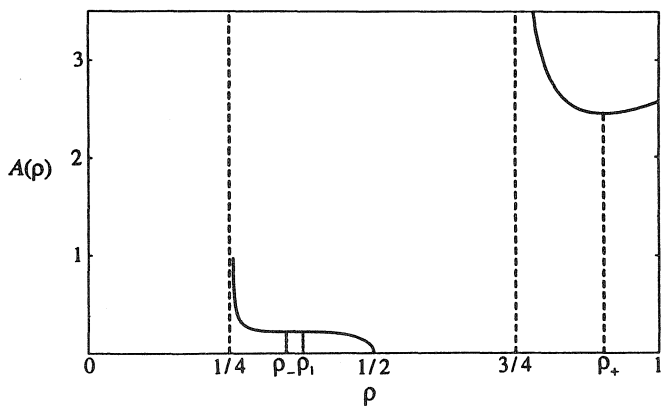

FIGURE 3. Graph of Ellipse Area

We can examine the meaning of these permissible intervals by considering what sort of ellipse will appear as the point of tangency moves along the right-hand cardioid. For the point $P(1,0)$ we have $\rho=1$ and, by (3.4), obtain an ellipse with $a=1$ (as expected) and $b=\sqrt{2/3}$. As our point moves (in the positive θ sense) along the cardioid, the ρ value decreases from unity and we approach the point of maximum vertical elevation. At this point, the tangent to the cardioid is horizontal and so the touching ellipse in this case will have an infinitely long horizontal axis and its area will be undefined. This point corresponds to $\rho=3/4$ and so, from (3.4), we see that a is undefined (as expected) and that $b=3\sqrt{3/8}$.

Before completing the problem, we pause briefly to dispose of the two cases not encompassed in the above derivation. These are the cases for which $a=b$ and $\rho=1$ ($\varphi=0$). In the former case, the point of intersection is $P(1,0)$ and the covering ellipse reduces to the unit circle centered at the origin. In the latter case, we obtain ellipses for which $b<a=1$. All such ellipses share a common vertical tangent with the cardioid at the point P, where their curvature is given by $1/b^2$. At P the cardioid has a curvature equal to 3/2 [12]. Only those ellipses whose curvature at P is less than 3/2 will lie completely outside the cardioid. Hence, we must have $b^2\geq 2/3$. The ellipse of least area satisfying this condition, E say, is obviously the one for which $b=\sqrt{2/3}$. The generic expressions (3.4) therefore reproduce this result for the $\rho=1$ case. In fact, the tangency condition (3.2) implies that the solutions ρ are double roots of (3.1), and $\rho=1$ is indeed such a double root for $a=1$ ($\lambda=\mu$) precisely when $b=\sqrt{2/3}$.

4. GENERIC SOLUTION

The final stage of the generic solution is to determine the geometrically reasonable ($\rho > 0$) critical points of the area function $A(\rho)$. The third condition is therefore the choosing of those ρ values for which the derivative $A'(\rho)$ becomes zero. By imposing this requirement, we will find the ρ value corresponding to the outer elliptical envelope of minimal area—that is, the optimal ellipse. However, due to the nature of this approach, in determining the critical points of $A(\rho)$ we shall find *another* tangent ellipse whose dimensions will also be of interest.

Differentiating the square of (3.5) with respect to ρ, simplifying, and then setting the result to zero, leads to the equation $\rho^4 g(\rho) = 0$, where

$$g(\rho) = 128\rho^3 - 208\rho^2 + 100\rho - 15. \tag{4.1}$$

The solutions of interest will naturally come from the zeros of the cubic polynomial $g(\rho)$ defined in (4.1). As we now show, these can all be determined exactly.

As $g(1)$ is nonzero and $0 < \rho \le 1$, there will be no integer solutions for $g(\rho) = 0$ (excluding the trivial case). Therefore, since the coefficients of $g(\rho)$ are all integral, any rational solutions will have the form p/q, where $p|15$, $q|128$, and $p < q$ [1]. The only nontrivial rational zero of $g(\rho)$ is found to be $\rho_1 = 3/8$. This leads to the factorization

$$g(\rho) = (8\rho - 3)(16\rho^2 - 20\rho + 5). \tag{4.2}$$

The remaining critical points are the zeros of the quadratic factor on the right-hand side of (4.2). These can be written as

$$\rho_+ = \frac{1}{4}(2+\tau), \quad \rho_- = \frac{1}{4}(3-\tau). \tag{4.3}$$

By considering the sign of the second derivative of $A(\rho)$ (or otherwise), it is readily seen that the rational solution ρ_1 corresponds to a local maximum for the area of the tangent ellipse while the remaining two conjugate solutions $\rho_\pm$ correspond to local minima for this area. This is also clear from the graph of $A(\rho)$ displayed in Figure 3. We shall denote by $E_\pm$ the ellipses corresponding to $\rho_\pm$, respectively. The larger of the two (conjugate) ellipses, E_+, is the desired unique optimal ellipse completely enclosing the cardioid pair. Its area is less than that of the two additional plane figures considered separately above, namely, the unit circle and the ellipse E.

The semi-axis lengths of the ellipse E_+ and the angular coordinate φ_+ of its point of intersection P_+ with the right-hand cardioid are found by substituting ρ_+ into (3.4) and (2.3), respectively. With the aid of (2.11) and the fact that τ satisfies (2.10), we obtain

$$a_+ = \frac{1}{4\sqrt{2}}(1+3\tau), \quad b_+ = \frac{\sqrt{5}}{4\sqrt{2}}\sqrt{2+\tau}, \tag{4.4}$$

and

$$\cos\varphi_+ = \frac{\tau}{2}, \tag{4.5}$$

or just (see [2]) $\varphi_+ = \pi/5$. We can immediately determine the focus F_+ and eccentricity e_+ of E_+ from (4.4):

$$F_+ = \frac{\sqrt{5}}{4}\sqrt{\tau}, \quad e_+ = \sqrt{2(2\tau - 3)}. \tag{4.6}$$

The relative area excess of E_+ over the double cardioid, DC, turns out to be

$$\frac{A_{E_+} - A_{DC}}{A_{DC}} = a_+ b_+ / (3/8 + 1/\pi) - 1 \approx .1223.$$

5. CONJUGACY RELATIONS

There is a whole series of interesting relations linking the dimensions of the two conjugate ellipses $E_\pm$. In these last two sections we present this material together with some related geometric constructions. We therefore begin by considering the tangent ellipse E_- and implicitly drop the restriction that $0 \le \theta \le \pi/2$. By substituting ρ_- into (3.4) and (2.3) and again using (2.10) and (2.11), we obtain the following semi-axis lengths for E_-:

$$a_- = \frac{1}{4\sqrt{2}}(3\tau - 4), \quad b_- = \frac{\sqrt{5}}{4\sqrt{2}}\sqrt{3-\tau}, \tag{5.1}$$

and the following angular coordinate φ_- for the point of intersection P_- of E_- with the right-hand cardioid:

$$\cos\varphi_- = -\frac{1}{2}(\tau - 1), \tag{5.2}$$

or just (see [2]) $\varphi_- = 3\pi/5$. The focus F_- and eccentricity e_- of E_- follow from (5.1):

$$F_- = \frac{\sqrt{5}}{4}\sqrt{\tau - 1}, \quad e_- = \sqrt{\frac{2}{5}}\sqrt{2\tau - 1}. \tag{5.3}$$

The ellipse E_- has its major axis lying along the y axis and actually *cuts* both cardioids. However, this ellipse can still be said to be tangential to the cardioid since, at the point of intersection, the curves have the same slope. In Figure 4 the two ellipses $E_\pm$ are shown superimposed onto the right-hand cardioid. We also make note of the angle φ_1 which corresponds to the rational zero ρ_1 of (4.1) and which, by (2.3), satisfies

$$\cos\varphi_1 = -\frac{1}{4}. \tag{5.4}$$

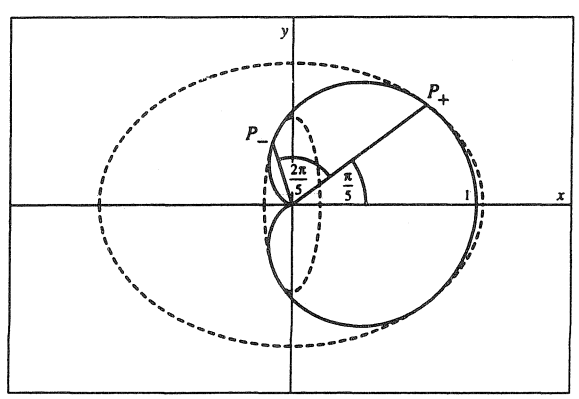

FIGURE 4. Right-Hand Cardioid with Conjugate Ellipses

In the following, we list several conjugacy results based on the values of the various ellipse parameters. Identities (2.10) and (2.11) for τ have been used where convenient to simplify the working.

Straightforward calculations employing (4.6) and (5.3) and then (4.5), (5.2), and (5.4) lead to

$$\frac{F_+}{F_-} = \tau, \quad \frac{1}{e_+^2} - \frac{1}{e_-^2} = 1, \quad \text{and} \quad \cos\varphi_+ \cos\varphi_- = \cos\varphi_1.$$

Using (4.4) and (5.1), it can also be shown that

$$\frac{1}{2}\left(1 + \frac{b_+ b_-}{a_+ a_-}\right) = \tau = \frac{1}{2}\left(\frac{a_+ b_-}{a_- b_+} - 1\right),$$

from which one immediately obtains

$$\frac{1}{4}\left(\frac{a_+ b_-}{a_- b_+} + \frac{b_+ b_-}{a_+ a_-}\right) = \tau.$$

By considering the ratios of corresponding quantities for the two conjugate ellipses $E_\pm$, it can be shown that the following set of quotients are all equal to τ:

$$\tau = \frac{b_+}{b_-} = \sqrt[5]{\frac{A_+}{A_-}} = \sqrt[4]{\frac{a_+}{a_-}} = \sqrt{\frac{\rho_+}{\rho_-}} = \sqrt[4]{\frac{x_+}{x_-}} = \frac{y_+}{y_-},$$

where the $A_\pm$ are the areas of the respective conjugate ellipses and the lengths $x_\pm$ and $y_\pm$ denote the Cartesian coordinates of the corresponding points $P_\pm$.

Another interesting result involves arc lengths along the (right-hand) cardioid. The expression for the arc length along the cardioid (2.3) from the point $P(1, 0)$ is $s(\theta) = 2\sin(\theta/2)$ (see [12]). At the points $P_\pm$ we therefore obtain

$$s(\varphi_+) = 2\sin\frac{\varphi_+}{2} = \tau - 1, \quad s(\varphi_-) = 2\sin\frac{\varphi_-}{2} = \tau,$$

where we have used (4.5) and (5.2). The arc length along the cardioid from P_+ to P_- is thus precisely one unit.

We can also consider curvatures. The curvature of the ellipse (2.4) at a point (r, θ) is given by (see [12])

$$K(r, \theta) = \frac{a^4 b}{(a^4 + (b^2 - a^2) r^2 \cos^2\theta)^{3/2}}. \tag{5.5}$$

Substituting the polar coordinates of the points $P_\pm$ into (5.5), using (4.4) and (5.1), and then taking the resulting ratio, leads to

$$\frac{K_+}{K_-} = \tau, \tag{5.6}$$

where the $K_\pm$ denote the curvatures of the ellipses $E_\pm$ at their respective points of intersection $P_\pm$ with the right-hand cardioid. A result analogous to (5.6) can also be shown to hold for the ratio of the curvatures of the right-hand cardioid at the points $P_\pm$.

6. PENROSE TILES

A routine calculation reveals that the slope of the tangent line at P_+ is equal to $\tan 4\pi/5$. By drawing in that part of this tangent line which lies in the first quadrant and then repeating the analogous procedure in the other three quadrants, we obtain a rhombus R with angles of $2\pi/5$ on the x axis and angles of $3\pi/5$ on the y axis. This quadrilateral is in fact known as a *Penrose rhombus* because it can be divided up to form two *Penrose tiles* [5]. This is shown in Figure 5. The two Penrose tiles with colored vertices (which do not concern us here) have been dubbed *darts* and *kites* (after John Horton Conway) [3]. The partition BGD (where $\angle BGD = 4\pi/5$) divides R into the dart $BCDG$ and the kite $ABGD$. Using some simple trigonometry, it can be shown that the length of OG is in fact equal to ρ_-. Also, the partitions formed by the rays OP_+ and OP'_+ form another Penrose rhombus $OP_+AP'_+$ which is one-quarter the size of R. Some elementary algebra also reveals that E_+ is in fact the ellipse of greatest area that can be inscribed within R.

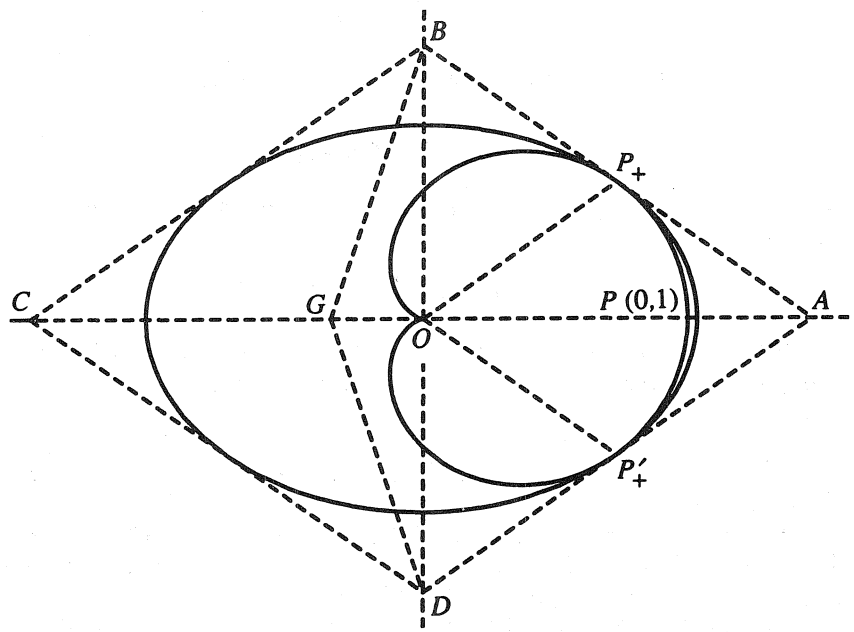

FIGURE 5. Right-hand Cardioid and Optimal Ellipse with Penrose Rhombus Divided into a Dart and a Kite

Another construction highlights the relationship between the darts and kites and the intersection points. It is possible to use intervals through these points to form new darts and kites. The upper and lower points of intersection of E_- with the left-hand cardioid will be denoted Q_- and Q'_-, respectively. We first produce both CB and OQ_- until they meet at H and then do the same with both CD and OQ'_- and the point I. In this way we form the dart $OHCI$. The contiguous quadrilateral $AB'OD'$ then turns out to be a kite. These structures are displayed in Figure 6. After some elementary geometric considerations, it can be shown that the ratio of the area of the dart $OHCI$ to that of the kite $AB'OD'$ is precisely equal to the golden section, τ.

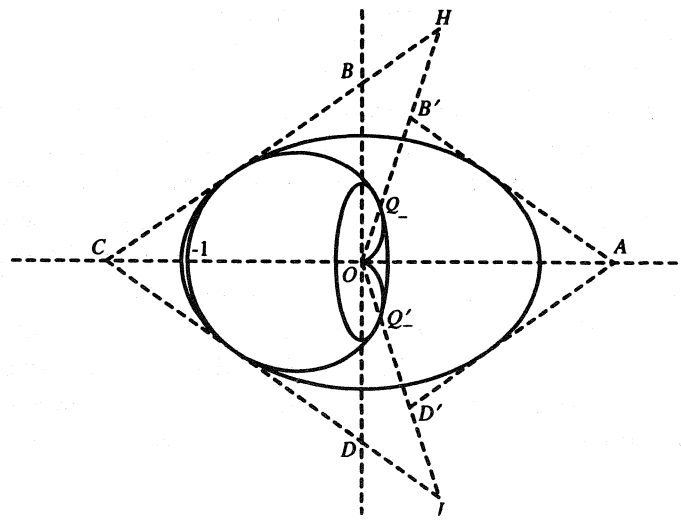

FIGURE 6. Additional Darts and Kites

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PRONIC FIBONACCI NUMBERS

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1. INTRODUCTION

"Pronic" is an old-fashioned term meaning "the product of two consecutive integers." (The reader will find the term indexed in [1], referring to some half-dozen articles.) In this paper we show that the only Fibonacci numbers that are the product of two consecutive integers are $F_0 = 0$ and $F_{\pm 3} = 2$.

The referee of this paper has called the author's attention to the prior publication (December 1996) of this result in Chinese (see Ming Luo [3]). However, because of the relative inaccessibility of the earlier result, the referee recommended publication of this article in the *Quarterly*.

If $F_n = r(r+1)$, then $4F_n + 1$ is a square. Our approach is to show that F_n, for $n \neq 0, \pm 3$, is not a pronic number by finding an integer $w(n)$ such that $4F_n + 1$ is a quadratic nonresidue modulo $w(n)$. There is a sense in which this paper may be considered a companion paper to Ming Luo's article on triangular numbers in the sequence of Fibonacci numbers: If F_n is a pronic number, then F_n is two times a triangular number. We shall use two results from Luo's paper, and take advantage of the periodicity of the sequence modulo an appropriate integer $w(n)$, enabling us to prove our result through use of the Jacobi symbol $(4F_n + 1 | w(n))$ in a finite number of cases. Our main result is the following theorem.

Main Theorem: The Fibonacci number F_n is the product of two consecutive integers if and only if $n = -3$, 0, or 3.

2. IDENTITIES AND PRELIMINARY LEMMAS

Let n and m be integers and $\{L_n\}$ be the sequence of Lucas numbers. Properties (1) through (4) are well known, and (5) was established in Luo's paper [2].

$$F_{-n} = (-1)^{n+1} F_n. \tag{1}$$

$$L_{2n} = L_n^2 - 2(-1)^n. \tag{2}$$

$$F_{m+n} = F_m L_n - (-1)^n F_{m-n}. \tag{3}$$

$$2F_{m+n} = F_m L_n + F_n L_m. \tag{4}$$

If k is even, $3 \nmid k$, and $(a, L_k) = 1$, then

$$(\pm 4aF_{2k} + 1 | L_{2k}) = -(8aF_k \pm L_k | 64a^2 + 5). \tag{5}$$

If the period of $\{F_n\}$ modulo Q is t and $n \equiv m \pmod{t}$, then $F_n \equiv F_m \pmod{Q}$. We will use this fact in our proofs for the following pairs: $(t, Q) = (8, 3)$, $(20, 5)$, $(16, 7)$, $(24, 9)$, $(10, 11)$, $(40, 41)$, $(50, 101)$, $(50, 151)$, and $(100, 3001)$.

It should be noted that we have given the least period t modulo Q in each of the above pairs; however, $F_n \equiv F_m \pmod{Q}$ if $n \equiv m \pmod{ht}$ for any integer h.

Finally, we comment that it is well known that F_n and L_n are even if and only if $3|n$.

Lemma 1: For all integers k and m, and g odd,

$$F_{2kg+m} \equiv \begin{cases} F_{2k+m} \pmod{L_{2k}}, & \text{if } g \equiv 1 \pmod 4, \\ -F_{2k+m} \pmod{L_{2k}}, & \text{if } g \equiv 3 \pmod 4. \end{cases}$$

Proof: By (3),

$$F_{2kg+m} = F_{2k(g-1)+m}L_{2k} - (-1)^{2k}F_{2k(g-2)+m} \equiv -F_{2k(g-2)+m} \pmod{L_{2k}};$$

clearly,

$$F_{2kg+m} \equiv -F_{2k(g-2)+m} \equiv +F_{2k(g-4)+m} \equiv \cdots \equiv \pm F_{2k+m} \pmod{L_{2k}},$$

where the positive sign occurs if and only if $g \equiv 1 \pmod 4$.

Lemma 2: If $3 \nmid k$, then $F_{2k+3} \equiv 2F_{2k} \pmod{L_{2k}}$.

Proof: By (4),

$$2F_{2k+3} = F_{2k}L_3 + F_3L_{2k} \equiv F_{2k} \cdot 4 \pmod{L_{2k}},$$

implying the lemma, since L_{2k} is odd.

Lemma 3: If F_n is pronic, then $n \equiv 0$ or $\pm 3 \pmod 8$.

Proof: Assume $4F_n + 1$ is a square. Then $4F_n + 1$ is a quadratic residue modulo 3 and modulo 7. However, $4F_n + 1$ is a quadratic nonresidue modulo 3 if $n \equiv 1$, 2, or 7 (mod 8), and a nonresidue modulo 7 if $n \equiv 4$ or 12 (mod 16). If $n \equiv 6 \pmod 8$, then $n \equiv 6$, 14, or 22 (mod 24); but, for each of these n's, $4F_n + 1$ is a quadratic nonresidue modulo 9, establishing the lemma.

3. PROOFS OF THE THEOREMS

Theorem 1: If n is odd and $n \neq \pm 3$, then F_n is not pronic.

Proof: Assume n is odd, $n \neq \pm 3$, and F_n is pronic. By Lemma 3, $n \equiv \pm 3 \pmod 8$. First, we assume that $n \equiv 3 \pmod 8$. Then $n \equiv 3$, 11, 19, 27, or 35 (mod 40); however, $(4F_m + 1|Q) = -1$ for $(m, Q) = (11, 5)$, (19, 41), (27, 5), and (35, 11), implying $n \equiv 3 \pmod{40}$. Then $n \equiv 3$, 23, 43, 63, or 83 (mod 100). Proceeding as before, we find that $(4F_m + 1|Q) = -1$ for $(m, Q) = (23, 3001)$, (43, 101), (63, 151), and (83, 101). Hence, if $n \equiv 3 \pmod 8$, then $n \equiv 3 \pmod{100}$. Let $n = 2 \cdot 2^u \cdot 5^2 t + 3$, $u \geq 1$. Now, if $n = 2kg + 3$, $3 \nmid k$, and g is odd, then, by Lemmas 1 and 2,

$$(4F_n + 1|L_{2k}) = (\pm 8F_{2k} + 1|L_{2k}).$$

By (5), if k is even and $3 \nmid k$, then

$$(\pm 8F_{2k} + 1|L_{2k}) = -(16F_k \pm L_k|261) = -(16F_k \pm L_k|29).$$

In the proof of Luo's Lemma 2 (see [2]), it is shown that this Jacobi symbol is -1 for

$$\begin{array}{lll} k = 2^u & \text{and } g = 5^2 t & \text{if } u \equiv 0 \pmod 3, \\ k = 2^u \cdot 5^2 & \text{and } g = t & \text{if } u \equiv 1 \pmod 3, \\ k = 2^u \cdot 5 & \text{and } g = 5t & \text{if } u \equiv 2 \pmod 3. \end{array}$$

Thus, F_n is not pronic if $n \equiv 3 \pmod 8$.

Assume now that $n \equiv -3 \pmod 8$. By (1), $F_n = F_{-n}$ and, since $-n \equiv 3 \pmod 8$,

$$(4F_{-n}+1|L_{2k}) = -1$$

by the above proof. □

Lemma 4: If $u \geq 4$, then

(a) $F_{2^u} \equiv (-1)^u \cdot 21 \pmod{69}$ and $L_{2^u} \equiv -1 \pmod{69}$,

(b) $F_{2^u \cdot 5} \equiv (-1)^{u+1} \cdot 21 \pmod{69}$ and $L_{2^u \cdot 5} \equiv -1 \pmod{69}$.

Proof: $L_2 = 3$, $L_4 = 7$, $L_8 = 47$, $L_{16} = 2207 \equiv -1 \pmod{69}$ and, using (2), it follows by induction that $L_{2^u} \equiv -1 \pmod{69}$ for $u \geq 4$, Hence,

$$F_{2^u} = F_2 L_2 L_4 L_8 \ldots L_{2^{u-1}} \equiv 1 \cdot 3 \cdot 7 \cdot 47 \cdot (-1)^u \equiv (-1)^u \cdot 21 \pmod{69}.$$

Similarly, $L_{10}, L_{20}, L_{40}, L_{80} \equiv 54, 16, 47, -1 \pmod{69}$, respectively, and (b) readily follows. □

Proof of the Main Theorem: If $n = 0$ or ± 3, F_n is clearly the product of consecutive integers. Assume that $n \neq 0, \pm 3$, and F_n is pronic. By Lemma 3 and Theorem 1, $n \equiv 0 \pmod 8$; so $n \equiv 0$, 8, 16, 24, or 32 (mod 40). But $(4F_m + 1|Q) = -1$ for $(m, Q) = (8, 11)$, (16, 41), (24, 5), or (32, 5), so $n \equiv 0 \pmod{40}$. Let $n = 2 \cdot 2^u \cdot 5t$, $u \geq 2$. By Lemma 1 and (5), if $n = 2kg$, $3 \nmid k$, k is even, and g is odd, then

$$(4F_n + 1|L_{2k}) = (4F_{2kg} + 1|L_{2k}) \equiv \begin{cases} -(8F_k + L_k|69), & \text{if } g \equiv 1 \pmod 4, \\ -(8F_k - L_k|69), & \text{if } g \equiv 3 \pmod 4. \end{cases}$$

Case 1: $t \equiv 1 \pmod 4$. Let

$$k = 2^u \quad \text{and} \quad g = 5t \equiv 1 \pmod 4, \quad \text{if } u \text{ is odd}, \quad u \neq 3 \text{ or } u = 2,$$
$$k = 2^u \cdot 5 \quad \text{and} \quad g = t \equiv 1 \pmod 4, \quad \text{if } u \text{ is even}, \quad u \neq 2 \text{ or } u = 3.$$

If $u = 2$, $-(8F_k + L_k|69) = -(31|69) = -1$; if $u = 3$, $-(8F_k + L_k|69) = -(17|69) = -1$; if $u \geq 4$ and u is odd $(k = 2^u)$ or if u is even $(k = 2^u \cdot 5)$, then, by Lemma 4,

$$8F_k + L_k \equiv 8(-21) + -1 \equiv -169 \pmod{69}.$$

Hence, $-(8F_k + L_k|69) = -(-169|69) = -1$.

Case 2: $t \equiv 3 \pmod 4$. Let

$$k = 2^u \quad \text{and} \quad g = 5t \equiv 3 \pmod 4, \quad \text{if } u \text{ is even or } u = 3,$$
$$k = 2^u \cdot 5 \quad \text{and} \quad g = t \equiv 3 \pmod 4, \quad \text{if } u \text{ is odd}, \quad u \neq 3.$$

If $u = 2$, $-(8F_k - L_k|69) = -(17|69) = -1$; if $u = 3$, $-(8F_k - L_k|69) = -(121|69) = -1$; if $u \geq 4$ and u is odd $(k = 2^u \cdot 5)$ or u is even $(k = 2^u)$, then, by Lemma 4,

$$8F_k - L_k \equiv 8 \cdot 21 - (-1) \equiv 169 \pmod{69}.$$

Hence, $-(8F_k - L_k|69) = -(169|69) = -1$. □

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The author wishes to express his thanks to the anonymous referee, and to the editor of this *Quarterly* who very graciously accepted the referee's recommendation to publish this paper in order to make the previously published results in Chinese more widely available. The author understands that the proofs in this paper and those in the earlier paper [3] (which he has not yet seen) are along similar lines but differ in detail.

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PRONIC LUCAS NUMBERS

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1. INTRODUCTION

An integer m is a pronic number if m is the product of two consecutive integers. We shall show that the only Lucas number which is a product of two consecutive integers is $L_0 = 2$.

The author has been informed by the referee that the results of this paper appeared recently in a Chinese journal (in Chinese) [2]; however, because of the relative inaccessibility of that article, the editor has accepted the referee's recommendation to publish the results in *The Fibonacci Quarterly*. The author has not yet seen the earlier publication, but understands that the proofs employ the same line of reasoning, although differing in details.

If $m = r(r+1)$, then $4m+1$ is a square. Our approach is to show that L_n, for $n > 0$, is not a pronic number by finding an integer $w(n)$ such that $4L_n + 1$ is a quadratic nonresidue modulo $w(n)$. It may be noted that if L_n is a pronic number, then L_n is two times a triangular number. Our interest in this problem was prompted by Ming Luo's very nice paper entitled "On Triangular Lucas Numbers," [2], and we employ an approach similar to that of Luo. We prove the following theorem.

Theorem: The Lucas number L_n is the product of two consecutive integers if and only if $n = 0$.

2. SOME IDENTITIES, SOME LEMMAS, AND THE PROOF

The Lucas numbers are defined by

$$L_0 = 2,\ L_1 = 1,\ \text{and } L_n = L_{n-1} + L_{n-2},\ \text{for } n \geq 2,$$

and the recursive relation holds for n negative if $L_{-n} = (-1)^n L_n$.

Let n and m be any integers, and $\{F_n\}$ be the Fibonacci sequence. We require the following well-known identities:

$$L_n^2 = 5F_n^2 + 4(-1)^n; \tag{1}$$

$$L_{2n} = L_n^2 - 2(-1)^n; \tag{2}$$

$$2L_{m+n} = L_m L_n + 5F_m F_n; \tag{3}$$

$$L_{m+n} = L_m L_n - (-1)^n L_{m-n} = 5F_m F_n + (-1)^n L_{m-n}. \tag{4}$$

Our proof makes use of the periodicity of the sequence of Lucas numbers modulo an odd integer. It is well known [and easily shown using (4)] that, if t_k is an odd divisor of $5F_k$ and $n \equiv m \pmod{2k}$, then $L_n \equiv L_m \pmod{t_k}$. The reader may readily verify this fact using a table of Lucas numbers for these pairs used in the proofs: $(2k, t_k) = (8,3)$, $(4,5)$, $(16,7)$, $(10,11)$, $(20,25)$, $(50,101)$, $(44,89)$. $(22,199)$, $(88,43)$, and $(88,307)$.

Lemma 1: If L_n is pronic, then $n \equiv 0 \pmod{100}$.

Proof: Assume that $4L_n+1$ is a square. Then $4L_n+1$ is a quadratic residue modulo 11 and modulo 25. However, we find that $4L_n+1$ is a quadratic residue modulo 11 only if $n\equiv 0$, 1, or 5 (mod 10), i.e., $n\equiv 0$, 1, 5, 10, 11, or 15 (mod 20), and modulo 25 only if $n\equiv 0$, 4, 8, 12, or 16 (mod 20). Hence, $n\equiv 0$ (mod 20), so $n\equiv 0$, ± 20, ± 40 (mod 100). Since $L_{-n}=L_n$ for n even, it suffices to show that $4L_n+1$ is not a quadratic residue modulo 101 for $n\equiv 20$ and 40 (mod 100). We find that the Jacobi symbol

$$(4L_{20}+1|101)=(10|101)=-1,$$

and

$$(4L_{40}+1|101)=(89|101)=-1.$$

Lemma 2: If L_n is pronic, then $n\equiv 0$ (mod 88).

Proof: Assume $4L_n+1$ is a square. Then $4L_n+1$ is a quadratic residue modulo t_k, for $t_k=3$, 5, and 7. However, the only integers n for which $4L_n+1$ is a quadratic residue modulo 3 *and* modulo 5 are $n\equiv 0$ and 5 (mod 8), and $4L_n+1$ is a quadratic nonresidue modulo 7 for $n\equiv 5$ and 13 (mod 16). Hence, $n\equiv 0$ (mod 8), so $n\equiv 0$, ± 8, ± 16, ± 24, ± 32, ± 40 (mod 88), and, as noted above, it suffices to show that $4L_n+1$ is not a quadratic residue for $n\equiv 8, 16, 24, 32$, and 40 (mod 88). We find that $(4L_8+1|307)=(189|307)$, $(4L_{16}+1|199)=(73|199)$, $(4L_{24}+1|43)=(37|43)$, $(4L_{32}+1|43)=(3|43)$, and $(4L_{40}+1|89)=(29|89)$. Each Jacobi symbol equals -1, implying that L_n is pronic only if $n\equiv 0$ (mod 88).

Lemma 3: If $n=kg$, g odd, then

$$L_n\equiv\begin{cases}L_k \pmod{L_{2k}}, & \text{if } g\equiv 1,3 \pmod 8,\\ -L_n \pmod{L_{2k}}, & \text{if } g\equiv 5,7 \pmod 8.\end{cases}$$

Proof: By (4),

$$L_n=L_{k(g-2)}L_{2k}-(-1)^{2k}L_{k(g-4)}\equiv -L_{k(g-4)} \pmod{L_{2k}};$$

hence,

$$L_n=L_{kg}\equiv -L_{k(g-4)}\equiv +L_{k(g-8)}\equiv\cdots\equiv\pm L_{\pm k}=\pm L_k \pmod{L_{2k}}.$$

It is readily seen that the positive sign occurs if and only if $g=1,3$ (mod 8). □

In the following proof, we shall use the facts that L_m is odd if and only if $3\nmid m$, and $L_{2^u m}\equiv -1$ (mod 8) if $u>1$ and $3\nmid m$.

Proof of the Theorem: If $n=0$, $L_n=L_0=2$, a pronic number. Conversely, assume L_n is a pronic number. By Lemmas 1 and 2, $n=2^u\cdot 5^2\cdot 11t$, $u\geq 3$. Now, if $n=kg$, $2^u|k$, $3\nmid k$, and g is odd, then, by Lemma 3,

$$\begin{aligned}(4L_n+1|L_{2k})&=(\pm 4L_k+1|L_{2k})=\pm(4L_k\pm 1|L_{2k})=(L_{2k}|4L_k\pm 1)\\&=(L_k^2-2|4L_k\pm 1)=(16L_k^2-32|4L_k\pm 1)\\&=((4L_k+1)(4L_k-1)-31|4L_k\pm 1)=(-31|4L_k\pm 1)\\&=\pm(31|4L_k\pm 1)=(4L_k\pm 1|31).\end{aligned}$$

Case 1: $t \equiv 5$ or 7 (mod 8). Let $k = 2^u \cdot 5^2$ and $g = 11t \equiv 7$ or 5 (mod 8). By Lemma 3, $L_n \equiv -L_k \pmod{L_{2k}}$. Now, $L_{2\cdot 5^2} \equiv -1 \pmod{31}$ and, by induction [using (2)], $L_{2^u\cdot 5^2} \equiv -1 \pmod{31}$. Hence,

$$(4L_n + 1 | L_{2k}) = (4L_k - 1 | 31) = (-5 | 31) = -1.$$

Case 2: $t \equiv 1$ or 3 (mod 8). If $4 \nmid u$, let $k = 2^u$ and $g = 5^2 \cdot 11t \equiv 3$ or 1 (mod 8); if $4 | u$, let $k = 2^u \cdot 11$ and $g = 5^2 t \equiv 1$ or 3 (mod 8). By Lemma 3, $L_n \equiv L_k \pmod{L_{2k}}$. Using (2), we find that $4L_{2^u} + 1 \equiv 25, 13, -2, 3 \pmod{31}$ for $u \equiv 0, 1, 2, 3 \pmod 4$, respectively. Then, if $4 \nmid u$,

$$(4L_n + 1 | L_{2k}) = (4L_{2^u} + 1 | 31) = (13 | 31), (-2 | 31), \text{ or } (3 | 31),$$

each of which equals -1.

Similarly, $4L_{2^u\cdot 11} + 1 \equiv -2, 3, 25, 13 \pmod{31}$ for $u \equiv 0, 1, 2, 3 \pmod 4$, respectively; hence, for $4 | u$,

$$(4L_n + 1 | L_{2k}) = (4L_{2^u\cdot 11} + 1 | 31) = (-2 | 31) = -1. \quad \square$$

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The author acknowledges the earlier publication of these results by Ming Luo [2] and wishes to express his appreciation to the anonymous referee and to the editor of this *Quarterly* for making them accessible to a wider audience.

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COMBINATORIAL EXPRESSIONS FOR LUCAS NUMBERS

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1. AIM OF THIS NOTE

Several closed-form expressions involving binomial coefficients exist for Lucas numbers. The most celebrated among them is the following specialization of *Waring's formula* (e.g., see (6) of [7]),

$$L_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i} \quad (n \geq 1), \tag{1.1}$$

where the symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function. Other combinatorial expressions for Lucas numbers are:

$$L_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 5^i \quad (n \geq 0) \quad \text{(e.g., see (4) of [7])}; \tag{1.2}$$

$$L_n = \sum_{i=-\lfloor (n+1)/5 \rfloor}^{\lfloor n/5 \rfloor} (-1)^i \frac{n + \lfloor (n-5i)/2 \rfloor}{n} \binom{n}{\lfloor (n-5i)/2 \rfloor} \quad (n \geq 1); \tag{1.3}$$

$$L_{2n} = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i}\binom{2n-i}{i} 5^{n-i} \quad (n \geq 1) \quad \text{(from (4.2) of [2])}. \tag{1.4}$$

A supposedly new combinatorial expression for odd-subscripted Lucas numbers is reported (without proof) in the Appendix.

Expression (1.3) was obtained by Robbins [7] on the basis of an analogous formula for Fibonacci numbers that was established by Andrews in [1].

As reported in (1.5) of [4], Jaiswal [3] discovered that

$$F_{n+3} = 1 + \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \binom{n-2i}{i} 2^{n-3i} \quad (n \geq 0). \tag{1.5}$$

The aim of this note is to parallel Robbins' work by using (1.5) to prove a new combinatorial expression for Lucas numbers that can be added to the above list.

2. ANOTHER COMBINATORIAL EXPRESSION FOR L_n

We discovered that

$$L_n = -1 + \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \frac{n}{n-2i}\binom{n-2i}{i} 2^{n-3i} \quad (n \geq 1). \tag{2.1}$$

Proof: Let us write

$$S_n \stackrel{\text{def}}{=} \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \frac{n}{n-2i}\binom{n-2i}{i} 2^{n-3i}$$

$$= \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \left[\binom{n-2i}{i} + 2\binom{n-1-2i}{i-1}\right] 2^{n-3i}$$

$$= F_{n+3} - 1 + 2\sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \binom{n-1-2i}{i-1} 2^{n-3i} \quad \text{[from (1.5)]}$$

$$= F_{n+3} - 1 + 2\sum_{j=-1}^{\lfloor n/3 \rfloor - 1} (-1)^{j+1} \binom{n-3-2j}{j} 2^{n-3-3j}. \tag{2.2}$$

Since $\lfloor n/3 \rfloor - 1 = \lfloor (n-3)/3 \rfloor$ and the binomial coefficient in (2.2) vanishes for $j = -1$ (see [6], p. 2), (2.2) can be rewritten as

$$S_n = F_{n+3} - 1 - 2\sum_{j=0}^{\lfloor (n-3)/3 \rfloor} (-1)^j \binom{n-3-2j}{j} 2^{n-3-3j}$$

$$= F_{n+3} - 2F_n + 1 = L_n + 1 \quad \text{[from (1.5)]}.$$

3. CONCLUDING REMARKS

Some simple divisibility and congruence properties of the Lucas numbers can be derived immediately from their closed-form expressions. For example, from (1.1), it can be seen that $L_p \equiv 1 \pmod{p}$ (p a prime), whereas, from (1.2), it is apparent that no Lucas number is divisible by 5.

From (2.1), it is evident that L_n is even iff $n \equiv 0 \pmod 3$. More precisely, it is not hard to see that

$$L_n \equiv 3^{1-r} x_r (-1)^{\lfloor n/3 \rfloor} - 1 \pmod{2^{r+3}}, \tag{3.1}$$

where r is the residue of n modulo 3, and

$$x_r = \begin{cases} 1 & \text{if } r = 0, \\ 2n(n+1)^{r-1} & \text{if } r \neq 0. \end{cases} \tag{3.2}$$

APPENDIX

The following combinatorial expression for odd-subscripted Lucas numbers emerges from a specialization of an expression for *generalized NSW numbers* (see [5], p. 288), a study of which is being undertaken by the author of this note. The interested reader might enjoy finding a proof for this expression:

$$L_{2n+1} = \frac{[1+(-1)^n](-1)^{n/2}}{2} + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n-1-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-2j} \quad (n \geq 0).$$

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A LAYMAN'S VIEW

of

Music of the Spheres

by Albert V. Carlin

When I contemplate
a page of symbols in mathematical array,
symmetrical and beautiful,
though I may not understand it all,
my mind rejoices
to think that here and now again
the human mind has come so far.
So far, to glimpse
the wondrous order
and balance of the Universe.

(Submitted by Herta T. Freitag, November 1997)

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A NOTE ON TWO THEOREMS OF MELHAM AND SHANNON

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1. INTRODUCTION AND PRELIMINARIES

In this note we use some properties of the Lucas sequences,

$$U_n(m, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(m, Q) = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha > \beta$, $m = \alpha + \beta$, and $Q = \alpha\beta$, to extend two theorems due to Melham and Shannon [3].

For the sequences defined above, it is known that

$$U_n[V_h(m, Q), Q^h] = U_{nh}(m, Q) / U_h(m, Q) \quad (h \neq 0) \tag{1.2}$$

and

$$V_n[V_h(m, Q), Q^h] = V_{nh}(m, Q). \tag{1.3}$$

In this note we are concerned with sequences where $Q = \pm 1$. In this case, for proofs of (1.2) and (1.3) in the literature see, for example, [1, p. 632]. In [3], Melham and Shannon proved that

$$\sum_{j=1}^{\infty} \frac{1}{U_{kj}(m, 1)U_{k(j+1)}(m, 1)} = \frac{1}{\alpha^k U_k^2(m, 1)} \quad (k \neq 0) \tag{1.4}$$

and

$$\sum_{j=0}^{\infty} \frac{1}{V_{kj}(m, 1)V_{k(j+1)}(m, 1)} = \frac{1}{2(\alpha - \beta)U_k(m, 1)}. \tag{1.5}$$

They evaluated analogous sums involving $U_n(m, -1)$ and $V_n(m, -1)$ only in the special case in which $m = 1$ (Fibonacci and Lucas numbers, see (3.9) and (3.10) of [3]). The aim of this note is to extend (1.4) and (1.5) to even-subscripted numbers $U_n(m, -1)$ and $V_n(m, -1)$, with m arbitrary, so that (3.9) and (3.10) of [3] will emerge as special cases of our results.

2. OUR RESULTS

Theorem 1:

$$\sum_{j=1}^{\infty} \frac{1}{U_{2kj}(m, -1)U_{2k(j+1)}(m, -1)} = \frac{1}{\alpha^{2k} U_{2k}^2(m, -1)} \quad (k \neq 0). \tag{2.1}$$

Theorem 2:

$$\sum_{j=0}^{\infty} \frac{1}{V_{2kj}(m, -1)V_{2k(j+1)}(m, -1)} = \frac{1}{2(\alpha - \beta)U_{2k}(m, -1)}. \tag{2.2}$$

Proof of Theorem 1: If we let $U_{kt}[V_2(m, -1), 1] = U_{kt}(\overline{m}, 1)$ with $\overline{m} = \gamma + \delta$, $\gamma\delta = 1$, $\gamma > \delta$, then (1.2) may be written as

$$U_{2kt}(m, -1) = U_2(m, -1) \cdot U_{kt}(\overline{m}, 1),$$

and it follows (for $t = 1, j$ and $j + 1$) that

$$\sum_{j=1}^{\infty} \frac{1}{U_{2kj}(m,-1)U_{2k(j+1)}(m,-1)} = \frac{1}{U_2^2(m,-1)} \sum_{j=1}^{\infty} \frac{1}{U_{kj}(\overline{m},1)U_{k(j+1)}(\overline{m},1)}$$

which, by (1.4) and (1.2),

$$= \frac{1}{U_2^2(m,-1)} \cdot \frac{1}{\gamma^k U_k^2(\overline{m},1)} = \frac{1}{\gamma^k U_{2k}^2(m,-1)}.$$

Now, since $\gamma + \delta = \overline{m} = V_2(m,-1) = \alpha^2 + \beta^2$, with $\alpha\beta = -1$, we have

$$\gamma + \frac{1}{\gamma} = \alpha^2 + \frac{1}{\alpha^2},$$

whence $\gamma = \alpha^2$. This completes the proof.

By using (1.3), the proof of Theorem 2 can be carried out in a similar way, so it is left as an exercise for the interested reader.

We shall conclude this note by working out some reciprocal sums emerging from particular choices of m in (2.1) and (2.2). If we let $m = 1$, we obtain (3.9) and (3.10) of [3], respectively. If we let $m = 2$, we obtain, respectively,

$$\sum_{j=1}^{\infty} \frac{1}{P_{2kj}P_{2k(j+1)}} = \frac{1}{P_{2k}^2(3+2\sqrt{2})^k} \tag{2.3}$$

and

$$\sum_{j=0}^{\infty} \frac{1}{Q_{2kj}Q_{2k(j+1)}} = \frac{1}{4\sqrt{2}P_{2k}}, \tag{2.4}$$

where P_k (resp. Q_k) denotes the kth Pell (resp. Pell-Lucas [2]) number.

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A CLASS OF SEQUENCES AND THE AITKEN TRANSFORMATION

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1. INTRODUCTION

In the notation of Horadam [1], let $W_n = W_n(a,b;\, p,q)$, where

$$W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2), \quad W_0 = a,\ W_1 = b. \tag{1.1}$$

If α and β are assumed distinct, then the roots of $\lambda^2 - p\lambda + q = 0$ have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.2}$$

in which $A = b - a\beta$ and $B = b - a\alpha$.

The nth terms of the Fibonacci and Lucas sequences are:

$$F_n = W_n(0,1;\, 1,-1); \quad L_n = W_n(2,1;\, 1,-1). \tag{1.3}$$

As usual, we write

$$U_n = W_n(0,1;\, p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = W_n(2,p;\, p,q) = \alpha^n + \beta^n, \tag{1.4}$$

where $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively, generated by (1.3). These sequences have been studied extensively, particularly by Lucas [3] and Horadam [1]. Throughout this paper, d is a natural number.

Define the Aitken transformation by

$$A(x, x', x'') = (xx'' - x'^2)/(x - 2x' + x''), \tag{1.5}$$

where the denominator is assumed to be nonzero.

In 1984, Phillips discovered the following relation between ratios of Fibonacci numbers and the Aitken transformation,

$$A(r_{n-t}, r_n, r_{n+t}) = r_{2n}, \tag{1.6}$$

where $r_n = F_{n+1}/F_n$. An account of this work is also given by Vajda in [3]. McCabe and Phillips [5] generalized this to show that (1.6) holds when $r_n = U_{n+1}/U_n$, and Muskat [7] showed that (1.6) holds for $r_n = U_{n+d}/U_n$. Jamieson [6] obtained the generalization

$$A(W_{i-t}^{(k)}, W_i^{(k)}, W_{i+t}^{(k)}) = \begin{cases} W_{2i}^{(2k)}, & 2k < p, \\ W_{2i}^{(2k-p)}, & 2k \geq p, \end{cases} \tag{1.7}$$

where $W_i^{(k)} = F_{p(i+1)-k}/F_{pi-k}$, $0 \leq k \leq p-1$.

The purpose of this paper is to establish a further generalization of these results.

2. THE MAIN RESULTS

First we introduce a new class of more general sequences that has not appeared previously in the literature.

Definition: The generalized Fibonacci sequence (GF-Sequence) is defined by

$$W_{n,d}^{(k)}(a,b;\,p,q)=\frac{A^k\alpha^{nk+d}-B^k\beta^{nk+d}}{\alpha-\beta}. \tag{2.1}$$

Thus, we have $F_n=W_{n,0}^{(1)}(0,1;\,1,-1)$, $U_n=W_{n,0}^{(1)}(0,1;\,p,q)$, and $W_n=W_{n,0}^{(1)}(a,b;\,p,q)$, and the GF-sequence $W_{n,d}^{(k)}(a,b;\,p,q)$ is seen to be an extension of these sequences.

We write $W_{n,d}^{(k)}$ for $W_{n,d}^{(k)}(a,b;\,p,q)$ and note that this sequence satisfies the recurrence relation

$$W_{n+1,d}^{(k)}=(\alpha^k+\beta^k)W_{n,d}^{(k)}-\alpha^k\beta^kW_{n-1,d}^{(k)},,$$

which has characteristic equation with roots α^k and β^k and generating function

$$\sum_{n=0}^{\infty}W_{n,d}^{(k)}t^n=\frac{A^k\alpha^d-B^k\beta^d-(A^k\alpha^d\beta^k-B^k\alpha^k\beta^d)t}{(\alpha-\beta)(1-(\alpha^k+\beta^k)t+\alpha^k\beta^kt^2)}=\frac{W_{0,d}^{(k)}-q^kW_{-1,d}^{(k)}t}{1-V_kt+q^kt^2}.$$

Introducing such a class of generalized Fibonacci sequences $W_{n,d}^{(k)}$, we can find a nice property between the appropriate ratios involving this sequence and Aitken acceleration.

If $W_{n,0}^{(k)}\neq 0$, we define the ratio

$$R_n^{(k)}=W_{n,d}^{(k)}/W_{n,0}^{(k)} \tag{2.2}$$

and state the main result of this paper.

Theorem:

$$A(R_{n-t}^{(k)},\,R_n^{(k)},\,R_{n+t}^{(k)})=R_n^{(2k)}. \tag{2.3}$$

3. LEMMA

For the proof of the Theorem, we introduce the following lemma.

Lemma:

(a) $W_{n+t,d}^{(k)}W_{n-t,d}^{(k)}-(W_{n,d}^{(k)})^2=-A^kB^kq^{(n-t)k+d}(U_{kt})^2,$ (3.1)

(b) $W_{n,0}^{(k)}W_{n-t,d}^{(k)}-W_{n,d}^{(k)}W_{n-t,0}^{(k)}=A^kB^kq^{(n-t)k}U_dU_{kt},$ (3.2)

(c) $W_{n,d}^{(k)}W_{n+t,0}^{(k)}-W_{n,0}^{(k)}W_{n+t,d}^{(k)}=A^kB^kq^{nk}U_dU_{kt},$ (3.3)

(d) $(W_{n,d}^{(k)})^2-q^d(W_{n,0}^{(k)})^2=U_dW_{n,d}^{(2k)},$ (3.4)

(e) $W_{n+t,0}^{(k)}-q^{kt}W_{n-t,0}^{(k)}=U_{kt}(A^k\alpha^{nk}+B^k\beta^{nk}).$ (3.5)

Proof: We prove only part (a) because the proofs of (b)-(e) are similar. Using the definition of $W_{n,d}^{(k)}$, we have

$$W_{n+t,d}^{(k)}W_{n-t,d}^{(k)}-(W_{n,d}^{(k)})^2$$

$$=\frac{A^k\alpha^{(n+t)k+d}-B^k\beta^{(n+t)k+d}}{\alpha-\beta}\,\frac{A^k\alpha^{(n-t)k+d}-B^k\beta^{(n-t)k+d}}{\alpha-\beta}-\left(\frac{A^k\alpha^{nk+d}-B^k\beta^{nk+d}}{\alpha-\beta}\right)^2$$

$$=-A^kB^kq^{(n-t)k+d}\left(\frac{\alpha^{tk}-\beta^{tk}}{\alpha-\beta}\right)^2=-A^kB^kq^{(n-t)k+d}(U_{kt})^2$$

and the proof of (a) is complete.

4. PROOF OF THE THEOREM

Using (1.5) and (2.2), we may write

$$A(R_{n-t}^{(k)},R_n^{(k)},R_{n+t}^{(k)})=\frac{R_{n-t}^{(k)}R_{n+t}^{(k)}-(R_n^{(k)})^2}{R_{n-t}^{(k)}-2R_n^{(k)}+R_{n+t}^{(k)}}=\frac{\dfrac{W_{n-t,d}^{(k)}W_{n+t,d}^{(k)}}{W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}}-\left(\dfrac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}\right)^2}{\dfrac{W_{n-t,d}^{(k)}}{W_{n-t,0}^{(k)}}-2\dfrac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}+\dfrac{W_{n+t,d}^{(k)}}{W_{n+t,0}^{(k)}}}$$

$$=\frac{(W_{n,0}^{(k)})^2W_{n-t,d}^{(k)}W_{n+t,d}^{(k)}-(W_{n,d}^{(k)})^2W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}}{(W_{n,0}^{(k)})^2W_{n-t,d}^{(k)}W_{n+t,0}^{(k)}-2W_{n,0}^{(k)}W_{n,d}^{(k)}W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}+(W_{n,0}^{(k)})^2W_{n+t,d}^{(k)}W_{n-t,0}^{(k)}}$$

$$=\frac{(W_{n,0}^{(k)})^2(W_{n-t,d}^{(k)}W_{n+t,d}^{(k)}-(W_{n,d}^{(k)})^2)-(W_{n,d}^{(k)})^2(W_{n-t,0}^{(k)}W_{n+t,0}^{(k)}-(W_{n,0}^{(k)})^2)}{W_{n,0}^{(k)}[W_{n+t,0}^{(k)}(W_{n,0}^{(k)}W_{n-t,d}^{(k)}-W_{n,d}^{(k)}W_{n-t,0}^{(k)})-W_{n-t,0}^{(k)}(W_{n,d}^{(k)}W_{n+t,0}^{(k)}-W_{n,0}^{(k)}W_{n+t,d}^{(k)})]}$$

$$=\frac{(W_{n,0}^{(k)})^2(-A^kB^kq^{(n-t)k+d})U_{kt}^2-(W_{n,d}^{(k)})^2(-A^kB^kq^{(n-t)k})U_{kt}^2}{W_{n,0}^{(k)}[W_{n+t,0}^{(k)}A^kB^kq^{(n-t)k}U_{kt}U_d-W_{n-t,0}^{(k)}A^kB^kq^{nk}U_{kt}U_d]}$$

$$=\frac{U_{kt}[(W_{n,d}^{(k)})^2-q^d(W_{n,0}^{(k)})^2]}{W_{n,0}^{(k)}U_d[W_{n+t,0}^{(k)}-q^{tk}W_{n-t,0}^{(k)}]},\text{ by (3.1) and (3.2),}$$

$$=\frac{U_{kt}U_dW_{n,d}^{(2k)}}{W_{n,0}^{(k)}U_dU_{kt}(A^k\alpha^{nk}+B^k\beta^{nk})},\text{ by (3.3), (3.4), and (3.5),}$$

$$=\frac{W_{n,d}^{(2k)}}{W_{n,0}^{(2k)}}=R_n^{(2k)}.$$

This completes the proof of the Theorem.

5. REMARK

There is a major difference between the result of this paper and those of other papers on this topic. In this paper, when the Aitken transformation is applied to the three numbers, $R_{n-t}^{(k)}$, $R_n^{(k)}$, and $R_{n+t}^{(k)}$, we obtain a doubling of k, giving $R_n^{(2k)}$. This contrasts with the results of all the other authors quoted, such as the relation $A(r_{n-t},r_n,r_{n+t})=r_{2n}$, where it is n that is doubled.

But, when $k = 1, a = 0$, and $b = 1$, we have $R_n^{(2)} = U_{2n+d} / U_{2n} = r_{2n}$. Thus, the result of this paper may be regarded as a further generalization of the former results.

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ASYMPTOTIC ESTIMATION OF A SUM OF DIGITS

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Let $s(k)$ denote the sum of the base 10 digits of $k \in \mathbf{N}$. For natural $x \geq 2$ and arbitrary fixed exponent $m \in \mathbf{N}$, it will be shown that

$$\frac{1}{x} \cdot \sum_{k=1}^{x-1} s(k)^m = \left(\frac{9}{2} \lg x\right)^m + O((\lg x)^{m-1}).$$

Here, "lg" denotes the base 10 log function. It is obvious that this formula can be generalized on arbitrary p-adic systems. The case $m = 1$ has been treated in [1], $m = 2$ in [2]; there the general case is exhibited as an *open problem*. The proof given now is based on induction.

I wish to thank Harald Scheid, University of Wuppertal, Germany, who drew my attention to certain unsolved arithmetical problems, the above among them.

1. THE ASSUMPTION

Let A_x for $x = 2, 3, \ldots$ be the arithmetic function

$$A_x(m) = \sum_{k=0}^{x-1} s(k)^m, \; m \in \mathbf{N}_0 \; (= \{0, 1, 2, \ldots\}).$$

I denote the above assertion in the following manner,

$$A_x(i) = x\left(\frac{9}{2} \lg x\right)^i + d_i(x) \cdot x(\lg x)^{i-1}, \; x \geq 2, \tag{1}$$

with certain bounded functions $d_i(x)$, i.e.,

$$|d_i(x)| \leq d_i \text{ for all } x, \tag{2}$$

and assume that it is valid for $i = 1, \ldots, m-1$. The validity for $i = 1$ is guaranteed in [1] and the validity for $i = m$ will be deduced now in several steps.

2. A REDUCTION FORMULA FOR A_{10x}

The *binomial product* $B * C$ of two arithmetical functions is defined by

$$B * C(m) = \sum_{k=0}^{m} \binom{m}{k} B(k) \cdot C(m-k).$$

First, I will show that

$$A_{10x} = A_{10} * A_x. \tag{3}$$

$$A_{10x}(m) = \sum_{k=0}^{x-1} \sum_{i=0}^{9} s(10k+i)^m = \sum_{k=0}^{x-1} \sum_{i=0}^{9} (s(k)+i)^m = \sum_{k=0}^{x-1} \sum_{i=0}^{9} \sum_{j=0}^{m} \binom{m}{j} s(k)^{m-j} i^j$$

$$= \sum_k \sum_j \left(s(k)^{m-j} \binom{m}{j} \sum_{i=0}^{9} i^j \right) = \sum_k \sum_j s(k)^{m-j} \binom{m}{j} A_{10}(j) = \sum_j \left(\binom{m}{j} A_{10}(j) \sum_k s(k)^{m-j} \right)$$

$$= \sum_j \binom{m}{j} A_{10}(j) \cdot A_x(m-j) = (A_{10} * A_x)(m).$$

3. ESTIMATION OF THE REMAINDER

Let x have the decomposition $10y+z$ with $z<10$. Suppose $R_x = A_x - A_{10y}$. In the case $z=0$ we have $R_x = 0$, otherwise

$$R_x(m) = \sum_{i=0}^{z-1} s(10y+i)^m.$$

If $n+1$ denotes the number of digits of x, then

$$R_x(m) \le z \cdot ((n+1)\cdot 9)^m \le 9^{m+1} \cdot (n+1)^m.$$

Let $(a_n \ldots a_0)$ be the decimal representation of x and $x_k = (a_n \ldots a_k)$ (especially $x_0 = x$, $x_n = a_n$), then, in particular,

$$R_{x_k}(m) \le 9^{m+1}(n-k+1)^m, \quad k = 0, 1, \ldots, n. \tag{4}$$

4. A DECOMPOSITION OF $A_x(m)$

One can verify immediately that

$$A_x = 10^n A_{x_n} + \sum_{k=1}^{n} (10^{k-1} A_{10x_k} - 10^k A_{x_k}) + \sum_{k=0}^{n-1} (10^k A_{x_k} - 10^k A_{10x_{k+1}})$$

$$\overset{(3)}{=} 10^n A_{a_n} + \sum_{k=1}^{n} 10^{k-1} (A_{10} * A_{x_k} - 10 A_{x_k}) + \sum_{k=0}^{n-1} 10^k R_{x_k},$$

$$A_x(m) = 10^n A_{a_n}(m) + \sum_{k=1}^{n} \left(10^{k-1} \sum_{i=1}^{m} \binom{m}{i} A_{10}(i) A_{x_k}(m-i) \right) + \sum_{k=0}^{n-1} 10^k R_{x_k}(m)$$

$$= \underbrace{10^n A_{a_n}(m)}_{U} + \underbrace{\sum_{i=1}^{m} \left(\underbrace{\binom{m}{i} \frac{A_{10}(i)}{10}}_{c_i} \underbrace{\sum_{k=1}^{n} 10^k A_{x_k}(m-i)}_{W_{m-i}} \right)}_{W} + \underbrace{\sum_{k=0}^{n-1} 10^k R_{x_k}(m)}_{V}$$

The expressions U, V, and W shall be treated now one after another.

5. ESTIMATION OF U AND V

$U = 10^n A_{a_n}(m) = 10^n R_{a_n}(m) \overset{(4)}{\le} 10^n \cdot 9^{m+1}$ and, since $10^n \le x < 10^{n+1}$, we have $U = O(x)$. Furthermore,

$$V = \sum_{k=0}^{n-1} 10^k R_{x_k}(m) \overset{(4)}{\le} 9^{m+1} \sum_{k=0}^{n-1} 10^k (n-k+1)^m.$$

Since the power series $\sum_k k^m z^k$ has radius of convergence 1, it is particularly convergent for $z = 1/10$; hence,

$$\sum_{k=0}^{n} 10^k (n-k+1)^m = 10^{n+1} \sum_{k=1}^{n+1} k^m \left(\frac{1}{10}\right)^k = O(x). \tag{5}$$

Thus, $V = O(x)$.

6. DECOMPOSITION AND ESTIMATION OF THE W_i

With respect to the assumption under induction, we obtain, for $i \le m-1$,

$$W_i = \sum_{i=1}^{n} 10^k A_{x_k}(i) = \sum_{k=1}^{n} 10^k \left(x_k \left(\frac{9}{2} \lg x_k\right)^i + d_i(x_k) \cdot x_k \cdot (\lg x_k)^{i-1} \right)$$

$$= \left(\frac{9}{2}\right)^i \underbrace{\sum_{k=1}^{n} 10^k x_k (\lg x_k)^i}_{G_i} + \underbrace{\sum_{k=1}^{n} d_i(x_k) \cdot 10^k x_k (\lg x_k)^{i-1}}_{G_i^*}.$$

Let $y_k = (a_k \ldots a_0)$. Then $10^k x_k = (\underbrace{a_n \ldots a_k 0 \ldots 0}_{n+1 \text{ digits}}) = x - (a_{k-1} \ldots a_0) = x - y_{k-1}$, so we have

$$G_i = \sum_{k=1}^{n} (x - y_{k-1})(\lg x_k)^i = x \sum_{k=1}^{n} (\lg x_k)^i - \sum_{k=1}^{n} y_{k-1} (\lg x_k)^i.$$

The two sums herein shall now be estimated separately:

a) We have $(n-k)^i = (\lg 10^{n-k})^i \le (\lg x_k)^i < (\lg 10^{n-k+1})^i = (n-k+1)^i$; hence,

$$\sum_{k=1}^{n} (n-k)^i \le \sum_{k} (\lg x_k)^i < \sum_{k=1}^{n} (n-k+1)^i = n^i + \sum_{k=1}^{n} (n-k)^i.$$

Since

$$\sum_{k=1}^{n} (n-k)^i = \sum_{k=1}^{n-1} k^i = \frac{n^{i+1}}{i+1} + O(n^i),$$

we see that, for arbitrary $i \in \mathbf{N}$,

$$\sum_{k=1}^{n} (\lg x_k)^i = \frac{n^{i+1}}{i+1} + O(n^i).$$

b) $\displaystyle\sum_{k=1}^{n} y_{k-1} (\lg x_k)^i \le \sum_{k} 10^k (\lg 10^{n-k+1})^i = \sum_{k} 10^k (n-k+1)^i \overset{(5)}{=} O(x).$

Putting the two parts together, we have

$$G_i = x \cdot \frac{n^{i+1}}{i+1} + O(x \cdot n^i),$$

particularly with respect to (2): $|G_i^*| \le d_i G_{i-1} = O(x \cdot n^i)$; therefore,

$$W_i = \left(\frac{9}{2}\right)^i G_i + G_i^* = \left(\frac{9}{2}\right)^i x \cdot \frac{n^{i+1}}{i+1} + O(x \cdot n^i) \text{ for all } i \le m-1.$$

Now it is easily seen that

$$W = \sum_{i=1}^{m} \binom{m}{i} \frac{A_{10}(i)}{10} W_{m-i} = m \frac{A_{10}(1)}{10} W_{m-1} + O(x \cdot n^{m-1})$$

$$= m \cdot \frac{9}{2} \cdot \left(\frac{9}{2}\right)^{m-1} x \cdot \frac{n^m}{m} + O(x \cdot n^{m-1}) = \left(\frac{9}{2}\right)^m x \cdot n^m + O(x \cdot n^{m-1}).$$

And, finally,

$$A_x(m) = \left(\frac{9}{2}\right)^m x \cdot n^m + O(x \cdot n^{m-1}).$$

From this, the initial assertion is deduced immediately.

Often a solved problem procreates a new problem. Here is an open question: Does the given asymptotic estimation hold even for arbitrary *real* $m \geq 1$? The reader is invited to prove or disprove this result.

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A SPARSE MATRIX AND THE CATALAN NUMBERS*

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1. INTRODUCTION

We shall consider a stack of r glass plates. A light ray comes from the upper left direction, reflecting at some inner boundary surfaces of the plates and passing through others. After repeated reflections and transmissions, the light ray goes away to the upper-right or the lower-right direction. How many possible paths are there in this case? The closed formulas for coefficients in the recurrent relations arising from the problem of enumeration of the possible reflection paths of light rays in the multiple glass plates were first given by J. A. Brooks (cf. [1, p. 271, eq. $T(n)$]). Using the signed ballot numbers $D(k, j)$, which are defined below, we can also obtain the formulas ([5, p. 385, eq. (3.17)]). A matrix $B = B^{(r)}$ constructed using the numbers $D(k, j)$ in a particular but natural manner indicates some interesting properties; for instance, "the sparseness" in the sense that the number of zero-elements of the matrix is maximum among the equivalent matrices. Let B^T be the transpose of B. Then the Catalan numbers (cf. [3]) appear in the matrix product of B^T and B.

The contents of this paper are regarded as continuations of [5]. For completeness, we will now summarize the results of [5].

Let A be an r by r matrix such that

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix}.$$

(This matrix arises when one enumerates the increased numbers of paths of light rays produced by an extra reflection from r plates, in an iterative scheme (cf. [5].)

Then we have

$$A^{-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

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Let $\underline{1}$ be a column vector of size r such that

$$\underline{1}^T = (1, 1, 1, \ldots, 1, 1, 1).$$

Then successive multiplications by A^{-1} give the following sequences:

$$\begin{aligned}
\underline{1}^T A^{-1} &= (0, \ldots, 0, D(1, 0)),\\
\underline{1}^T A^{-2} &= (D(2, 0), 0, \ldots, 0),\\
&\ldots,\\
&\ldots,\\
\underline{1}^T A^{-2m+1} &= (0, \ldots, 0, D(2m-1, 0), D(2m-1, 1), \ldots, D(2m-1, m-1)),\\
\underline{1}^T A^{-2m} &= (D(2m, m-1), D(2m, m-2), \ldots, D(2m, 0), 0, \ldots, 0),\\
&\ldots,
\end{aligned}$$

where

$$D(1, 0); D(2, 0); D(3, 0), D(3, 1); D(4, 0), D(4, 1); D(5, 0), D(5, 1), D(5, 2);$$
$$\ldots = 1; 1; -1, 1; -1, 2; 1, -3, 2; \ldots, \text{ respectively.}$$

From the process used to produce $D(k, j)$, we can obtain the following recurrence relations (cf. [5, p. 382, eqs. (2.1)-(2.3)]):

$$D(k, j) = \begin{cases} (-1)^k \{D(k-1, j) - D(k-1, j-1)\} & \text{for } 1 \le j \le \lfloor \frac{k-1}{2} \rfloor, \\ (-1)^{\lfloor \frac{k-1}{2} \rfloor} & \text{for } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ is the floor function of D. Knuth and represents the greatest integer less than or equal to x (see [4]). Hence, we can get a closed expression for the numbers $D(k, j)$ $(1 \le k; 0 \le j \le \lfloor (k-1)/2 \rfloor)$, namely,

$$D(k, j) = (-1)^{\lfloor \frac{k-1}{2} \rfloor + j} \frac{k-2j}{k} \binom{k}{j} \tag{1}$$

(cf. [5, p. 382, eq. (2.6)]). The ballot numbers can be expressed as

$$\text{bal}(k, j) = \frac{k-2j}{k} \binom{k}{j}$$

(cf. [2, p. 73]). So our numbers are called "signed ballot numbers." The Catalan numbers c_n are usually defined as

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

In particular, for both even and odd cases, if $k = 2k'$ and $k = 2k' + 1$, respectively, we have

$$D(2k', k'-1) = D(2k'+1, k') = \frac{1}{k'+1} \binom{2k'}{k'} = c_{k'}.$$

Hence, we can regard our numbers $\{D(k, j)\}$ as signed ballot numbers and, simultaneously, as a generalization of the Catalan numbers.

Let B be a matrix such that

$$B = (A^{-1}\underline{1}, A^{-2}\underline{1}, \ldots, A^{-(r-1)}\underline{1}, A^{-r}\underline{1}). \tag{2}$$

(In [5], we use the symbol B^T in place of B (see [5, p. 381]).

It can be shown that the Catalan numbers c_n and zeros appear alternately in the first row and the last row of B (cf. [5, p. 382, eq. (2.7)], and see B below for the case $r = 9$).

For $m = \ldots, -2, -1, 0, 1, 2, \ldots$, let us consider an associated set of linear equations, that is, $(A^{m-1}\underline{1}, A^{m-2}\underline{1}, \ldots, A^{m-r}\underline{1})\underline{x} = A^m\underline{1}$. (This $\underline{x}$ is the coefficient vector of the recurrent relations arising from the problem of light rays in multiple glass plates (cf. [5]).) Then the matrix B is the coefficients matrix for the case $m = 0$, from which we can obtain the solution $\underline{x} = B^{-1}\underline{1}$, where B is a nonsingular matrix because of (7) below.

Let $T_n = T_n^{(r)}$ be the total number of ray paths formed by the r plates after n reflections, and let $\underline{t} = \underline{t}^{(r)} = (T_{n-1}, T_{n-2}, \ldots, T_{n-r})^T$. It is shown in [1, p. 271] and [5, p. 385, eq. (3.17)] that

$$T_n = (B^{-1}\underline{1})^T\underline{t} = \sum_{j=1}^{r} (-1)^{\lfloor \frac{j-1}{2} \rfloor} \binom{\lfloor \frac{r-j}{2} \rfloor + j}{j} T_{n-j}.$$

For the (p, q) element $z_{p,q}$ of B^{-1}, we notice that the following are also valid:

$$\begin{cases} z_{2p',\, p'-1+m} = (-1)^{p'-1}\binom{2p'-2+m}{2p'-1} \cdots 1 \le p' \le \lfloor r/2 \rfloor, 1 \le m \le \lfloor r/2 \rfloor - p' + 1, \\ z_{2p'+1,\, \lfloor r/2 \rfloor + m} = (-1)^{p'}\binom{\lfloor (r+1)/2 \rfloor + p' - m}{2p'} \ldots 0 \le p' \le \lfloor (r-1)/2 \rfloor, 1 \le m \le \lfloor (r+1)/2 \rfloor - p', \\ z_{p,q} = 0 \ldots \text{otherwise.} \end{cases} \tag{3}$$

(See [5, eqs. (3.8)-(3.10)].) An algebraic manipulation yields

$$z_{p,q} = \begin{cases} (-1)^{p/2-1}\binom{p/2+q-1}{p-1} & \text{for } p \text{ even;} \quad p/2 \le q \le \lfloor r/2 \rfloor, \\ (-1)^{\lfloor p/2 \rfloor}\binom{r+\lfloor p/2 \rfloor - q}{p-1} & \text{for } p \text{ odd;} \quad \lfloor r/2 \rfloor + 1 \le q \le r - \lfloor p/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

For example, in the case $r = 9$, we have

$$B = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ 0 & 0 & 0 & -1 & 0 & -4 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 20 \\ 0 & 0 & -1 & 0 & -3 & 0 & -9 & 0 & -28 \\ 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \end{pmatrix}.$$

and

$$B^{-1}=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & -6 & -3 & -1 & 0 \\ 0 & -1 & -4 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. CATALAN NUMBERS IN B^TB

Now we will discuss further properties of B. For matrix B, computing B^TB, we have the Catalan numbers and zeros that run parallel to the skew-diagonal line. From the lower-left to the upper-right of B^TB, the numbers $c_0, c_1, \ldots, c_n$ appear on the first, the third, ..., and the $(2n+1)^{\text{st}}$ line, respectively; i.e., we have

$$B^TB=\begin{pmatrix} c_0 & 0 & c_1 & 0 & c_2 & \cdots & 0 & c_{r'} \\ 0 & c_1 & 0 & c_2 & 0 & \cdots & c_{r'} & 0 \\ c_1 & 0 & c_2 & 0 & c_3 & \cdots & 0 & c_{r'+1} \\ 0 & c_2 & 0 & c_3 & 0 & \cdots & c_{r'+1} & 0 \\ c_2 & 0 & c_3 & 0 & c_4 & \cdots & 0 & c_{r'+2} \\ 0 & c_3 & 0 & c_4 & 0 & \cdots & c_{r'+2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{r'} & 0 & c_{r'+1} & 0 & \cdots & c_{2r'-1} & 0 \\ c_{r'} & 0 & c_{r'+1} & 0 & c_{r'+2} & \cdots & 0 & c_{2r'} \end{pmatrix},$$

where $r=2r'+1$. In the case $r=2r'$, to obtain the expression B^TB, we have to delete the last row and the last column from the one above. All the odd skew-diagonal elements of order $2n+1$ running from the lower-left to the upper-right of B^TB are the Catalan number c_n, while those of even order are zero. Namely, we have the following theorem.

Theorem 1: For every k $(1 \le k \le r)$, it holds that

$$(B^TB)_{i,j}=\begin{cases} c_{k-1} & \text{for } (i,j)=(k+m,k-m) \text{ and } (k-m,k+m), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$m=\begin{cases} 0,1,\ldots,k-2 & \text{for } 2 \le k \le \lfloor \frac{r+1}{2} \rfloor, \\ 0,1,\ldots,r-k & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r. \end{cases}$$

Proof: From (2), consider an odd-skew-diagonal matrix element, we deal with the two cases simultaneously:

$$\begin{aligned}(B^TB)_{k\pm m,\,k\mp m} &= \underline{1}^T A^{-k\mp m} A^{-k\pm m}\underline{1} = \underline{1}^T A^{-2k}\underline{1} = \underline{1}^T A^{-2k+1}A^{-1}\underline{1} \\ &= (0,\ldots,0,D(2k-1,0),\ldots,D(2k-1,k-2),D(2k-1,k-1))(0,\ldots,0,c_0)^T \\ &= D(2k-1,k-1)c_0 = c_{k-1}.\end{aligned}$$

Next, consider an even-skew-diagonal matrix element:

$$(B^TB)_{k\pm(m+1),\,k\mp m} = \underline{1}^T A^{-k\mp(m+1)} A^{-k\pm m}\underline{1} = \underline{1}^T A^{-2k\mp 1}\underline{1}.$$

In the upper sign case, we have

$$\begin{aligned}
&= \underline{1}^T A^{-2k} A^{-1}\underline{1}\\
&= (D(2k, k-1), D(2k, k-2), \ldots, D(2k, 0), 0, \ldots, 0)(0, \ldots, 0, c_0)^T\\
&= 0,
\end{aligned}$$

where

$$m = \begin{cases} 0, 1, \ldots, (\cdot) & \text{for } 1 \le k \le \lfloor \frac{r+1}{2} \rfloor,\\ 0, 1, \ldots, r-k-1 & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r-1, \end{cases}$$

where

$$(\cdot) = \begin{cases} k-2 & \text{for } r = 2r'+1,\\ k-1 & \text{for } r = 2r'. \end{cases}$$

In the lower sign case, we have

$$\begin{aligned}
&= \underline{1}^T A^{-2k+2} A^{-1}\underline{1}\\
&= (D(2k-2, k-2), D(2k-2, k-3), \ldots, D(2k-2, 0), 0, \ldots, 0)(0, \ldots, 0, c_0)^T\\
&= 0,
\end{aligned}$$

where

$$m = \begin{cases} 0, 1, \ldots, k-2 & \text{for } 2 \le k \le \lfloor \frac{r+1}{2} \rfloor,\\ 0, 1, \ldots, r-k & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r. \end{cases}$$

This establishes Theorem 1.

As a corollary, we also have, from (1):

$$\begin{aligned}
(B^TB)_{k\pm m,\,k\mp m} &= \sum_{j=0}^{\lfloor (k-m-1)/2 \rfloor} D(k \pm m, m+j) D(k \mp m, j)\\
&= \frac{1}{k^2 - m^2} \sum_{j=0}^{\lfloor (k-m-1)/2 \rfloor} (k \mp m - 2j)^2 \binom{k \pm m}{k-j} \binom{k \mp m}{j}\\
&\equiv c_{k-1},
\end{aligned}$$

where

$$m = \begin{cases} 0, 1, \ldots, k-1 & \text{for } 1 \le k \le \lfloor \frac{r+1}{2} \rfloor,\\ 0, 1, \ldots, r-k & \text{for } \lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r. \end{cases}$$

This is a binomial identity for the Catalan numbers.

For example, in the case $r = 9$, we have

$$B^TB=\begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \\ 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 \\ 0 & 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 \\ 2 & 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 \\ 0 & 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 \\ 5 & 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 \\ 0 & 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 \\ 14 & 0 & 42 & 0 & 132 & 0 & 429 & 0 & 1430 \end{pmatrix}.$$

We may remark here that BB^T is a particular kind of block matrix, with symmetric blocks in the main diagonal. For example, in the case $r = 9$, we have

$$BB^T=\begin{pmatrix} 226 & -218 & 89 & -14 & 0 & 0 & 0 & 0 & 0 \\ -218 & 213 & -88 & 14 & 0 & 0 & 0 & 0 & 0 \\ 89 & -88 & 37 & -6 & 0 & 0 & 0 & 0 & 0 \\ -14 & 14 & -6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -7 & 20 & -28 & 14 \\ 0 & 0 & 0 & 0 & -7 & 50 & -145 & 205 & -103 \\ 0 & 0 & 0 & 0 & 20 & -145 & 426 & -608 & 307 \\ 0 & 0 & 0 & 0 & -28 & 205 & -608 & 875 & -444 \\ 0 & 0 & 0 & 0 & 14 & -103 & 307 & -444 & 227 \end{pmatrix}.$$

3. SPARSENESS OF B AND B^{-1}

We may call B "a sparse matrix" (for A) in the sense that, for a regular matrix A, it holds that

$$\max_{m:\,\text{integer}} n\{A^mB\} = n\{B\}$$

and, simultaneously, that

$$\max_{m:\,\text{integer}} n\{B^{-1}A^m\} = n\{B^{-1}\}$$

where $n\{M\}$ is the number of zero-elements of a matrix (or vector) M. We shall establish below that

$$n\{B\} = n\{B^{-1}\} = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right),$$
$$|B| = |B^{-1}| = (-1)^{\lfloor r/2 \rfloor}.$$

To prove these statements, we need the following lemma.

Lemma: For nonnegative integers $m \geq 0$, we have

$$n\{A^{-m}B\} = \begin{cases} \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right) - r\left\lfloor \tfrac{m}{2} \right\rfloor & \text{for } m = 2m', \\ \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 2\right) - r\left\lfloor \tfrac{m}{2} \right\rfloor & \text{for } m = 2m'+1. \end{cases} \tag{5}$$

Proof of the Lemma: From the expression $\underline{1}^TA^{-m}$ in Section 1, it follows by inspection that

$$n\{A^n\underline{1}\} = 0 \qquad \text{for all } n \geq 0,$$
$$n\{A^{-1}\underline{1}\} = r-1,$$
$$n\{A^{-2}\underline{1}\} = r-1,$$

$$n\{A^{-m}\underline{1}\} = r - \left\lfloor \tfrac{m+1}{2} \right\rfloor,$$
$$n\{A^{-m^*}\underline{1}\} = r - \left\lfloor \tfrac{m^*+1}{2} \right\rfloor = 0,$$

where $m^* = 2r - 1$. Hence, for $m \geq 0$, we have

$$\begin{aligned} n\{A^{-m}B\} &= n\{(A^{-m-1}\underline{1}, A^{-m-2}\underline{1}, \ldots, A^{-m-r}\underline{1})\} \\ &= \sum_{k=1}^{r} n\{A^{-m-k}\underline{1}\} = \sum_{k=1}^{r} \left(r - \left\lfloor \tfrac{m+k+1}{2} \right\rfloor\right). \end{aligned}$$

To establish the Lemma, we may calculate the last summation separately for the even and odd cases of both m and r.

First, in the case in which $m = 2m'$, we obtain the following results:

(i) When $r = 2r' + 1$, we get

$$\begin{aligned} n\{A^{-m}B\} &= r^2 + m' + r' + 1 - 2(m' + 1 + \cdots + m' + r' + m' + r' + 1) \\ &= 3r'^2 + 2r' - m'(2r' + 1) = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right) - r\left\lfloor \tfrac{m}{2} \right\rfloor. \end{aligned}$$

(ii) When $r = 2r'$, we get

$$\begin{aligned} n\{A^{-m}B\} &= r^2 - 2(m' + 1 + \cdots + m' + r') \\ &= 3r'^2 - r' - 2r'm' = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right) - r\left\lfloor \tfrac{m}{2} \right\rfloor. \end{aligned}$$

The case in which $m = 2m' + 1$ is derived in an analogous fashion, so we omit the discussion for brevity. This proves the Lemma.

We now have the following theorem.

Theorem 2:

(a) For the r by r matrix B, we have

$$n\{B\} = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right), \tag{6}$$

$$|B| = (-1)^{\left\lfloor \frac{r}{2} \right\rfloor}, \tag{7}$$

$$\max_{m:\,\text{integer}} n\{A^m B\} = n\{B\}. \tag{8}$$

(b) For the matrix B^{-1}, we have

$$n\{B^{-1}\} = \left\lfloor \tfrac{r}{2} \right\rfloor \left(3\left\lfloor \tfrac{r+1}{2} \right\rfloor - 1\right), \tag{9}$$

$$|B^{-1}| = (-1)^{\left\lfloor \frac{r}{2} \right\rfloor}, \tag{10}$$

$$\max_{m:\,\text{integer}} n\{B^{-1}A^m\} = n\{B^{-1}\}. \tag{11}$$

Proof of (a): In (5) of the Lemma, putting $m = 0$, we immediately have (6).

For (7), the proof is by induction. If $r = 2$, $B^{(2)}$ is a skew unit matrix of order 2. Hence, we have $|B^{(2)}| = -1$. Here, we note that in order to construct $B^{(r+1)}$ of order $r + 1$ from $B^{(r)}$ of order

r, we must affix the column vector $A^{-r-1}\underline{1}$ of size $r+1$ to $B^{(r)}$ as the last column, and also affix the row vector $(0, 0, \ldots, 0, D(r+1, 0))$ of size $r+1$ to $B^{(r)}$ as the central row. Using Laplace's expansion theorem, we have

$$|B^{(r+1)}| = (-1)^{\lfloor (r+2)/2 \rfloor + r+1} D(r+1, 0)|B^{(r)}| = (-1)^{\lfloor (r+1)/2 \rfloor}.$$

Thus, we have the desired result.

For (8), from (2) for l $(1 \le l \le r)$, we have

$$n\{A^l B\} = n\{(A^{l-1}\underline{1}, A^{l-2}\underline{1}, \ldots, \underline{1}, \ldots, A^{l-r}\underline{1})\}.$$

Since columns of $A^l B$ for which the exponent of A^{l-s} is nonnegative $(1 \le s \le l)$ have no zero-elements, we have

$$\begin{aligned} n\{B\} &> n\{AB\} > n\{A^2 B\} > \cdots > n\{A^r B\} \\ &= n\{A^{r+1}B\} = n\{A^{r+2}B\} = \cdots = 0. \end{aligned}$$

On the other hand, by virtue of the Lemma, we have

$$\begin{aligned} n\{B\} &> n\{A^{-1}B\} > n\{A^{-2}B\} > \cdots > n\{A^{-r^*+1}B\} \\ &= n\{A^{-r^*}B\} = n\{A^{-r^*-1}B\} = \cdots = 0, \end{aligned}$$

where

$$r^* = \begin{cases} 4r'+1 \ldots & \text{for } r = 2r'+1, \\ 4r'-1 \ldots & \text{for } r = 2r'. \end{cases}$$

This proves (a).

Proof of (b): For (9), from the available range of each subscript in the expression for the elements of B^{-1} [see (4) above], we can count the number $n\{B^{-1}\}$ of zero-elements of B^{-1}.

The validity of (10) follows from (7).

To establish (11), we must count the number of zero-elements of $B^{-1}A^m$. Let L, C, and R be the number of zero-elements of $B^{-1}A^m$ $(0 \le m \le r-1)$ in the left parts $(1 \le j \le \lfloor (r-m)/2 \rfloor)$, in the central parts $(\lfloor (r-m)/2 \rfloor + 1 \le j \le \lfloor (r-m)/2 \rfloor + m)$, and in the right parts $(\lfloor (r-m)/2 \rfloor + m + 1 \le j \le r)$, respectively, where j is a column number. Then we can easily obtain

$$\begin{aligned} L &= r\left\lfloor \tfrac{r-m}{2} \right\rfloor - \tfrac{1}{2}\left\lfloor \tfrac{r-m}{2} \right\rfloor\left(\left\lfloor \tfrac{r-m}{2} \right\rfloor + 1\right), \\ C &= 0, \\ R &= r\left\lfloor \tfrac{r-m+1}{2} \right\rfloor - \tfrac{1}{2}\left\lfloor \tfrac{r-m+1}{2} \right\rfloor\left(\left\lfloor \tfrac{r-m+1}{2} \right\rfloor + 1\right). \end{aligned}$$

Since, for a natural number n (see [4]), $n = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor$, we obtain

$$\begin{aligned} n\{B^{-1}A^{-m}\} &= L + C + R \\ &= \tfrac{1}{2}(r-m)(r+m-1) + \left\lfloor \tfrac{r-m}{2} \right\rfloor \left\lfloor \tfrac{r-m+1}{2} \right\rfloor. \end{aligned} \tag{12}$$

It is easy to observe that $n\{B^{-1}A^{-m}\}$ is a strictly decreasing function of m. On the other hand, it can be shown that

$$n\{B^{-1}A\} = \left\lfloor \tfrac{r+1}{2} \right\rfloor \left\lfloor \tfrac{r}{2} \right\rfloor + \tfrac{1}{2}\left\lfloor \tfrac{r-1}{2} \right\rfloor\left(\left\lfloor \tfrac{r-1}{2} \right\rfloor + 1\right) < n\{B^{-1}\}$$

and

$$n\{B^{-1}A^2\} = n\{B^{-1}A^3\} = \cdots = 0.$$

Hence, we get the following relation:

$$n\{B^{-1}\} > n\{B^{-1}A\} > n\{B^{-1}A^2\} = n\{B^{-1}A^3\} = \cdots = 0.$$

Thus, (11) is obtained. This completes the proof of (b).

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4. R. L. Graham, D. E. Knuth, & O. Patashnik. *Concrete Mathematics*. New York: Addison-Wesley Publishing Company, 1989.
5. Naotaka Imada. "A Sequence Arising from Reflections in Multiple Glass Plates." In *Applications of Fibonacci Numbers* **5**:379-86. Dordrecht: Kluwer, 1993.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stan@wwa.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

The Fibonacci polynomials, $F_n(x)$, and the Lucas polynomials, $L_n(x)$, satisfy

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x),\quad F_0(x) = 0,\quad F_1(x) = 1;$$
$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x),\quad L_0(x) = 2,\quad L_1(x) = x.$$

Also,

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} \text{ and } L_n(x) = \alpha(x)^n + \beta(x)^n,$$

where $\alpha(x) = (x + \sqrt{x^2+4})/2$ and $\beta(x) = (x - \sqrt{x^2+4})/2$.

PROBLEMS PROPOSED IN THIS ISSUE

Today's column is all about Fibonacci and Lucas polynomials, $F_n(x)$ and $L_n(x)$, which are defined above. For more information about Fibonacci polynomials, see Marjorie Bicknell, "A Primer for the Fibonacci Numbers: Part VII—An Introduction to Fibonacci Polynomials and Their Divisibility Properties," The Fibonacci Quarterly **8.4** *(1970):407-420.*

B-842 ***Proposed by the editor***

Prove that no Lucas polynomial is exactly divisible by $x-1$.

B-843 ***Proposed by R. Horace McNutt, Montreal, Canada***

Find the last three digits of $L_{1998}(114)$.

B-844 ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***

If $a+b$ is even and $a > b$, show that $[F_a(x)+F_b(x)][F_a(x)-F_b(x)] = F_{a+b}(x)F_{a-b}(x)$.

B-845 ***Proposed by Gene Ward Smith, Brunswick, ME***

Show that, if m and n are odd positive integers, then $L_n(L_m(x)) = L_m(L_n(x))$.

B-846 ***Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy***

Show that

$$\sum_{n=1}^{5} \frac{F_n(40k+1)}{n!}$$

is an integer for all integral k. Generalize.

B-847 ***Proposed by Gene Ward Smith, Brunswick, ME***

Find the greatest common polynomial divisor of $F_{n+4k}(x)+F_n(x)$ and $F_{n+4k-1}(x)+F_{n-1}(x)$.

B-837 (corrected) ***Proposed by Joseph J. Koštál, Chicago IL***

Let

$$P(x) = x^{1997} + x^{1996} + x^{1995} + \cdots + x^2 + x + 1$$

and let $R(x)$ be the remainder when $P(x)$ is divided by $x^2 - x - 1$. Show that $R(x)$ is divisible by L_{999}.

NOTE: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.

SOLUTIONS

It Keeps on Growing

B-826 ***Proposed by the editor***
(Vol. 35, no. 2, May 1997)

Find a recurrence consisting of positive integers such that each positive integer n occurs exactly n times.

Solution

All solvers selected the monotone sequence $\langle a_n \rangle = 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$, whose n^{th} term they found to be $a_n = \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor$.

The recurrences found were:

$$a_n = 1 + a_{n-a_{n-1}}; \qquad \textit{L. A. G. Dresel}$$

$$a_n = a_{n-1} + \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor - \left\lfloor \frac{1+\sqrt{8n-15}}{2} \right\rfloor; \qquad \textit{Gerald A. Heuer}$$

$$a_{n+1} = a_n + \left\lfloor \frac{1}{2n} \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor \left\lfloor \frac{\sqrt{8n+1}+1}{2} \right\rfloor \right\rfloor; \qquad \textit{H.-J. Seiffert}$$

$$a_n = 1 + a_{\lfloor (2n+3-\sqrt{8n+9})/2 \rfloor}; \qquad \textit{Reginald H. McNutt}$$

$$a_{n+1} = a_n + \begin{cases} 1, & \text{if } n \text{ is triangular,} \\ 0, & \text{otherwise;} \end{cases} \qquad \textit{Paul S. Bruckman}$$

each with initial condition $a_1 = 1$.

A Simple Third-Order Recurrence

B-827 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland (Vol. 35, no. 2, May 1997)***

Find a solution to the recurrence

$$A_{n+3} = A_n - 2A_{n+2},\ A_0 = 0,\ A_1 = 1,\ A_2 = -2,$$

in terms of F_n and L_n.

Solution by Graham Lord, Princeton, NJ

That $A_n = (-1)^{n-1}(F_{n+2} - 1)$ satisfies the recurrence is verified by substitution:

$$\begin{aligned}
A_{n+3} + 2A_{n+2} &= (-1)^{n+2}(F_{n+5} - 1) + 2(-1)^{n+1}(F_{n+4} - 1)\\
&= (-1)^{n+2}(F_{n+5} - 2F_{n+4}) - (-1)^{n+2} - 2(-1)^{n+1}\\
&= (-1)^{n+2}(F_{n+3} - F_{n+4}) + (-1)^{n+1} - 2(-1)^{n+1}\\
&= (-1)^{n+1}F_{n+2} - (-1)^{n+1}\\
&= (-1)^{n+1}(F_{n+2} - 1)\\
&= (-1)^{n-1}(F_{n+2} - 1)\\
&= A_n.
\end{aligned}$$

Also solved by Mohammad K. Azarian, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Gerald A. Heuer, Harris Kwong, Bob Prielipp, Maitland A. Rose, James A. Sellers, H.-J. Seiffert, I. Strazdins, and the proposer.

Semi Fibonacci

B-828 ***Proposed by Piero Filipponi, Rome, Italy (Vol. 35, no. 2, May 1997)***

For n a positive integer, prove that

$$\sum_{r=0}^{\lfloor \frac{n-1}{4} \rfloor} \binom{n-1-2r}{2r}$$

is within 1 of $F_n/2$.

Solution by H.-J. Seiffert, Berlin, Germany

Let n be a positive integer. It is well known ([2], p. 50) that

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} = F_n. \tag{1}$$

The formula ([1], p. 33)

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} (2\cos x)^{n-1-2k} = \frac{\sin nx}{\sin x}$$

when letting $x = \pi/3$ gives

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-1-k}{k} = \frac{2}{\sqrt{3}} \sin\left(\frac{n\pi}{3}\right), \tag{2}$$

since $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$. Adding equations (1) and (2) and dividing the resulting equation by 2 yields

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1+(-1)^k}{2}\binom{n-1-k}{k} = \frac{1}{2}F_n + \frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right)$$

or, equivalently,

$$\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n-1-2r}{2r} = \frac{1}{2}F_n + \frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right).$$

Thus, the desired sum differs from $F_n/2$ by at most $\frac{1}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right)$, which is less than 1.

The proposer also found the corresponding result for Lucas numbers:

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \frac{n}{n-2r}\binom{n-2r}{2r} = \begin{cases} (L_n+2)/2, & \text{if } n \equiv 0 \pmod 6, \\ (L_n+1)/2, & \text{if } n \equiv \pm 1 \pmod 6, \\ (L_n-1)/2, & \text{if } n \equiv \pm 2 \pmod 6, \\ (L_n-2)/2, & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

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1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products.* 5th ed. San Diego, CA: Academic Press, 1994.
2. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by David M. Bloom, Paul S. Bruckman, Leonard A. G. Dresel, Indulis Strazdins, and the proposer.

Powers of 2

B-829 ***Proposed by Jack G. Segers, Liège, Belgium (Vol. 35, no. 2, May 1997)***

For n a positive integer, let $P_n = F_{n+1}F_n$, $A_n = P_{n+1} - P_n$, $B_n = A_n - A_{n-1}$, $C_n = B_{n+1} - B_n$, $D_n = C_n - C_{n-1}$, and $E_n = D_{n+1} - D_n$. Show that $|P_n - B_n|$, $|A_n - C_n|$, $|B_n - D_n|$, and $|C_n - E_n|$ are successive powers of 2.

Solution by Harris Kwong, SUNY College at Fredonia, Fredonia, NY

We generalize the result as follows. Define an array of integers $S_{i,n}$ by $S_{0,n} = F_{n+1}F_n$, and, for $k \geq 1$,

$$S_{2k-1,n} = S_{2k-2,n+1} - S_{2k-2,n},$$
$$S_{2k,n} = S_{2k-1,n} - S_{2k-1,n-1}.$$

Note that $S_{0,n} = P_n$ and $S_{i,n}$, $1 \leq i \leq 5$, equals A_n, B_n, C_n, D_n, and E_n, respectively. We shall prove, by induction on i, that

$$S_{i,n} - S_{i+2,n} = (-1)^{n-1+\lceil i/2 \rceil}2^i.$$

We have

$$S_{1,n} = S_{0,n+1} - S_{0,n} = F_{n+2}F_{n+1} - F_{n+1}F_n = F_{n+1}(F_{n+2} - F_n) = F_{n+1}^2.$$

It follows that

$$S_{0,n} - S_{2,n} = F_{n+1}F_n - (F_{n+1}^2 - F_n^2) = F_n^2 - F_{n+1}(F_{n+1} - F_n) = F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1},$$

hence the assertion holds for $i = 0$. In general, assume it holds for some $i \geq 0$. If $i+1$ is odd, then

$$\begin{aligned} S_{i+1,n} - S_{i+3,n} &= (S_{i,n+1} - S_{i,n}) - (S_{i+2,n+1} - S_{i+2,n}) \\ &= (S_{i,n+1} - S_{i+2,n+1}) - (S_{i,n} - S_{i+2,n}) \\ &= (-1)^{n+\lceil i/2 \rceil} 2^i - (-1)^{n-1+\lceil i/2 \rceil} 2^i \\ &= (-1)^{n+\lceil i/2 \rceil} 2^{i+1} \\ &= (-1)^{n-1+\lceil (i+1)/2 \rceil} 2^{i+1}. \end{aligned}$$

The induction is completed by proving the case of even $i+1$ in a similar manner. Therefore, the absolute differences stated in the problem are 1, 2, 4, and 8, respectively.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Herta T. Freitag, Graham Lord, Bob Prielipp, H.-J. Seiffert, and the proposer.

Offset Entries

B-830 ***Proposed by Al Dorp, Edgemere, NY***
(Vol. 35, no. 2, May 1997)

(a) Prove that, if $n = 84$, then $(n+3) | F_n$.

(b) Find a positive integer n such that $(n+19) | F_n$.

(c) Is there an integer a such that $n + a$ never divides F_n?

Solution by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY

We use the following results:

Result 1 ([1], p. 37): If $d | n$, then $F_d | F_n$.

Result 2 ([1], p. 44): Every positive integer a divides some Fibonacci number F_n $(n > 0)$.

Result 3 ([2], p. 21): If $a | n$ and $b | n$, where a and b are relatively prime, then $ab | n$.

Result 4 ([2], p. 24): If a and b are relatively prime positive integers, then the arithmetic progression $\langle an + b \rangle$, $n = 1, 2, 3, \ldots$ contains infinitely many primes (Dirichlet's Theorem).

Result 5 ([3], p. 79): If the prime p is of the form $5t \pm 1$, then $p | F_{p-1}$.

(a) Since $3 | F_4$ and $29 | F_{14}$, we must have $3 | F_{84}$ and $29 | F_{84}$ by result 1. Thus, $87 | F_{84}$ by result 3.

(b) The integer $n = 2052 = 19 \cdot 108$ meets the conditions of part (b). For $19 | F_{18}$ and $109 | F_{27}$, so $19 | F_{19 \cdot 109}$ and $109 | F_{19 \cdot 109}$ by result 1. Thus, $(19 \cdot 108 + 19) | F_{19 \cdot 109}$ by result 3.

(c) The answer to part (c) is that, if a is any integer, then there must be a positive integer n such that $(n+a) | F_n$. For $a = 0$, $n = 5$ works. For $a < 0$, $n = 1 - a$ works.

If $a > 0$, there must be a positive integer b such that $a | F_b$ by result 2. By result 4, the arithmetic progression $10b + 1$, $20b + 1$, $30b + 1$, ... contains infinitely many primes, so there exists a prime $p = 10kb + 1$ such that $p > a$. Since $p > a$, it must be relatively prime to a. Since $p \equiv 1 \pmod{10}$, p divides $F_{p-1} = F_{10kb}$ by result 5. Thus, $p | F_n$, where $n = 10kba$ by result 1.

Likewise, $a|F_b$ implies $a|F_n$ by result 1. Finally, $n+1=(10kb+1)a=pa$, which divides F_n by result 3.

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.
2. Don Redmond. *Number Theory: An Introduction.* New York: Marcel Dekker, 1996.
3. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

For part (b), Bloom found $n=19\cdot 108$ *and Bruckman found* $n=19\cdot 180$. *Dresel removed the "0", finding that* $n=19\cdot 18$ *satisfies part (b).*

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, and the proposer.

Minimal Polynomial

B-831 ***Proposed by the editor***
(Vol. 35, no. 3, August 1997)

Find a polynomial $f(x, y)$ with integer coefficients such that $f(F_n, L_n)=0$ for all integers n.

Solution

All solvers came up with

$$f(x,y)=(y^2-5x^2-4)(y^2-5x^2+4)=25x^4-10x^2y^2+y^4-16$$

essentially by the same method; namely, squaring the fundamental identity

$$L_n^2-5F_n^2=4(-1)^n,$$

which is Hoggatt's identity (I_{12}) from [1].

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Bob Prielipp, H.-J. Seiffert, Indulis Strazdins, and the proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-536 ***Proposed by Paul S. Bruckman, Highwood, IL***

Given an odd prime p, integers n and r with $n \geq 1$, let $m = 2\left[\frac{1}{2}n\right] - 1$,

$$S_{n,r,p} = \sum_{k=1}^{p-1} F_m^k \cdot \frac{F_{nk+r}}{k}, \quad T_{n,r,p} = \sum_{k=1}^{p-1} F_m^k \cdot \frac{L_{nk+r}}{k}.$$

Prove the following congruences:

(a) $S_{n,r,p} \equiv \dfrac{F_n^p F_{mp+r} - F_m^p F_{np+r} + F_r}{p} \pmod{p}$;

(b) $T_{n,r,p} \equiv \dfrac{F_n^p L_{mp+r} - F_m^p L_{np+r} + L_r}{p} \pmod{p}$.

H-537 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let $\langle w_n \rangle$ be any sequence satisfying the recurrence

$$w_{n+2} = Pw_{n+1} - Qw_n.$$

Let $e = w_0 w_2 - w_1^2$ and assume $e \neq 0$ and $Q \neq 0$.

Computer experiments suggest the following formula, where k is an integer larger than 1:

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_{k-i} \binom{k}{i} (-1)^i w_n^i w_{n+1}^{k-i},$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}.$$

Prove or disprove this conjecture.

H-538 ***Proposed by Paul S. Bruckman, Highwood, IL***

Define the sequence of integers $(B_k)_{k \geq 0}$ by the generating function:

$$(1-x)^{-1}(1+x)^{-\frac{1}{2}} = \sum_{k\geq 0} B_k \frac{\left(\frac{1}{2}x\right)^k}{k!}, \quad |x|<1 \text{ (see [1])}.$$

Show that

$$\sum_{k\geq 0} B_k^2 \cdot \frac{1}{(2k+2)!} = \frac{\pi^2}{8} - \frac{1}{4}\log^2 u, \text{ where } u = 1+\sqrt{2}.$$

Reference

1. P. S. Bruckman. "An Interesting Sequence of Numbers Derived from Various Generating Functions." *The Fibonacci Quarterly* **10.2** (1972):169-81.

SOLUTIONS

Find Your Identity

H-518 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 34, no. 5, November 1996)***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(y) = (x-y)^{n-1}F_n\left(\frac{xy+4}{x-y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$\sum_{\substack{k=0\\ 5\nmid 2n-k-1}}^{2n-1}(-1)^{[(2n-k+1)/5]}\binom{4n-2}{k} = 5^{n-1}L_{2n-1}; \tag{2}$$

$$\sum_{\substack{k=0\\ 5\nmid 2n-k}}^{2n}(-1)^{[(2n-k+2)/5]}\binom{4n}{k} = 5^n F_{2n}; \tag{3}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{3k}P_k = 2^n F_n(6), \text{ where } P_k = F_k(2) \text{ is the } k^{\text{th}} \text{ Pell number}; \tag{4}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)F_k(x+1) = F_n(x^2+x+4); \tag{5}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)F_k(4/x) = \frac{1-(-1)^n}{2}\left(\frac{x^2+4}{x}\right)^{n-1}, \quad x\neq 0; \tag{6}$$

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_k(x)^2 = (x^2+4)^{n-1}; \tag{7}$$

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(x)^2 = \frac{4^n-(-x^2)^n}{4+x^2}; \tag{8}$$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{2n}{n-2k-1}F_{2k+1}(x) = x^{n-1}F_n(4/x). \tag{9}$$

The latter equation is the one given in H-500. **Hint:** Deduce (1) from the main identity of H-492.

Solution by the proposer

Proof of (1): From H-492, we know that

$$\sum_{k=0}^{[n/2]}\binom{n}{k}F_{n-2k}(x)F_{n-2k}(y)=z^{n-1}F_n(xy/z),$$

where $z=\sqrt{x^2+y^2+4}$. Replacing n by $2n$ and substituting k by $n-k$ gives

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_{2k}(x)F_{2k}(y)=z^{2n-1}F_{2n}(xy/z).$$

Using $F_{2k}(x)=i^{1-k}xF_k(i(x^2+2))$, $i=\sqrt{(-1)}$, we get

$$xy\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_k(i(x^2+2))\,F_k(i(y^2+2))=i^{1-n}xyz^{2n-2}F_n(i((xy/z)^2+2)).$$

Now, we replace x by $i\sqrt{2+ix}$ and y by $i\sqrt{2-iy}$, so that z becomes $\sqrt{i(y-x)}$. Then, using $(-1)^{k+1}F_k(-y)=F_k(y)$ and some elementary calculations, we obtain (1).

Proof of (2) and (3): Let $x=i\alpha$ and $y=i\beta$. In [1] it was shown that

$$F_k(i\alpha)F_k(i\beta)=\begin{cases}(-1)^{[(k+2)/5]} & \text{if } 5\nmid k,\\ 0 & \text{if } 5\mid k,\end{cases}$$

so that by (1),

$$\sum_{\substack{k=0\\5\nmid k}}^{n}(-1)^{[(k+2)/5]}\binom{2n}{n-k}=(i\sqrt{5})^{n-1}F_n(-i\sqrt{5}).$$

Replacing n by $2n-1$, using $F_{2n-1}(-i\sqrt{5})=(-1)^{n-1}L_{2n-1}$, and reindexing k by $2n-k-1$, we find (2).

Replacing n by $2n$, using $iF_{2n}(-i\sqrt{5})=(-1)^{n-1}\sqrt{5}F_{2n}$, and substituting k by $2n-k$ gives (3).

Proof of (4): This follows from (1) by taking $x=4$, $y=2$, and using $F_k(4)=F_{3k}/2$.

Proof of (5): Take $y=x+1$. We note the particular case,

$$\sum_{k=0}^{n}\binom{2n}{n-k}F_kP_k=F_n(6),$$

obtained when $x=1$.

Proof of (6): Take $y=-4/x$, use $F_k(-4/x)=(-1)^{k+1}F_k(4/x)$ and $F_n(0)=(1-(-1)^n)/2$. Then, with $x=1$, we obtain

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}F_kF_{3k}=(1-(-1)^n)5^{n-1}.$$

Proof of (7): Take $y=x$.

Proof of (8): Take $y=-x$ and use

$$(2x)^{n-1}F_n\left(\frac{4-x^2}{2x}\right)=\frac{4^n-(-x^2)^n}{4+x^2},$$

which easily follows from the well-known Binet form of the Fibonacci polynomials. With $x=2$, we get

$$\sum_{k=0}^{n}(-1)^{k+1}\binom{2n}{n-k}P_k^2=(1-(-1)^n)2^{2n-3}.$$

Proof of (9): Take $y=0$.

Reference

1. N. Jensen. "Solution of H-492." *The Fibonacci Quarterly* **34.1** (1996):91-96.

Also solved by P. Bruckman.

Squares among US

H-520 ***Proposed by Andrej Dujella, University of Zagreb, Croatia (Vol. 34, no. 5, November 1996)***

Let n be an integer. Prove that there exists an infinite set $D\subseteq\mathbf{N}$ with the property that, for all $c,d\in D$, the integer $cd+n$ is not squarefree.

Solution by David Terr, University of California at Berkeley, CA

We claim that, for all n, an arithmetic sequence

$$D=\{kp^2+a\,|\,k\in\mathbf{N}\}$$

satisfying the above property exists, where p is a prime and $a<p^2/2$ is a nonnegative integer. If $4|n$, we may choose $p=2$ and $a=0$. If $n\equiv 3 \pmod 4$, we may choose $p=2$ and $a=1$. Finally, if $n\equiv 1$ or $2 \pmod 4$, we choose p to be an odd prime such that $\left(\frac{-n}{p}\right)=1$ and find a nonnegative integer $a<p^2/2$ such that $a^2\equiv -n(p^2)$. By Hensel's lemma, such an a exists and is unique.

To see that D satisfies the above property, first consider the case in which $4|n$. In this case, $D=\{4k\,|\,k\in\mathbf{N}\}$, so if $c,d\in D$, we have $c=4k$ and $d=4l$ for some $k,l\in\mathbf{N}$, whence $cd+n=16kl+n$, which is divisible by 4 and, thus, not squarefree.

Next, consider the case in which $n\equiv 3 \pmod 4$. In this case, $D=\{4k+1\,|\,k\in\mathbf{N}\}$, so if $c,d\in D$, we have $c=4k+1$ and $d=4l+1$ for some $k,l\in\mathbf{N}$, whence $cd+n=16kl+4(k+l)+1+n$, which is again divisible by 4 and, thus, not squarefree.

Finally, consider the case in which $n\equiv 1$ or $2 \pmod 4$. In this case, $D=\{kp^2+a\,|\,k\in\mathbf{N}\}$ for some odd prime p and some nonnegative integer $a<p^2/2$ such that $p^2|(a^2+n)$. If $c,d\in D$, we have $c=kp^2+a$ and $d=lp^2+a$ for some $k,l\in\mathbf{N}$, whence $cd+n=klp^4+a(k+l)p^2+a^2+n$, which is divisible by p^2 and, thus, not squarefree. □

The following table lists the values of p and a found by this method for $|n|\le 10$.

n	p	a
−10	3	1
−9	2	1
−8	2	0
−7	3	4
−6	5	9
−5	2	1
−4	2	0
−3	11	27
−2	7	10
−1	2	1

n	p	a
0	2	0
1	5	7
2	3	4
3	2	1
4	2	0
5	3	2
6	5	12
7	2	1
8	2	0
9	5	4
10	7	23

Also solved by B. Beasley, P. Bruckman, and the proposer.

Zeroing In

H-521 ***Proposed by Paul S. Bruckman, Highland, IL***
(Vol. 35, no. 1, February 1997)

Let ρ denote any zero of the Riemann Zeta Function $\zeta(z)$ lying in the strip

$$S = \{z \in C : 0 < \mathrm{Re}(z) < 1\}.$$

Prove the following:

(1) $\displaystyle\sum_{\rho \in S} \left(\rho - \frac{1}{2}\right)^{-1} = 0;$

(2) $\displaystyle\sum_{\rho \in S} \rho^{-1} = 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 4\pi$, where γ is Euler's Constant.

Solution by Kee-Wai Lau, Hong Kong

Proof of (1): It is well known that the zeros are in conjugate pairs. They either lie on the line $\mathrm{Re}\, z = \frac{1}{2}$ or occur in pairs symmetrical about this line. If $\mathrm{Re}\,\rho = \frac{1}{2}$, we have

$$\frac{1}{\rho - \frac{1}{2}} + \frac{1}{\overline{\rho} - \frac{1}{2}} = 0.$$

If $\mathrm{Re}\,\rho \neq \frac{1}{2}$, then ρ is a zero if and only if $\overline{\rho}$, $1-\rho$, and $1-\overline{\rho}$ are zeros, and we have

$$\frac{1}{\rho - \frac{1}{2}} + \frac{1}{\overline{\rho} - \frac{1}{2}} + \frac{1}{(1-\rho) - \frac{1}{2}} + \frac{1}{(1-\overline{\rho}) - \frac{1}{2}} = 0. \quad \square$$

Proof of (2): It is known (see [1], Formula 2.12.7, p. 31) that

$$\frac{\zeta'(z)}{\zeta(z)\gamma} = \log 2\pi - 1 - \frac{1}{2}\gamma - \frac{1}{z-1} - \frac{1}{2}\frac{\Gamma'((z/2)+1)}{\Gamma((z/2)+1)} + \sum_{\rho \in S}\left(\frac{1}{z-\rho} + \frac{1}{\rho}\right), \qquad (*)$$

where Γ is the Gamma function.

It is also known (see [1], p. 20) that

$$-\frac{\zeta'(1-z)}{\zeta(1-z)} = -\log 2\pi - \frac{1}{2}\pi\tan\frac{1}{2}z\pi + \frac{\Gamma'(z)}{\Gamma(z)} + \frac{\zeta'(z)}{\zeta(z)}. \qquad (**)$$

By substituting $z = \frac{1}{2}$ into $(*)$ and $(**)$ and making use of (1) we obtain, after some algebra,

$$\sum_{\rho\in S}\frac{1}{\rho} = \frac{1}{2}\frac{\Gamma'(5/4)}{\Gamma(5/4)} - 1 + \frac{1}{2}\gamma - \frac{1}{2}\log 2\pi + \frac{\pi}{4} - \frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)}.$$

Since

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\gamma - 2\log 2,$$

in order to prove (2) it remains to show that

$$\frac{\Gamma'(5/4)}{\Gamma(5/4)} = -\gamma - 3\log 2 - \frac{\pi}{2} + 4.$$

In fact, by substituting $z = \frac{1}{4}$ into the duplication formula

$$\frac{\Gamma'(2z)}{\Gamma(2z)} = \frac{1}{2}\frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{2}\frac{\Gamma'(z+(1/2))}{\Gamma(z+(1/2))} + \log 2$$

and into the reflection formula

$$\frac{\Gamma'(1-z)}{\Gamma(1-z)} = \frac{\Gamma'(z)}{\Gamma(z)} + \pi\cot\pi z,$$

we easily obtain

$$\frac{\Gamma'(1/4)}{\Gamma(1/4)} = -\gamma - 3\log 2 - \frac{\pi}{2}.$$

The result for $\frac{\Gamma'(5/4)}{\Gamma(5/4)}$ now follows by substituting $z = \frac{1}{4}$ into the recurrence formula

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z}. \quad \square$$

This completes the solution of the problem.

Reference

1. E. C. Titchmarch. *The Theory of the Riemann Zeta-Function*. 2nd ed. Oxford: Clarendon Press, 1986.

Also solved by.-J. Seiffert and the proposer.

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