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ON THE SQUARE ROOTS OF TRIANGULAR NUMBERS

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1. BALANCING NUMBERS

We call an integer $n \in \mathbf{Z}^+$ a *balancing number* if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1}$$

for some $r \in \mathbf{Z}^+$. Here r is called the *balancer* corresponding to the balancing number n.

For example, 6, 35, and 204 are balancing numbers with balancers 2, 14, and 84, respectively. It follows from (1) that, if n is a balancing number with balancer r, then

$$n^2 = \frac{(n+r)(n+r+1)}{2} \tag{2}$$

and thus

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 1}}{2}. \tag{3}$$

It is clear from (2) that n is a balancing number if and only if n^2 is a triangular number (cf. [2], p. 3). Also, it follows from (3) that n is a balancing number if and only if $8n^2 + 1$ is a perfect square.

2. FUNCTIONS GENERATING BALANCING NUMBERS

In this section we introduce some functions that generate balancing numbers. For any balancing number x, we consider the following functions:

$$F(x) = 2x\sqrt{8x^2 + 1}, \tag{4}$$

$$G(x) = 3x + \sqrt{8x^2 + 1}, \tag{5}$$

$$H(x) = 17x + 6\sqrt{8x^2 + 1}. \tag{6}$$

First, we prove that the above functions always generate balancing numbers.

Theorem 2.1: For any balancing number x, $F(x)$, $G(x)$, and $H(x)$ are also balancing numbers.

Proof: Since x is a balancing number, $8x^2 + 1$ is a perfect square, and

$$\frac{8x^2(8x^2 + 1)}{2} = 4x^2(8x^2 + 1)$$

is a triangular number which is also a perfect square; therefore, its square root $2x\sqrt{8x^2 + 1}$ is a (an even) balancing number. Thus, for any given balancing number x, $F(x)$ is an even balancing number. Since $8x^2 + 1$ is a perfect square, it follows that

$$8(G(x))^2 + 1 = (8x + 3\sqrt{8x^2 + 1})^2$$

is also a perfect square; hence, $G(x)$ is a balancing number. Again, since $G(G(x)) = H(x)$, it follows that $H(x)$ is also a balancing number. This completes the proof of Theorem 2.1.

It is important to note that, if x is any balancing number, then $F(x)$ is always even, whereas $G(x)$ is even when x is odd and $G(x)$ is odd when x is even. Thus, if x is any balancing number, then $G(F(x))$ is an odd balancing number. But

$$G(F(x)) = 6x\sqrt{8x^2+1} + 16x^2 + 1.$$

The above discussion proves the following result.

Theorem 2.2: If x is any balancing number, then

$$K(x) = 6x\sqrt{8x^2+1} + 16x^2 + 1 \qquad (7)$$

is an odd balancing number.

3. FINDING THE NEXT BALANCING NUMBER

In the previous section, we showed that $F(x)$ generates only even balancing numbers, whereas $K(x)$ generates only odd balancing numbers. But $H(x)$ and $K(x)$ generate both even and odd balancing numbers. Since $H(6) = 204$ and there is a balancing number 35 between 6 and 204, it is clear that $H(x)$ does not generate the next balancing number for any given balancing number x. Now the question arises: "Does $G(x)$ generate the next balancing number for any given balancing number x?" The answer to this question is affirmative. More precisely, if x is any balancing number, then the next balancing number is $3x + \sqrt{8x^2+1}$ and, consequently, the previous one is $3x - \sqrt{8x^2+1}$.

Theorem 3.1: If x is any balancing number, then there is no balancing number y such that $x < y < 3x + \sqrt{8x^2+1}$.

Proof: The function $G:[0,\infty) \to [1,\infty)$, defined by $G(x) = 3x + \sqrt{8x^2+1}$, is strictly increasing since

$$G'(x) = 3 + \frac{8x}{\sqrt{8x^2+1}} > 0.$$

Also, it is clear that G is bijective and $x < G(x)$ for all $x \ge 0$. Thus, G^{-1} exists and is also strictly increasing with $G^{-1}(x) < x$. Let $u = G^{-1}(x)$. Then $G(u) = x$ and $u = 3x \pm \sqrt{8x^2+1}$. Since $u < x$, we have $u = 3x - \sqrt{8x^2+1}$. Also, since $8(G^{-1}(x))^2 + 1 = (8x - 3\sqrt{8x^2+1})^2$ is a perfect square, it follows that $G^{-1}(x)$ is also a balancing number.

Now we can complete the proof in two ways. The first is by the *method of induction*; the second is by the *method of infinite descent* used by Fermat ([2], p. 228).

By induction: We define $B_0 = 1$ (the reason is that $8 \cdot 1^2 + 1 = 9$ is a perfect square) and $B_n = G(B_{n-1})$ for $n = 1, 2, \ldots$. Thus, $B_1 = 6$, $B_2 = 35$, and so on. Let H_i be the hypothesis that there is no balancing number between B_{i-1} and B_i. Clearly, H_1 is true. Assume H_i is true for $i = 1, 2, \ldots, n$. We shall prove that H_{n+1} is true, i.e., there is no balancing number y such that $B_n < y < B_{n+1}$. Assume, to the contrary, that such a y exists. Then $G^{-1}(y)$ is a balancing number, and since G^{-1} is strictly increasing, it follows that $G^{-1}(B_n) < G^{-1}(y) < G^{-1}(B_{n+1})$, i.e., $B_{n-1} < G^{-1}(y) < B_n$, which is a contradiction to the assumption that H_n is true. So H_{n+1} is also true. Thus, if x is a balancing number, then $x = B_n$ for some n and there is no balancing number between x and $G(x)$.

By the method of infinite descent: Here assume H_n is false for some n. Then there exists a balancing number y such that $B_{n-1} < y < B_n$, and this implies that $B_{n-2} < G^{-1}(y) < B_{n-1}$. Finally, this would imply that there exists a balancing number B between B_0 and B_1, which is false. Thus, H_n is true for $n = 1, 2, \dots$.

This completes the proof of Theorem 3.1.

Corollary 3.2: If x is any balancing number, then its previous balancing number is $3x - \sqrt{8x^2 + 1}$.

Proof: $G(3x - \sqrt{8x^2 + 1}) = x$.

4. ANOTHER FUNCTION GENERATING BALANCING NUMBERS

In this section we develop a function $f(x, y)$ of two variables generating balancing numbers such that all the functions $F(x)$, $G(x)$, $H(x)$, and $K(x)$ are obtained as particular cases of this function.

Let x be any balancing number. We try to find balancing numbers of the form

$$B = px + q\sqrt{8x^2 + 1},$$

where $p, q \in \mathbf{Z}^+$. In the previous section we have seen that most of the balancing numbers are of this form. Since B is a balancing number, $8B^2 + 1 = (8qx + p\sqrt{8x^2 + 1})^2 + 8q^2 - p^2 + 1$ must be a perfect square; this happens if $8q^2 - p^2 + 1 = 0$, i.e., $p = \sqrt{8q^2 + 1}$. Since $p \in \mathbf{Z}^+$, it follows that $8q^2 + 1$ must be a perfect square, and this is possible if q is a balancing number.

The above discussion proves the following theorem.

Theorem 4.1: If x and y are balancing numbers, then

$$f(x, y) = x\sqrt{8y^2 + 1} + y\sqrt{8x^2 + 1} \tag{8}$$

is also a balancing number.

Remark 4.2: *(a)* $f(x, x) = F(x)$; *(b)* $f(x, 1) = G(x)$; *(c)* $f(x, 6) = H(x)$; *(d)* $f(x, G(x)) = K(x)$.

5. RECURRENCE RELATIONS FOR BALANCING NUMBERS

We know that $B_1 = 6$, $B_2 = 35$, $B_3 = 204$, and so on. We have already assumed that $B_0 = 1$. In Section 3 we proved that, if B_n is the nth balancing number, then

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1} \quad \text{and} \quad B_{n-1} = 3B_n - \sqrt{8B_n^2 + 1}.$$

It is clear that the balancing numbers obey the following recurrence relation:

$$B_{n+1} = 6B_n - B_{n-1}. \tag{9}$$

Using the recurrence relation (9), we can obtain some other interesting relations concerning balancing numbers.

Theorem 5.1:

(a) $B_{n+1} \cdot B_{n-1} = (B_n + 1)(B_n - 1)$.

(b) $B_n = B_k \cdot B_{n-k} - B_{k-1} \cdot B_{n-k-1}$ for any positive integer $k < n$.

(c) $B_{2n} = B_n^2 - B_{n-1}^2$.

(d) $B_{2n+1} = B_n(B_{n+1} - B_{n-1})$.

Proof: From (9), it follows that

$$\frac{B_{n+1} + B_{n-1}}{B_n} = 6. \tag{10}$$

Replacing n by $n-1$ in (10), we get

$$\frac{B_{n-1} + B_{n-2}}{B_{n-1}} = 6. \tag{11}$$

From (10) and (11), we obtain $B_n^2 - B_{n-1} \cdot B_{n+1} = B_{n-1}^2 - B_{n-2} \cdot B_n$. Now, iterating recursively, we see that $B_n^2 - B_{n-1} \cdot B_{n+1} = B_1^2 - B_0 \cdot B_2 = 36 - 1 \cdot 35 = 1$. Thus, $B_n^2 - 1 = B_{n+1} \cdot B_{n-1}$, from which (a) follows.

The proof of (b) is based on induction. Clearly, (b) is true for $n > 1$ and $k = 1$. Assume that (b) is true for $k = r$, i.e., $B_n = B_r \cdot B_{n-r} - B_{r-1} \cdot B_{n-r-1}$. Thus,

$$\begin{aligned} B_{r+1} \cdot B_{n-r-1} - B_r \cdot B_{n-r-2} &= (6B_r - B_{r-1})B_{n-r-1} - B_r \cdot B_{n-r-2} \\ &= 6B_r \cdot B_{n-r-1} - B_{r-1} \cdot B_{n-r-1} - B_r \cdot B_{n-r-2} \\ &= B_r(6B_{n-r-1} - B_{n-r-2}) - B_{r-1} \cdot B_{n-r-1} \\ &= B_r \cdot B_{n-r} - B_{r-1} \cdot B_{n-r-1} = B_n, \end{aligned}$$

showing that (b) is true for $k = r+1$. This completes the proof of (b).

The proof of (c) follows by replacing n by $2n$ and k by n in (b). Similarly, the proof of (d) follows by replacing n by $2n+1$ and k by n in (b). This completes the proof of Theorem 5.1.

6. GENERATING FUNCTION FOR BALANCING NUMBERS

In Section 5 we obtained some recurrence relations for the sequence of balancing numbers. In this section our aim is to find a nonrecursive form for B_n, $n = 0, 1, 2, \ldots$, using the generating function for the sequence B_n.

Recall that the generating function for a sequence $\{x_n\}$ of real numbers is defined by

$$g(s) = \sum_{n=0}^{\infty} x_n s^n.$$

Thus,

$$x_n = \frac{1}{n!} \frac{d^n}{ds^n} g(s) \Big|_{s=0} \quad \text{(see [5], p. 29)}.$$

Theorem 6.1: The generating function of the sequence B_n of balancing numbers is $g(s) = \frac{1}{1-6s+s^2}$ and, consequently,

$$\begin{aligned} B_n &= 6^n - \binom{n-1}{1}6^{n-2} + \binom{n-2}{2}6^{n-4} - \cdots + (-1)^{\left[\frac{n}{2}\right]}\binom{n-\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]}6^{n-\left[\frac{n}{2}\right]} \\ &= \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n-k}{k}6^{n-2k}, \end{aligned} \tag{12}$$

where [ ] denotes the greatest integer function.

Proof: From (9) for $n = 1, 2, \ldots$, we have $B_{n+1} - 6B_n + B_{n-1} = 0$. Multiplying each term by s^n and taking summation over $n = 1$ to $n = \infty$, we obtain

$$\frac{1}{s}\sum_{n=1}^{\infty} B_{n+1}s^{n+1} - 6\sum_{n=1}^{\infty} B_n s^n + s\sum_{n=1}^{\infty} B_{n-1}s^{n-1} = 0$$

which, in terms of $g(s)$, yields

$$\frac{1}{s}(g(s) - 1 - 6s) - 6(g(s) - 1) + sg(s) = 0.$$

Thus,

$$g(s) = \frac{1}{1 - 6s + s^2} = (1 - (6s - s^2))^{-1} \tag{13}$$
$$= 1 + (6s - s^2) + (6s - s^2)^2 + (6s - s^2)^3 + \cdots.$$

When n is even, the terms containing s^n in (13) are $(6s - s^2)^{n/2}, (6s - s^2)^{(n/2)+1}, \ldots, (6s - s^2)^n$, and in this case the coefficient of s^n in $g(s)$ is

$$6^n - \binom{n-1}{1}6^{n-2} + \binom{n-2}{2}6^{n-4} - \cdots + (-1)^{n/2}. \tag{14}$$

When n is odd, the terms containing s^n in (13) are $(6s - s^2)^{(n+1)/2}, (6s - s^2)^{(n+3)/2}, \ldots, (6s - s^2)^n$, and in this case the coefficient of s^n in $g(s)$ is

$$6^n - \binom{n-1}{1}6^{n-2} + \binom{n-2}{2}6^{n-4} - \cdots + (-1)^{(n-1)/2}\binom{\frac{n+1}{2}}{\frac{n-1}{2}}6. \tag{15}$$

It is clear that (14) represents the right-hand side of (12) when n is even and (15) represents the right-hand side of (12) when n is odd. This completes the proof of Theorem 6.1.

7. ANOTHER NONRECURSIVE FORM FOR BALANCING NUMBERS

In Section 6 we obtained a nonrecursive form for B_n, $n = 0, 1, 2, \ldots$, using the generating function. In this section we shall obtain another nonrecursive form for B_n by solving the recurrence relation (9) as a difference equation.

We rewrite (9) in the form

$$B_{n+1} - 6B_n + B_{n-1} = 0, \tag{16}$$

which is a second-order linear homogeneous difference equation whose auxiliary equation is

$$\lambda^2 - 6\lambda + 1 = 0. \tag{17}$$

The roots $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ of (17) are real and unequal. Thus,

$$B_n = A\lambda_1^n + B\lambda_2^n, \tag{18}$$

where A and B are determined from the values of B_0 and B_1. Substituting $B_0 = 1$ and $B_1 = 6$ into (18), we get

$$A + B = 1, \tag{19}$$
$$A\lambda_1 + B\lambda_2 = 6. \tag{20}$$

Solving (19) and (20) for A and B, we obtain

$$A = \frac{\lambda_2 - 6}{\lambda_2 - \lambda_1} = \frac{\lambda_1}{\lambda_1 - \lambda_2}; \quad B = \frac{6 - \lambda_1}{\lambda_2 - \lambda_1} = -\frac{\lambda_2}{\lambda_1 - \lambda_2}.$$

Substituting these values into (18), we get

$$B_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \quad n = 0, 1, 2, \ldots.$$

Theorem 7.1: If B_n is the nth balancing number, then

$$B_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \quad n = 0, 1, 2, \ldots,$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

8. LIMIT OF THE RATIO OF THE SUCCESSIVE TERMS

The Fibonacci numbers ([1], p. 6) are defined as follows: $F_0 = 1, F_1 = 1, \ F_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, \ldots$. It is well known that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2},$$

which is called the *golden ratio* [1]. We prove a similar result concerning balancing numbers.

Theorem 8.1: If B_n is the nth balancing number, then

$$\lim_{n \to \infty} \frac{B_{n+1}}{B_n} = 3 + \sqrt{8}.$$

Proof: From the recurrence relation (9), we have

$$\frac{B_{n+1}}{B_n} + \frac{B_{n-1}}{B_n} = 6. \tag{21}$$

Putting $\lambda = \lim_{n \to \infty} \frac{B_{n+1}}{B_n}$ in (21), we get $\lambda^2 - 6\lambda + 1 = 0$, i.e., $\lambda = 3 \pm \sqrt{8}$. Since $B_{n+1} > B_n$, we must have $\lambda \geq 1$. Thus, $\lambda = 3 + \sqrt{8}$. This completes the proof of Theorem 8.1.

An alternative proof of Theorem 8.1 can be obtained by considering the relation

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}$$

and using the fact that $B_n \to \infty$ as $n \to \infty$.

It is important to note that the limit ratio $3 + \sqrt{8}$ represents the *simple periodic continued fraction* ([4], Ch. X)

$$[\dot{6}, -\dot{6}] = 6 + \cfrac{1}{-6 + \cfrac{1}{6 + \cfrac{1}{-6 + \cdots}}}, \tag{22}$$

and from Theorem 178 ([4], p. 147) it follows that, if C_n is the nth *convergent* of (22), then

$$C_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1^{n+1} - \lambda_2^{n+1}},$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$. An application of Theorem 7.1 shows that $C_n = \frac{B_{n+1}}{B_n}$; thus, $B_0 = 1$ and $B_{n+1} = B_n C_n$, $n = 0, 1, 2, \ldots$.

9. AN APPLICATION OF BALANCING NUMBERS TO A DIOPHANTINE EQUATION

It is quite well known that the solutions of the Diophantine equation

$$x^2 + y^2 = z^2, \quad x, y, z \in \mathbf{Z}^+ \tag{23}$$

are of the form

$$x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2,$$

where $u, v \in \mathbf{Z}^+$ and $u > v$ ([3], [4], [7]). The solution (x, y, z) is called a *Pythagorean triplet*. We consider the solutions of (23) in a particular case, namely,

$$x^2 + (x+1)^2 = y^2. \tag{24}$$

In this section we relate the solutions of (24) with balancing numbers.

Let (x, y) be a solution of (24). Hence, $2y^2 - 1 = (2x+1)^2$. Thus,

$$\frac{(2y^2 - 1) \cdot 2y^2}{2} = y^2 \cdot (2y^2 - 1)$$

is a triangular number as well as a perfect square. Therefore,

$$B = \sqrt{y^2(2y^2 - 1)} \tag{25}$$

is an odd balancing number (since y^2 and $2y^2 - 1$ are odd). Since $y^2 \geq 1$, it follows from (25) that

$$y^2 = \frac{1 + \sqrt{8B^2 + 1}}{4}. \tag{26}$$

Again, since y is positive by assumption, we have

$$y = \frac{1}{2}\sqrt{1 + \sqrt{8B^2 + 1}}.$$

From (24) and (26), we obtain

$$2x^2 + 2x + 1 = \frac{1 + \sqrt{8B^2 + 1}}{4}.$$

Since x is positive, it follows that

$$x = \frac{\sqrt{\frac{1}{2}\left(\sqrt{8B^2 + 1} - 1\right)} - 1}{2}.$$

For example, if we take $B = 35$ (an odd balancing number), then we have

$$x = \frac{\sqrt{\frac{1}{2}\left(\sqrt{8 \cdot 35^2 + 1} - 1\right)} - 1}{2} = 3,$$

$$y = \frac{1}{2}\sqrt{1+\sqrt{8 \cdot 35^2 + 1}} = 5,$$

and

$$3^2 + (3+1)^2 = 5^2,$$

i.e.,

$$x^2 + (x+1)^2 = y^2.$$

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GENERALIZATIONS OF SOME IDENTITIES OF LONG

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(*Submitted April 1997*)

1. INTRODUCTION

Long [4] considered the identity

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{1.1}$$

and noted that the left side consists of terms of the second degree. He gave numerous variations of (1.1) by varying the terms that make up the products and also the subscripts. For example, he obtained

$$L_n F_m - L_{n+d} F_{m+d} = \begin{cases} F_{-d} L_{m+n+d}, & d \text{ even}, \\ L_{-d} F_{m+n+d} - 2(-1)^n F_{m-n}, & d \text{ odd}. \end{cases} \tag{1.2}$$

Long noticed that the replacement of the minus sign on the left with a plus sign simply reversed the even and odd cases on the right side, so that a counterpart to (1.2) is

$$L_n F_m + L_{n+d} F_{m+d} = \begin{cases} L_{-d} F_{m+n+d} - 2(-1)^n F_{m-n}, & d \text{ even}, \\ F_{-d} L_{m+n+d}, & d \text{ odd}. \end{cases} \tag{1.3}$$

In this paper we generalize all the results of Long that focus on the difference of products, and we produce many more. A pleasing feature of the identities contained here is that while being more general than those of Long, they maintain the elegant properties which Long observed.

2. THE SEQUENCES

Define the sequences $\{U_n\}$, $\{V_n\}$, $\{W_n\}$, and $\{X_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0, \ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, & V_0 = 2, \ V_1 = p, \\ W_n = pW_{n-1} - qW_{n-2}, & W_0 = a, \ W_1 = b, \\ X_n = W_{n+1} - qW_{n-1}. \end{cases} \tag{2.1}$$

Here a, b, p, and q are any complex numbers with $\Delta = p^2 - 4q \neq 0$. Then the roots α and β of $x^2 - px + q = 0$ are distinct. Hence, the Binet form (see [2] and [3]) for W_n is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$. It can also be shown that $X_n = A\alpha^n + B\beta^n$. The sequences $\{U_n\}$ and $\{V_n\}$ generalize $\{F_n\}$ and $\{L_n\}$, respectively. Also, since $\{W_n\}$ generalizes $\{U_n\}$, then $\{X_n\}$ generalizes $\{V_n\}$ by virtue of the fact that $V_n = U_{n+1} - qU_{n-1}$, which can be proved using Binet forms.

We consider a second group of sequences obtained from (2.1) by putting $q = -1$. In the obvious order, we name these sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{x_n\}$. The sequences $\{u_n\}$ and $\{v_n\}$ also generalize $\{F_n\}$ and $\{L_n\}$, respectively. Furthermore, $\{w_n\}$ and $\{x_n\}$ generalize $\{u_n\}$ and $\{v_n\}$, respectively. We write $D = p^2 + 4$.

Finally, our third group of sequences is obtained from (2.1) by putting $q = 1$. In order, we name these sequences $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, and $\{S_n\}$.

3. THE FIRST SET OF IDENTITIES

For the sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{x_n\}$, we have found the following:

$$x_n v_m - x_{n+d} v_{m+d} = \begin{cases} v_{-d} x_{m+n+d} + 2(-1)^m x_{n-m}, & d \text{ odd}, \\ D u_{-d} w_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.1}$$

$$x_n v_m - x_{m+d} v_{n+d} = \begin{cases} v_{-d} x_{m+n+d} + (-1)^m x_0 v_{n-m}, & d \text{ odd}, \\ D(u_{-d} w_{m+n+d} + (-1)^m w_0 u_{n-m}), & d \text{ even}, \end{cases} \tag{3.2}$$

$$x_n u_m - x_{n+d} u_{m+d} = \begin{cases} v_{-d} w_{m+n+d} + 2(-1)^{m+1} w_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.3}$$

$$x_n u_m - x_{m+d} u_{n+d} = \begin{cases} v_{-d} w_{m+n+d} + (-1)^{m+1} w_0 v_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + (-1)^{m+1} x_0 u_{n-m}, & d \text{ even}, \end{cases} \tag{3.4}$$

$$x_n u_m - v_{n+d} w_{m+d} = \begin{cases} v_{-d} w_{m+n+d} + (-1)^{m+1} x_0 u_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + (-1)^{m+1} w_0 v_{n-m}, & d \text{ even}, \end{cases} \tag{3.5}$$

$$x_n u_m - v_{m+d} w_{n+d} = \begin{cases} v_{-d} w_{m+n+d}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + 2(-1)^{m+1} w_{n-m}, & d \text{ even}, \end{cases} \tag{3.6}$$

$$x_n v_m - D w_{n+d} u_{m+d} = \begin{cases} v_{-d} x_{m+n+d}, & d \text{ odd}, \\ D u_{-d} w_{m+n+d} + 2(-1)^m x_{n-m}, & d \text{ even}, \end{cases} \tag{3.7}$$

$$x_n v_m - D w_{m+d} u_{n+d} = \begin{cases} v_{-d} x_{m+n+d} + (-1)^m D w_0 u_{n-m}, & d \text{ odd}, \\ D u_{-d} w_{m+n+d} + (-1)^m x_0 v_{n-m}, & d \text{ even}, \end{cases} \tag{3.8}$$

$$w_n u_m - w_{n+d} u_{m+d} = \begin{cases} \frac{1}{D}(v_{-d} x_{m+n+d} + 2(-1)^{m+1} x_{n-m}), & d \text{ odd}, \\ u_{-d} w_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.9}$$

$$w_n u_m - w_{m+d} u_{n+d} = \begin{cases} \frac{1}{D}(v_{-d} x_{m+n+d} + (-1)^{m+1} x_0 v_{n-m}), & d \text{ odd}, \\ u_{-d} w_{m+n+d} + (-1)^{m+1} w_0 u_{n-m}, & d \text{ even}, \end{cases} \tag{3.10}$$

$$w_n v_m - w_{n+d} v_{m+d} = \begin{cases} v_{-d} w_{m+n+d} + 2(-1)^m w_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d}, & d \text{ even}, \end{cases} \tag{3.11}$$

$$w_n v_m - w_{m+d} v_{n+d} = \begin{cases} v_{-d} W_{m+n+d} + (-1)^m w_0 v_{n-m}, & d \text{ odd}, \\ u_{-d} x_{m+n+d} + (-1)^m x_0 u_{n-m}, & d \text{ even}. \end{cases} \quad (3.12)$$

If on the left side, in each case, we replace the minus sign with a plus sign, the identities are exactly as stated but with the even and odd cases reversed. This parallels the observations of Long for his Fibonacci-Lucas identities. For example, as a counterpart to (3.6), we have

$$x_n u_m + v_{m+d} w_{n+d} = \begin{cases} u_{-d} x_{m+n+d} + 2(-1)^{m+1} w_{n-m}, & d \text{ odd}, \\ v_{-d} W_{m+n+d}, & d \text{ even}. \end{cases} \quad (3.13)$$

The proofs of (3.1)-(3.12) and their counterparts with a plus sign on the left are similar. For the proofs, we require the following:

$$q^n U_{-n} = -U_n, \quad (3.14)$$

$$q^n V_{-n} = V_n, \quad (3.15)$$

$$W_{n+d} + q^d W_{n-d} = W_n V_d, \quad (3.16)$$

$$W_{n+d} - q^d W_{n-d} = X_n U_d, \quad (3.17)$$

$$X_{n+d} + q^d X_{n-d} = X_n V_d, \quad (3.18)$$

$$X_{n+d} - q^d X_{n-d} = \Delta W_n U_d. \quad (3.19)$$

Identities (3.14) and (3.15) can be proved using Binet forms, while (3.16)-(3.19) occur in Bergum and Hoggatt [1].

As an example, we prove (3.1).

Proof of (3.1): With $q = -1$ and using the Binet forms in Section 2, we have

$$x_n v_m - x_{n+d} v_{m+d} = (A\alpha^n + B\beta^n)(\alpha^m + \beta^m) - (A\alpha^{n+d} + B\beta^{n+d})(\alpha^{m+d} + \beta^{m+d})$$

$$= (A\alpha^{m+n} + B\beta^{m+n}) - (A\alpha^{m+n+2d} + B\beta^{m+n+2d})$$

$$+ (A\alpha^n \beta^m + B\alpha^m \beta^n) - (A\alpha^{n+d}\beta^{m+d} + B\alpha^{m+d}\beta^{n+d})$$

$$= x_{m+n} - x_{m+n+2d} + (\alpha\beta)^m (A\alpha^{n-m} + B\beta^{n-m}) - (\alpha\beta)^{m+d}(A\alpha^{n-m} + B\beta^{n-m})$$

$$= -(x_{(m+n+d)+d} - x_{(m+n+d)-d}) + (1 - (\alpha\beta)^d)(\alpha\beta)^m x_{n-m}.$$

Now $q = -1$ implies $\alpha\beta = -1$. From (3.18) and (3.19), the right side becomes

$$\begin{cases} -x_{m+n+d} v_d + 2(-1)^m x_{n-m}, & d \text{ odd}, \\ -D w_{m+n+d} u_d, & d \text{ even}, \end{cases}$$

and the use of (3.14) and (3.15) gives the result.

4. THE SECOND SET OF IDENTITIES

We now consider the sequences $\{P_n\}$, $\{Q_n\}$, $\{R_n\}$, and $\{S_n\}$. For these sequences, we have derived twenty-four identities that parallel (3.1)-(3.12). Twelve identities have a minus sign connecting the two products on the left, and twelve have a plus sign. Each can be obtained by looking at its counterpart in the list (3.1)-(3.12) and using the following rules:

(i)   Replace u by P, v by Q, w by R, and x by S.

(ii) Replace any occurrence of $(-1)^m$ on the right side by 1.

(iii) Then the difference of the two products is equal to the even case, and the sum of the two products is equal to the odd case.

For example, using (3.3), we have

$$S_n P_m - S_{n+d} P_{m+d} = P_{-d} S_{m+n+d}, \tag{4.1}$$

$$S_n P_m + S_{n+d} P_{m+d} = Q_{-d} R_{m+n+d} - 2R_{n-m}. \tag{4.2}$$

These can be proved in the same manner shown previously, and because of the above rules of formation we refrain from listing the others.

5. THE THIRD SET OF IDENTITIES

The identities of Long [4, (18)-(24)] are generalizations and variations of Simson's identity

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n. \tag{5.1}$$

They share the common thread of being differences of products the sum of whose subscripts are equal. For example,

$$L_n F_m - L_{n-d} F_{m+d} = (-1)^n F_{-d} L_{m-n+d}. \tag{5.2}$$

We have found similar identities for mixtures of terms from the sequences $\{U_n\}$, $\{V_n\}$, $\{W_n\}$, and $\{X_n\}$. Following our notation in Section 2, we write $e = pab - qa^2 - b^2 = -AB$, which is essentially the notation of Horadam [2]. The first group of identities is:

$$X_n X_m - X_{n-d} X_{m+d} = -eq^m \Delta U_d U_{n-m-d}, \tag{5.3}$$

$$X_n V_m - X_{n-d} V_{m+d} = q^m \Delta U_d W_{n-m-d}, \tag{5.4}$$

$$X_n V_m - X_{m+d} V_{n-d} = q^m \Delta W_d U_{n-m-d}. \tag{5.5}$$

If we replace the minus sign connecting the two products on the left with a plus sign, then identity (5.3) does not have an interesting counterpart. But in (5.4) and (5.5) we modify the right side by replacing U with V, W with X, dividing by Δ, and then adding $2X_{m+n}$.

The second group is:

$$W_n W_m - W_{n-d} W_{m+d} = eq^m U_d U_{n-m-d}, \tag{5.6}$$

$$W_n U_m - W_{n-d} U_{m+d} = -q^m U_d W_{n-m-d}, \tag{5.7}$$

$$W_n U_m - W_{m+d} U_{n-d} = -q^m W_d U_{n-m-d}. \tag{5.8}$$

As before, if we replace the minus sign on the left with a plus sign, then (5.6) does not have an interesting counterpart. However, we change the right sides of (5.7) and (5.8) by replacing U with V, W with X, adding $2X_{m+n}$, and then dividing by Δ.

The third group is:

$$X_n X_m - \Delta W_{n-d} W_{m+d} = -eq^m V_d V_{n-m-d}, \tag{5.9}$$

$$X_n V_m - \Delta W_{n-d} U_{m+d} = q^m V_d X_{n-m-d}, \tag{5.10}$$

$$X_n V_m - \Delta W_{m+d} U_{n-d} = q^m X_d V_{n-m-d}. \tag{5.11}$$

Again, (5.9) yields nothing interesting after replacing the minus sign on the left with a plus sign. However, we change the right sides of (5.10) and (5.11) by replacing V with U, X with W, multiplying by Δ, and adding $2X_{m+n}$. This should be compared with the processes in the previous groups of identities.

The last group of identities is:

$$X_n W_m - X_{n-d} W_{m+d} = e q^m U_d V_{n-m-d}, \qquad (5.12)$$

$$X_n W_m - X_{m+d} W_{n-d} = e q^m V_d U_{n-m-d}, \qquad (5.13)$$

$$X_n U_m - X_{n-d} U_{m+d} = -q^m U_d X_{n-m-d}, \qquad (5.14)$$

$$X_n U_m - X_{m+d} U_{n-d} = -q^m X_d U_{n-m-d}, \qquad (5.15)$$

$$V_n W_m - V_{n-d} W_{m+d} = q^n U_{-d} X_{m-n+d}, \qquad (5.16)$$

$$V_n W_m - V_{m+d} W_{n-d} = q^n X_{-d} U_{m-n+d}, \qquad (5.17)$$

$$X_n U_m - V_{n-d} W_{m+d} = -q^m W_d V_{n-m-d}, \qquad (5.18)$$

$$X_n U_m - V_{m+d} W_{n-d} = -q^m V_d W_{n-m-d}. \qquad (5.19)$$

In (5.12) and (5.13), replacing the minus sign on the left with a plus sign yields identities which are not interesting. However, in (5.14)-(5.19), we change the right side by replacing $U(V)$ with $V(U)$ and $W(X)$ with $X(W)$ and adding $2W_{m+n}$.

We refrain from giving proofs of identities (5.3)-(5.19) and their counterparts because they are similar to the proof of (3.1) demonstrated earlier.

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LUCAS SEQUENCES AND FUNCTIONS OF A 3-BY-3 MATRIX

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1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} - qU_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} - qV_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases} \tag{1.1}$$

where p and q are real numbers with $q(p^2 - 4q) \neq 0$. These sequences were studied originally by Lucas [9], and have subsequently been the subject of much attention.

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α / β is not an nth root of unity for any n. Write

$$\Delta = (\alpha - \beta)^2 = p^2 - 4q.$$

A well-known relationship between U_n and V_n is

$$V_n = U_{n+1} - qU_{n-1}, \tag{1.2}$$

which we use subsequently.

Barakat [2] considered the matrix exponential, $\exp(X)$, for the 2-by-2 matrix

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where he took $\text{trace}(X) = p$ and $\det(X) = q$. In so doing, he established various infinite sums involving terms from $\{U_n\}$ and $\{V_n\}$.

Following Barakat, Walton [13] evaluated the series for the sine and cosine functions at the matrix X and obtained further infinite sums involving terms from $\{U_n\}$ and $\{V_n\}$. Extensions of these ideas to higher-order recurrences have been given by Shannon and Horadam [12] and Pethe [11]. Recently, many papers have appeared which have followed the theme of these writers. See, for example, Brugia and Filipponi [5], Filipponi and Horadam [6], Horadam and Filipponi [8], and Melham and Shannon [10].

In this paper we apply the techniques of the above writers to a 3-by-3 matrix to obtain new infinite sums involving *squares* of terms from the sequences $\{U_n\}$ and $\{V_n\}$.

2. THE MATRIX $R_{k,x}$

Berzsenyi [4] has shown that the matrix

$$R = \begin{pmatrix} 0 & 0 & q^2 \\ 0 & -q & -2pq \\ 1 & p & p^2 \end{pmatrix}$$

(2.1)

is such that, for nonnegative integers n,

$$R^n = \begin{pmatrix} q^2 U_{n-1}^2 & q^2 U_{n-1} U_n & q^2 U_n^2 \\ -2q U_{n-1} U_n & -q(U_n^2 + U_{n-1} U_{n+1}) & -2q U_n U_{n+1} \\ U_n^2 & U_n U_{n+1} & U_{n+1}^2 \end{pmatrix}.$$

(2.2)

The characteristic equation of R is

$$\lambda^3 + (q - p^2)\lambda^2 + q(p^2 - q)\lambda - q^3 = 0.$$

(2.3)

Since $p = \alpha + \beta$ and $q = \alpha\beta$, it is readily verified that α^2, β^2, and $\alpha\beta$ are the eigenvalues of R. These eigenvalues are nonzero and distinct because of our assumptions in Section 1.

Associated with R, we define the matrix $R_{k,x}$ by

$$R_{k,x} = xR^k = x\begin{pmatrix} q^2 U_{k-1}^2 & q^2 U_{k-1} U_k & q^2 U_k^2 \\ -2q U_{k-1} U_k & -q(U_k^2 + U_{k-1} U_{k+1}) & -2q U_k U_{k+1} \\ U_k^2 & U_k U_{k+1} & U_{k+1}^2 \end{pmatrix},$$

(2.4)

where x is an arbitrary real number and k is a nonnegative integer. From the definition of an eigenvalue, it follows immediately that $x\alpha^{2k}, x\beta^{2k}$, and xq^k are the eigenvalues of $R_{k,x}$. Again, they are nonzero and distinct.

3. THE MAIN RESULTS

Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series whose domain of convergence includes $x\alpha^{2k}$, $x\beta^{2k}$, and xq^k. Then we have, from (2.4),

$$f(R_{k,x}) = \sum_{n=0}^{\infty} a_n R_{k,x}^n = \sum_{n=0}^{\infty} a_n x^n R^{kn}$$

$$= \begin{pmatrix} q^2 \sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 & q^2 \sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn} & q^2 \sum_{n=0}^{\infty} a_n x^n U_{kn}^2 \\ -2q \sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn} & -q \sum_{n=0}^{\infty} a_n x^n (U_{kn}^2 + U_{kn-1} U_{kn+1}) & -2q \sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1} \\ \sum_{n=0}^{\infty} a_n x^n U_{kn}^2 & \sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1} & \sum_{n=0}^{\infty} a_n x^n U_{kn+1}^2 \end{pmatrix}.$$

(3.1)

On the other hand, from the theory of matrices ([3] and [7]), it is known that

$$f(R_{k,x}) = c_0 I + c_1 R_{k,x} + c_2 R_{k,x}^2,$$

(3.2)

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where I is the 3-by-3 identity matrix, and where c_0, c_1, and c_2 can be obtained by solving the system

$$\begin{cases} c_0 + c_1 x \alpha^{2k} + c_2 x^2 \alpha^{4k} = f(x\alpha^{2k}), \\ c_0 + c_1 x \beta^{2k} + c_2 x^2 \beta^{4k} = f(x\beta^{2k}), \\ c_0 + c_1 x \alpha^k \beta^k + c_2 x^2 \alpha^{2k} \beta^{2k} = f(x\alpha^k \beta^k). \end{cases}$$

If we use Cramer's rule and observe, using the Binet form for U_n, that

$$(\alpha^{2k} - \beta^{2k})(\beta^{2k} - \alpha^k \beta^k)(\alpha^k \beta^k - \alpha^{2k}) = q^k U_{2k} U_k^2 (\alpha - \beta)^3,$$

we obtain

$$\begin{cases} c_0 = \dfrac{f(x\alpha^{2k})q^k \beta^{3k}(\alpha^k - \beta^k) + f(x\beta^{2k})q^k \alpha^{3k}(\alpha^k - \beta^k) + f(xq^k)q^{2k}(\beta^{2k} - \alpha^{2k})}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}, \\[2mm] c_1 = \dfrac{f(x\alpha^{2k})\beta^{2k}(\beta^{2k} - \alpha^{2k}) + f(x\beta^{2k})\alpha^{2k}(\beta^{2k} - \alpha^{2k}) + f(xq^k)(\alpha^{4k} - \beta^{4k})}{xq^k U_{2k} U_k^2 (\alpha - \beta)^3}, \\[2mm] c_2 = \dfrac{f(x\alpha^{2k})\beta^k(\alpha^k - \beta^k) + f(x\beta^{2k})\alpha^k(\alpha^k - \beta^k) + f(xq^k)(\beta^{2k} - \alpha^{2k})}{x^2 q^k U_{2k} U_k^2 (\alpha - \beta)^3}. \end{cases}$$

Now, equating lower left entries in (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 = c_1 x U_k^2 + c_2 x^2 U_{2k}^2. \tag{3.3}$$

With the values of c_1 and c_2 obtained above, the right side of (3.3) is

$$\frac{f(x\alpha^{2k})(\beta^{2k}(\beta^{2k} - \alpha^{2k})U_k^2 + \beta^k(\alpha^k - \beta^k)U_{2k}^2)}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}$$

$$+ \frac{f(x\beta^{2k})(\alpha^{2k}(\beta^{2k} - \alpha^{2k})U_k^2 + \alpha^k(\alpha^k - \beta^k)U_{2k}^2)}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}$$

$$+ \frac{f(xq^k)((\alpha^{4k} - \beta^{4k})U_k^2 + (\beta^{2k} - \alpha^{2k})U_{2k}^2)}{q^k U_{2k} U_k^2 (\alpha - \beta)^3}.$$

If we note that $U_{2k} = U_k V_k$ and use Binet forms, we obtain finally

$$\sum_{n=0}^{\infty} a_n x^n U_{kn}^2 = \frac{f(x\alpha^{2k}) + f(x\beta^{2k}) - 2f(xq^k)}{\Delta}. \tag{3.4}$$

In precisely the same manner, we equate appropriate entries in (3.1) and (3.2) to obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1}^2 = \frac{\alpha^2 f(x\alpha^{2k}) + \beta^2 f(x\beta^{2k}) - 2qf(xq^k)}{\Delta}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn-1} = \frac{\alpha^{-1} f(x\alpha^{2k}) + \beta^{-1} f(x\beta^{2k}) - \frac{p}{q} f(xq^k)}{\Delta}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} U_{kn+1} = \frac{\alpha f(x\alpha^{2k}) + \beta f(x\beta^{2k}) - p f(xq^k)}{\Delta},$$ (3.7)

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1}^2 = \frac{\beta^2 f(x\alpha^{2k}) + \alpha^2 f(x\beta^{2k}) - 2q f(xq^k)}{q^2 \Delta},$$ (3.8)

$$\sum_{n=0}^{\infty} a_n x^n (U_{kn}^2 + U_{kn-1} U_{kn+1}) = \frac{2f(x\alpha^{2k}) + 2f(x\beta^{2k}) - \frac{p^2}{q} f(xq^k)}{\Delta}.$$ (3.9)

From (3.4) and (3.9), we obtain

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} U_{kn+1} = \frac{f(x\alpha^{2k}) + f(x\beta^{2k}) + \left(2 - \frac{p^2}{q}\right) f(xq^k)}{\Delta}.$$ (3.10)

Finally, from (1.2), we have

$$V_{kn}^2 = U_{kn+1}^2 + q^2 U_{kn-1}^2 - 2q U_{kn+1} U_{kn-1}.$$

This, together with (3.5), (3.8), and (3.10), yields

$$\sum_{n=0}^{\infty} a_n x^n V_{kn}^2 = f(x\alpha^{2k}) + f(x\beta^{2k}) + 2f(xq^k).$$ (3.11)

In contrast to our approach, Brugia and Filipponi [5] used the Kronecker square of a 2-by-2 matrix to obtain similar sums for the Fibonacci numbers. For the function f they took the exponential function, and remarked that analogous results could be obtained by using the circular and hyperbolic functions. Identities (3.4), (3.5), (3.8), (3.10), and (3.11), respectively, generalize identities (12)-(16) of [5].

4. APPLICATIONS

We now specialize (3.4) and (3.11) to the Chebyshev polynomials to obtain some attractive sums involving the squares of the sine and cosine functions.

Let $\{T_n(t)\}_{n=0}^{\infty}$ and $\{S_n(t)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{array}{l} S_n(t) = \dfrac{\sin n\theta}{\sin \theta} \\[2mm] T_n(t) = \cos n\theta \end{array}\right\}, \quad t = \cos\theta, \ n \geq 0.$$

Indeed, $\{S_n(t)\}_{n=0}^{\infty}$ and $\{2T_n(t)\}_{n=0}^{\infty}$ are the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$, respectively, generated by (1.1), where $p = 2\cos\theta$ and $q = 1$. Thus, $\alpha = e^{i\theta}$ and $\beta = e^{-i\theta}$, which are obtained by solving $x^2 - 2\cos\theta x + 1 = 0$. Further information about the Chebyshev polynomials can be found, for example, in [1].

We use the following well-known power series, each of which has the complex plane as its domain of convergence:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$ (4.1)

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \tag{4.2}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \tag{4.3}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{4.4}$$

Now, in (3.4), taking $U_n = \sin n\theta / \sin \theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, after replacing all occurrences of $k\theta$ by ϕ,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sin^2 (2n+1)\phi}{(2n+1)!} = \frac{\sin x - \sin(x \cos 2\phi) \cosh(x \sin 2\phi)}{2}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \sin^2 2n\phi}{(2n)!} = \frac{\cos x - \cos(x \cos 2\phi) \cosh(x \sin 2\phi)}{2}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \sin^2 (2n+1)\phi}{(2n+1)!} = \frac{\sinh x - \sinh(x \cos 2\phi) \cos(x \sin 2\phi)}{2}, \tag{4.7}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \sin^2 2n\phi}{(2n)!} = \frac{\cosh x - \cosh(x \cos 2\phi) \cos(x \sin 2\phi)}{2}. \tag{4.8}$$

Finally, in (3.11), taking $V_n = 2\cos n\theta$ and replacing f by the functions in (4.1)-(4.4), we obtain, respectively,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cos^2 (2n+1)\phi}{(2n+1)!} = \frac{\sin x + \sin(x \cos 2\phi) \cosh(x \sin 2\phi)}{2}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cos^2 2n\phi}{(2n)!} = \frac{\cos x + \cos(x \cos 2\phi) \cosh(x \sin 2\phi)}{2}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \cos^2 (2n+1)\phi}{(2n+1)!} = \frac{\sinh x + \sinh(x \cos 2\phi) \cos(x \sin 2\phi)}{2}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \cos^2 2n\phi}{(2n)!} = \frac{\cosh x + \cosh(x \cos 2\phi) \cos(x \sin 2\phi)}{2}. \tag{4.12}$$

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BOOK REVIEW

Richard A. Dunlap, *The Golden Ratio and Fibonacci Numbers*
(River Edge, NJ: World Scientific, 1997).

This attractive and carefully written book addresses the general reader with interest in mathematics and its application to the physical and biological sciences. In addition, it provides supplementary reading for a lower division university course in number theory or geometry and introduces basic properties of the golden ratio and Fibonacci numbers for researchers working in fields where these numbers have found applications.

An extensive collection of diagrams illustrate geometric problems in two and three dimensions, quasi-crystallography, Penrose tiling, and biological applications. Appendices list the first 100 Fibonacci and Lucas numbers, a collection of equations involving the golden ratio and generalized Fibonacci numbers, and a diverse list of references.

A new book on Fibonacci-related topics is published infrequently; this one will make a valuable addition to academic and personal libraries, and the many diagrams will knock your socks off.

Reviewed by Marjorie Bicknell-Johnson

ON THE 2-ADIC VALUATIONS OF THE TRUNCATED POLYLOGARITHM SERIES

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The aim of this paper is to prove the following theorem which was conjectured in [1] and [2] (and originated in a work of Yu [3]).

Theorem 1: Set

$$S_1(N) = \sum_{j=1}^{N} \frac{2^j}{j}.$$

Then, if $v(x)$ denotes the highest exponent of 2 that divides x (i.e., the 2-adic valuation), we have

$$v(S_1(2^m)) = 2^m + 2m - 2 \quad \text{for } m \geq 4.$$

For the sake of completeness, note that a direct computation shows that

$$v(S_1(2^m)) = 2^m + 2m + d_1(m),$$

with $d_1(0) = 0$, $d_1(1) = -2$, $d_1(2) = -3$, and $d_1(3) = -1$, the theorem claiming that $d_1(m) = -2$ for $m \geq 4$.

Before proving this theorem, we will need a few lemmas. In this paper, we will work entirely in the field $\mathbb{Q}_2$ of 2-adic numbers, on which the valuation v can be extended.

Lemma 2: We have

$$\sum_{j=1}^{\infty} \frac{2^j}{j} = 0 \quad \text{in } \mathbb{Q}_2.$$

Proof: Since the function

$$\text{Li}_1(x) = -\log(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j}$$

converges in $\mathbb{Q}_2$ for $v(x) \geq 1$, and satisfies

$$\text{Li}_1(x) + \text{Li}_1(y) = -\log((1-x)(1-y)) = \text{Li}_1(x+y-xy)$$

for all x and y such that $v(x) \geq 1$ and $v(y) \geq 1$, we deduce that our sum is equal to $\text{Li}_1(2)$ and that

$$2\text{Li}_1(2) = \text{Li}_1(0) = 0,$$

so $\text{Li}_1(2) = 0$ as claimed. $\square$

Lemma 3: We have

$$\sum_{j=1}^{\infty} \frac{2^j}{j^2} = 0 \quad \text{in } \mathbb{Q}_2.$$

Proof: This time we set

$$\text{Li}_2(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^2}.$$

This is the 2-adic dilogarithm, and converges in $\mathbb{Q}_2$ for $v(x) \geq 1$. Most of the usual complex functional equations for the dilogarithm are still valid in the *p*-adic case. The one we will need here is the following:

$$\text{Li}_2(x) + \text{Li}_2\left(\frac{-x}{1-x}\right) = -\frac{1}{2}\log^2(1-x),$$

valid for $v(x) > 1$. This can be proved by differentiation, or simply by noting that it is a formal identity valid over the field $\mathbb{C}$, hence also over any field of characteristic zero.

Setting $x = 2$, we obtain

$$2\text{Li}_2(2) = -\log(-1)^2 / 2 = -\text{Li}_1(2)^2 / 2 = 0$$

by Lemma 2, thus proving Lemma 3. $\square$

Remark: Lemmas 2 and 3 cannot be generalized immediately to polylogarithms. For example, an easy computation shows that $\text{Li}_3(2) \neq 0$, and in fact that $v(\text{Li}_3(2)) = -2$ (this is the explanation of $d_1(m) = -2$, as we will see below). I do not know if the value (in $\mathbb{Q}_2$) of $\text{Li}_3(2)$ can be computed explicitly. See also Theorem 8 below.

We can now prove the following.

Lemma 4: For all $N \geq 0$, we have

$$S_1(N) = \sum_{j=1}^{N} \frac{2^j}{j} = -N^2 2^N \sum_{j=1}^{\infty} \frac{2^j}{j^2(j+N)}.$$

Proof: From Lemma 2, we deduce that

$$S_1(N) = -\sum_{j=N+1}^{\infty} \frac{2^j}{j} = -\sum_{j=1}^{\infty} \frac{2^{N+j}}{N+j} = -2^N \sum_{j=1}^{\infty} \frac{2^j}{j+N}.$$

Applying Lemma 2 again, we deduce that

$$S_1(N) = S_1(N) + 2^N \sum_{j=1}^{\infty} \frac{2^j}{j} = N 2^N \sum_{j=1}^{\infty} \frac{2^j}{j(j+N)}.$$

Finally, applying Lemma 3, we obtain

$$S_1(N) = S_1(N) - N 2^N \sum_{j=1}^{\infty} \frac{2^j}{j^2} = -N^2 2^N \sum_{j=1}^{\infty} \frac{2^j}{j^2(j+N)}$$

as claimed. $\square$

We can now prove Theorem 1. It follows from Lemma 4 that

$$v(S_1(2^m)) = 2^m + 2m + v(T_1(2^m)) \quad \text{with} \quad T_1(2^m) = \sum_{j=1}^{\infty} \frac{2^j}{j^2(j+2^m)}.$$

Thus, Theorem 1 is equivalent to showing that $v(T_1(2^m)) = -2$ for $m \geq 4$. This will immediately follow from Lemma 5.

Lemma 5: Set

$$w_1(j, m) = v\left(\frac{2^j}{j^2(j + 2^m)}\right).$$

Then, for $m \geq 4$, we have $w_1(j, m) \geq -1$ for all j except for $j = 4$ for which $w_1(4, m) = -2$.

Since there is a unique term in the sum defining $T_1(2^m)$ having minimal valuation, it follows that the valuation of $T_1(2^m)$ is equal to that minimum; therefore, Theorem 1 clearly follows from Lemma 5.

Proof: Set $j = 2^a i$ with i odd. If $a < m$, we have $w_1(j, m) = 2^a i - 3a \geq 2^a - 3a$, with equality only if $i = 1$. Clearly, the function $2^a - 3a$ attains a unique minimum on the integers for $a = 2$, where its value is equal to -2; hence, if $a < m$, $w_1(j, m) \geq -1$ except for $a = 2$ and $i = 1$, i.e., for $j = 4$ for which $w_1(j, m) = -2$. Note that this value can be attained only if $2 < m$, i.e., if $m \geq 3$.

If $a < m$, we have $w_1(j, m) = 2^a i - 2a - m \geq 2^a i - 3a + 1 \geq -1$ for all j by what we have just proved.

Finally, if $a = m$, we have $w_1(j, m) = 2^m i - 3m - v(i + 1)$. We note that, for all i, we have $v(i + 1) \leq i$. Thus,

$$w_1(j, m) \geq (2^m - 1)i - 3m \geq 2^m - 3m - 1 \geq -1 \text{ for } m \geq 4.$$

Note that this is the only place where it is necessary to assume that $m \geq 4$ (for $m = 3$ the minimum would be -2, so we could not conclude that the valuation of the sum is equal to -2, and in fact it is not). This proves Lemma 5, hence Theorem 1. □

Remark: Lemma 4 and suitable generalizations of Lemma 5 allow us more generally to compute $v(S_1(h2^m))$ for $m \geq 4$ and a fixed odd h. I leave the details to the reader.

In view of Lemma 3, it is natural to ask if there is a generalization of Theorem 1 to the dilogarithm. This is indeed the case.

Theorem 6: Set

$$S_2(N) = \sum_{j=1}^{N} \frac{2^j}{j^2}.$$

Then we have

$$v(S_2(2^m)) = 2^m + m - 1 \text{ for } m \geq 4.$$

For the sake of completeness, note that a direct computation shows that

$$v(S_2(2^m)) = 2^m + m + d_2(m),$$

with $d_2(0) = 0$, $d_2(1) = -3$, $d_2(2) = -4$, and $d_2(3) = -3$, the theorem claiming that $d_2(m) = -1$ for $m \geq 4$.

Proof: By Lemma 3, we have

$$S_2(N) = -\sum_{j=N+1}^{\infty} \frac{2^j}{j^2} = -2^N \sum_{j=1}^{\infty} \frac{2^j}{(j + N)^2}.$$

Applying Lemma 3 once again, we have

$$S_2(N) = S_2(N) + 2^N \sum_{j=1}^{\infty} \frac{2^j}{j^2} = N 2^N \sum_{j=1}^{\infty} \frac{2^j(2j+N)}{j^2(j+N)^2}.$$

The proof is now nearly identical to that of Theorem 1. We have

$$v(S_2(2^m)) = 2^m + m + v(T_2(2^m)),$$

with

$$T_2(2^m) = \sum_{j=1}^{\infty} \frac{2^j(2j+2^m)}{j^2(j+2^m)^2}.$$

Further, we have

Lemma 7: Set

$$w_2(j,m) = v\left(\frac{2^j(2j+2^m)}{j^2(j+2^m)^2}\right).$$

Then, for $m \geq 4$, we have $w_2(j,m) \geq 0$ for all j except $j = 4$ for which $w_2(4,m) = -1$.

Since there is a unique term in the sum defining $T_2(2^m)$ having minimal valuation, it follows as before that the valuation of $T_2(2^m)$ is equal to that minimum; hence, Theorem 6 clearly follows from Lemma 7.

Proof: Set $j = 2^a i$ with i odd. If $a < m-1$, we have $w_2(j,m) = 2^a i - 3a + 1 \geq 2^a - 3a + 1$, with equality only if $i = 1$. The function $2^a - 3a + 1$ attains a unique minimum on the integers for $a = 2$, where its value is equal to -1. Thus, if $a < m-1$, $w_2(j,m) \geq 0$ except for $a = 2$ and $i = 1$, i.e., for $j = 4$ for which $w_2(j,m) = -1$. Note that this value can be attained only if $2 < m-1$, i.e., if $m \geq 4$.

If $a = m-1$, we have $w_2(j,m) \geq 2^a i - 3a + 1 \geq 2^a - 3a + 1$. Now, since $m \geq 4$, we have $a \geq 3$, hence $w_2(j,m) \geq 8 - 9 + 1 = 0$.

If $a > m$, we have $w_2(j,m) = 2^a i - 2a - m \geq 2^a i - 3a + 1 \geq 2^a - 3a + 1 \geq 0$ for all j, since $m \geq 2$.

Finally, if $a = m$, we have $w_2(j,m) = 2^m i - 3m - 2v(i+1)$. We note that, for all i, we have $v(i+1) \leq i$; thus,

$$w_2(j,m) \geq (2^m - 2)i - 3m \geq 2^m - 3m - 2 \geq 0 \quad \text{for } m \geq 4.$$

This proves Lemma 7, hence Theorem 6. □

Of course, once again this can be generalized to the computation of $v(S_2(h2^m))$ for a fixed odd h.

As already mentioned, the polylogarithms of order k at 2 do not vanish if $k \geq 3$; therefore, the corresponding sums $S_k(2^m)$ have a bounded valuation. Using the same methods, one can prove the following theorem.

Theorem 8: Denote by $\lg k$ the base 2 logarithm of k, set $e(k) = \lceil \lg k \rceil$ and $\delta(k) = 1$ if k is a power of 2, and $\delta(k) = 0$, otherwise. Then, for $k \geq 3$, we have $\text{Li}_k(2) \neq 0$, and in fact

$$v(\text{Li}_k(2)) = 2^{3(k)} - ke(k) + \delta(k).$$

More precisely (still for $k \geq 3$), if

$$S_k(N) = \sum_{j=1}^{N} \frac{2^j}{j^k},$$

then

$$v(S_k(N)) = 2^{e(k)} - ke(k) + \delta(k) \text{ for } N \geq 2^{e(k)+\delta(k)}.$$

Proof: It is clear that all the statements of the theorem follow from the last. Assume first that k is not a power of 2. Then, if we set $w_k(j) = v(2^j / j^k)$ and $j = 2^a i$ with i odd, we have $w_k(j) = 2^a i - ka$. For fixed a, this is minimal for $i = 1$. Furthermore, if we set $f(a) = 2^a - ka$, it is clear that f attains its minimum on the integers for $a = e(k)$, and that this minimum is unique if a is not a power of 2. Hence, there is a single term with minimum valuation for $j = 2^{e(k)} \leq N$, by assumption, so $v(S_k(N)) = 2^{e(k)} - ke(k)$, as claimed.

Assume now that a is a power of 2. Then the minimum of f is attained for $a = e(k)$ and for $a = e(k) + 1$. The corresponding terms in the sum not only have the same valuation, but are in fact equal, hence the valuation w of their sum is simply 1 more than usual. We now notice that $f(a+1) - f(a) = 2^a - 2^{e(k)}$. Therefore, since we have assumed $k \geq 3$, hence $e(k) \geq 2$, we have $|f(a+1) - f(a)| \geq 2$ for $a \neq e(k)$, so all the other terms have a valuation that is strictly larger than w, so $v(S_k(N)) = w = 2^{e(k)} - ke(k) + $ for $N \geq 2^{e(k)+1}$, as claimed. $\square$

Remark: One can generalize the above results to other primes than $p = 2$, but the results are much less interesting. For example, it is easy to show, using similar methods, that the 3-adic valuation of

$$\sum_{j=1}^{3^m} (2 + (-1)^{j-1}) \frac{3^j}{j}$$

is equal to $3^m + 1$ for all m.

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ALGORITHMIC SUMMATION OF RECIPROCALS OF PRODUCTS OF FIBONACCI NUMBERS

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1. INTRODUCTION

There is no known simple form for the following summations:

$$\mathbb{F}_N = \sum_{n=1}^{N} \frac{1}{F_n}, \quad \mathbb{G}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_n}, \quad \text{and} \quad \mathbb{K}_N = \sum_{n=1}^{N} \frac{1}{F_n F_{n+1}}. \tag{1}$$

It is our purpose to show that all other indefinite summations of reciprocals of products of Fibonacci numbers can be expressed in terms of these forms. More specifically, we will give an algorithm for expressing

$$S_N(a_1, a_2, \ldots, a_r) = \sum_{n=1}^{N} \frac{1}{F_{n+a_1} F_{n+a_2} \cdots F_{n+a_r}} \tag{2}$$

and

$$T_N(a_1, a_2, \ldots, a_r) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a_1} F_{n+a_2} \cdots F_{n+a_r}} \tag{3}$$

in terms of $\mathbb{F}_N, \mathbb{G}_N$, and $\mathbb{K}_N$, where $a_1, a_2, \ldots, a_r$ are distinct integers. Since $a_1, a_2, \ldots, a_r$ are constants, these symbols may appear in the limits of the summations, but it is our objective to find formulas in which N does not appear in any of the summation limits.

Expressions of the form $S_N(a_1, a_2, \ldots, a_r)$ and $T_N(a_1, a_2, \ldots, a_r)$ will be called reciprocal sums of order r. Those of the second form are also called alternating reciprocal sums.

Without loss of generality, we may assume that the a_i are ordered so that $a_1 < a_2 < \cdots < a_r$. Furthermore, we may assume that $a_1 = 0$, because a change of the index of summation allows us to compute those sums where $a_1 \neq 0$. For example, if $a_1 > 0$, then we have

$$S_N(a_1, a_2, \ldots, a_r) = S_{N+a_1}(0, a_2 - a_1, \ldots, a_r - a_1) - S_{a_1}(0, a_2 - a_1, \ldots, a_r - a_1).$$

2. REDUCTION FORMULAS

We start by showing that reciprocal sums of order r can be expressed in terms of reciprocal sums of order $r - 2$ for all integers $r > 2$.

The following identity is easily proved (e.g., by using algorithm `FibSimplify` from [8]).

Theorem 1 (The Partial Fraction Decomposition Formula):

Let a, b, and c be distinct integers. Then, for all integers n,

$$\frac{(-1)^n}{F_{n+a} F_{n+b} F_{n+c}} = \frac{A}{F_{n+a}} + \frac{B}{F_{n+b}} + \frac{C}{F_{n+c}}, \tag{4}$$

where

$$A = \frac{(-1)^a}{F_{b-a}F_{c-a}}, \quad B = \frac{(-1)^b}{F_{c-b}F_{a-b}}, \quad \text{and} \quad C = \frac{(-1)^c}{F_{a-c}F_{b-c}}. \tag{5}$$

Theorem 2 (The Reduction Algorithm): If $r > 2$, then any reciprocal sum of order r can be expressed in terms of reciprocal sums of order $r - 2$.

Proof: If $f(n)$ is any expression involving n, we see from Theorem 1 that

$$\sum_{n=1}^{N} \frac{1}{f(n)F_{n+a}F_{n+b}F_{n+c}} = \sum_{n=1}^{N} \frac{A(-1)^n}{f(n)F_{n+a}} + \sum_{n=1}^{N} \frac{B(-1)^n}{f(n)F_{n+b}} + \sum_{n=1}^{N} \frac{C(-1)^n}{f(n)F_{n+c}}, \tag{6}$$

with A, B, and C as given in equation (5). If $f(n)$ is the product of $r - 3$ factors, each of the form F_{n+c}, then this shows that a reciprocal sum of order r can be expressed in terms of reciprocal sums of order $r - 2$ for any integer $r > 2$. (If $r = 3$, then $f(n) = 1$.) Note that $f(n)$ may contain $(-1)^n$ as a factor to allow us to handle alternating reciprocal sums. □

Since we can repeatedly reduce the order of any reciprocal sum by 2, this shows that any reciprocal sum can be expressed in terms of reciprocal sums of orders 1 and 2.

3. RECIPROCAL SUMS OF ORDER 1

Any reciprocal sum of order 1 differs by a constant from expressions of the form $\mathbb{F}_{N+c}$ or $\mathbb{G}_{N+c}$. Specifically, if $a > 0$, then

$$\sum_{n=1}^{N} \frac{1}{F_{n+a}} = \sum_{n=1}^{a+N} \frac{1}{F_n} - \sum_{n=1}^{a} \frac{1}{F_n} = \mathbb{F}_{N+a} - \mathbb{F}_a \tag{7}$$

and

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \sum_{n=1}^{a+N} \frac{(-1)^n}{F_n} - \sum_{n=1}^{a} \frac{(-1)^n}{F_n} = \mathbb{G}_{N+a} - \mathbb{G}_a. \tag{8}$$

Thus, reciprocal sums of order 1 are readily computed in terms of $\mathbb{F}$'s and $\mathbb{G}$'s.

4. ALTERNATING RECIPROCAL SUMS OF ORDER 2

As has been pointed out, for reciprocal sums of order 2, we may assume that the denominator is of the form $F_n F_{n+a}$ with $a > 0$ for if not, the reciprocal sum differs by only a finite number of terms from one of this form.

There are two cases to consider, depending on whether the reciprocal sum is alternating or not.

In the alternating case, an explicit closed form can be found. The following result was proven by Brousseau [3] and Carlitz [5].

Theorem 3 (Computation of Alternating Reciprocal Sums of Order 2): If $a > 0$, then

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_n F_{n+a}} = \frac{1}{F_a}\left[\sum_{j=1}^{a} \frac{F_{j-1}}{F_j} - \sum_{j=1}^{a} \frac{F_{j+N-1}}{F_{j+N}}\right]. \tag{9}$$

Good [6] has found a different, but equivalent, expression for this reciprocal sum. He has shown that for $a > 0$,

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_n F_{n+a}} = \frac{F_N}{F_a} \sum_{n=1}^{a} \frac{(-1)^n}{F_n F_{n+N}}. \tag{10}$$

Another equivalent formulation is the following. We omit the proof.

$$\sum_{n=1}^{N} \frac{(-1)^n}{F_n F_{n+a}} = \frac{1}{F_a} \left[\sum_{j=1}^{a} \frac{F_{j+1}}{F_j} - \sum_{j=1}^{a} \frac{F_{j+N+1}}{F_{j+N}} \right]. \tag{11}$$

5. NONALTERNATING RECIPROCAL SUMS OF ORDER 2

We start with a preliminary result.

Theorem 4: Let H_n be any sequence of nonzero terms that satisfies the recurrence $H_{n+2} = H_{n+1} + H_n$. If $b \geq 0$, then

$$\sum_{n=1}^{N} \frac{1}{H_{n+b} H_{n+b+2}} = \frac{1}{H_{b+1} H_{b+2}} - \frac{1}{H_{N+b+1} H_{N+b+2}}. \tag{12}$$

Proof: We have

$$\frac{1}{H_{n+b} H_{n+b+2}} = \frac{H_{n+b+1}}{H_{n+b} H_{n+b+1} H_{n+b+2}} = \frac{H_{n+b+2} - H_{n+b}}{H_{n+b} H_{n+b+1} H_{n+b+2}}$$

$$= \frac{1}{H_{n+b} H_{n+b+1}} - \frac{1}{H_{n+b+1} H_{n+b+2}}.$$

Summing from 1 to N, we find that the right side telescopes, and we get the desired result. $\square$

Theorem 5: For $a > 0$, let

$$\mathbb{F}_N(a) = \sum_{n=1}^{N} \frac{1}{F_n F_{n+a}}. \tag{13}$$

If we can find a closed form expression for $\mathbb{F}_N(a - 2)$, then we can also find a closed form expression for $\mathbb{F}_N(a)$.

Proof: The following identity is well known (see equation (9) in [3]):

$$F_a F_{n+a-2} - F_{a-2} F_{n+a} = (-1)^a F_n. \tag{14}$$

Thus, we find that

$$\frac{F_a}{F_n F_{n+a}} - \frac{F_{a-2}}{F_n F_{n+a-2}} = \frac{(-1)^a}{F_{n+a-2} F_{n+a}}.$$

If we now sum as n goes from 1 to N, we get

$$F_a \mathbb{F}_N(a) - F_{a-2} \mathbb{F}_N(a - 2) = (-1)^a \sum_{n=1}^{N} \frac{1}{F_{n+a-2} F_{n+a}}.$$

Applying Theorem 4 gives

$$F_a \mathbb{F}_N(a) - F_{a-2} \mathbb{F}_N(a - 2) = (-1)^a \left[\frac{1}{F_{a-1} F_a} - \frac{1}{F_{N+a-1} F_{N+a}} \right]. \tag{15}$$

Solving for $\mathbb{F}_N(a)$ gives

$$\mathbb{F}_N(a) = \frac{F_{a-2}}{F_a}\mathbb{F}_N(a-2) + \frac{(-1)^a}{F_a}\left[\frac{1}{F_{a-1}F_a} - \frac{1}{F_{N+a-1}F_{N+a}}\right] \tag{16}$$

which shows that we can find $\mathbb{F}_N(a)$ if we know $\mathbb{F}_N(a-2)$. $\square$

By induction, we see that any expression of the form

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+a}},$$

with $a > 0$, can be expressed in terms of either

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+1}} \quad \text{or} \quad \sum_{n=1}^{N}\frac{1}{F_n F_{n+2}}.$$

The first form is known as $\mathbb{K}_N$. The second form is easily evaluated by setting $b = 0$ in Theorem 4 to get

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+2}} = 1 - \frac{1}{F_{N+1}F_{N+2}}. \tag{17}$$

We have just shown how to find a formula for any reciprocal sum of order 2 in terms of $\mathbb{K}_n$. We can also find a more explicit formula. If we let $a = 2c+1$ in formula (15), we get

$$F_{2c+1}\mathbb{F}_N(2c+1) - F_{2c-1}\mathbb{F}_N(2c-1) = (-1)^{2c+1}\left[\frac{1}{F_{2c}F_{2c+1}} - \frac{1}{F_{N+2c}F_{N+2c+1}}\right]. \tag{18}$$

Now sum as c goes from 1 to a. The left side telescopes, and we get

$$F_{2a+1}\mathbb{F}_N(2a+1) - \mathbb{K}_N = \sum_{c=1}^{a}\left[\frac{1}{F_{N+2c}F_{N+2c+1}} - \frac{1}{F_{2c}F_{2c+1}}\right]$$

so that

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+2a+1}} = \frac{1}{F_{2a+1}}\left\{\mathbb{K}_N + \sum_{c=1}^{a}\left[\frac{1}{F_{N+2c}F_{N+2c+1}} - \frac{1}{F_{2c}F_{2c+1}}\right]\right\}. \tag{19}$$

Similarly, if $a = 2c$, we can sum as c goes from 1 to a to get

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+2a}} = \frac{1}{F_{2a}}\sum_{c=1}^{a}\left[\frac{1}{F_{2c-1}F_{2c}} - \frac{1}{F_{N+2c-1}F_{N+2c}}\right]. \tag{20}$$

We can summarize these results with the following theorem.

Theorem 6: If a is a positive integer, then

$$\sum_{n=1}^{N}\frac{1}{F_n F_{n+a}} = \begin{cases}\frac{1}{F_a}\sum_{i=1}^{\lfloor a/2\rfloor}\left(\frac{1}{F_{N+2i}F_{N+2i+1}} - \frac{1}{F_{2i}F_{2i+1}}\right) + \frac{\mathbb{K}_N}{F_a}, & \text{if } a \text{ is odd,} \\ \frac{1}{F_a}\sum_{i=1}^{a/2}\left(\frac{1}{F_{2i-1}F_{2i}} - \frac{1}{F_{N+2i-1}F_{N+2i}}\right), & \text{if } a \text{ is even.}\end{cases} \tag{21}$$

These formulas give us the following values for $\mathbb{F}_N(a)$ for small a:

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+3}} = \frac{1}{2}\left[\mathbb{K}_N + \frac{1}{F_{N+2}F_{N+3}} - \frac{1}{2}\right]; \tag{22}$$

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+4}} = \frac{1}{3}\left[\frac{7}{6} - \frac{1}{F_{N+1}F_{N+2}} - \frac{1}{F_{N+3}F_{N+4}}\right]; \tag{23}$$

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+5}} = \frac{1}{5}\left[\mathbb{K}_N + \frac{1}{F_{N+2}F_{N+3}} + \frac{1}{F_{N+4}F_{N+5}} - \frac{17}{30}\right]; \tag{24}$$

$$\sum_{n=1}^{N} \frac{1}{F_n F_{n+6}} = \frac{1}{8}\left[\frac{143}{120} - \frac{1}{F_{N+1}F_{N+2}} - \frac{1}{F_{N+3}F_{N+4}} - \frac{1}{F_{N+5}F_{N+6}}\right]. \tag{25}$$

As $N \to \infty$ in formula (21), we get

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+a}} = \begin{cases} \dfrac{1}{F_a}\mathbb{K} - \dfrac{1}{F_a}\displaystyle\sum_{i=1}^{\lfloor a/2 \rfloor} \dfrac{1}{F_{2i}F_{2i+1}}, & \text{if } a \text{ is odd}, \\[3mm] \dfrac{1}{F_a}\displaystyle\sum_{i=1}^{a/2} \dfrac{1}{F_{2i-1}F_{2i}}, & \text{if } a \text{ is even}, \end{cases} \tag{26}$$

where $\mathbb{K} = \lim_{n\to\infty} \mathbb{K}_n$. For small values of a, these formulas yield the results found by Brousseau in [3].

6. SUMMARY

We have just shown that any reciprocal sum of order 1 can be expressed in terms of $\mathbb{F}_N$ and $\mathbb{G}_N$, and that any reciprocal sum of order 2 can be expressed in terms of $\mathbb{K}_N$. Thus, we can conclude that all reciprocal sums are expressible in terms of $\mathbb{F}_N, \mathbb{G}_N$, and $\mathbb{K}_N$. We also have presented a mechanical algorithm for finding all such representations.

Open Question 1: Is there a simple algebraic relationship between $\mathbb{L}_n = \sum_{n=1}^{N}(1/L_n)$ and any of $\mathbb{F}_N$, $\mathbb{G}_N$, and $\mathbb{K}_N$?

Open Question 2: Can we find the value of $\sum_{n=1}^{N}(1/F_n^2)$?

7. GOING TO INFINITY

If we take the limit as N goes to infinity, we can express many infinite sums in terms of

$$\mathbb{F} = \sum_{n=1}^{\infty} \frac{1}{F_n}, \quad \mathbb{G} = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}, \quad \mathbb{K} = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}}, \quad \mathbb{L} = \sum_{n=1}^{\infty} \frac{1}{L_n}, \text{ and } \mathbb{J} = \sum_{n=1}^{\infty} \frac{(-1)^n}{L_n}. \tag{27}$$

No simple expressions for these infinite sums are known; however, they have been expressed in terms of Elliptic Functions [4], Theta Series [7], [1], and Lambert Series [2].

For example, we get results of Brousseau [3], such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+a}} = \frac{1}{F_a}\left[\sum_{j=1}^{a} \frac{F_{j-1}}{F_j} - \frac{a}{\alpha}\right] \tag{28}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5} F_{n+6} F_{n+7} F_{n+8}} = \frac{319}{16380} \left(\mathbb{F} - \frac{46816051}{13933920} \right). \tag{29}$$

Carlitz has also found some pretty results for certain r^{th}-order reciprocal sums in terms of Fibonomial coefficients (see formulas (5.6), (5.7), and (6.7) in [5]).

Open Question 3: Are any of $\mathbb{F}, \mathbb{G}, \mathbb{K}, \mathbb{L}, \mathbb{J}$ connected by a simple algebraic relation?

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DIVISION OF FIBONACCI NUMBERS BY k

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1. AIM AND SCOPE OF THE PAPER

Notation

(i) F_n and L_n denote the nth Fibonacci and Lucas number, respectively.

(ii) r and s denote the residue of the subscript n [e.g., see (1.1)] modulo 2 and 4, respectively.

(iii) The symbol $\lfloor \cdot \rfloor$ denotes the greatest integer function.

(iv) $Z(k)$ denotes the so-called *entry point* of k in the Fibonacci sequence, that is, the smallest subscript m for which F_m is divisible by k.

(v) $P(k)$ denotes the repetition period of the Fibonacci sequence reduced modulo k.

The aim of this paper is to extend some results concerning the *Zeckendorf decomposition* (ZD, in brief) [6] of integers of the form F_n / d that have been established in [4]. More precisely, we shall determine the ZD of the integers of the form

$$F_{Z(k)n} / k \quad (n = 1, 2, 3, \ldots) \tag{1.1}$$

for certain values of the integer $k \geq 2$. The integrality of the numbers (1.1) is ensured by the definition of $Z(k)$ and by the well-known property $F_{nm} \equiv 0 \pmod{F_m}$.

This kind of study is, obviously, "endless" so that the choice of the values of k in (1.1) posed some problems for us. After some thinking, we decided to consider all values of $k \leq 11$ and the prime values of $k \leq 23$ (Section 2). More interesting results emerge from the ZD of (1.1) for special values of k, as shown in Sections 4, 5, and 6. The numerical values of $Z(k)$ have been taken from the list available in [1, pp. 33-41].

All of the results have been established by proving conjectures based on the behavior which became apparent through a study of early cases of n, once k was chosen. Conjecturing these results is, in most cases, a laborious task involving the use of a software package that can handle large-subscripted Fibonacci numbers, and a computer program for the ZD of large integers. On the other hand, once the conjectures are made, the proofs are not difficult but, in general, they are very lengthy and tedious so that giving them for all the results is unreasonable. In fact, only the (partial) proof of (2.4) and the complete proof of (5.4) are given (resp. Sections 3 and 5), just to show the technique we used. The main mathematical tools used in the proofs are the identities (1.4)-(1.6) of [3].

Note that $F_1 = 1$ and $F_2 = 1$ are used indifferently in the ZDs. Moreover, as usual, we assume that a sum vanishes whenever its upper range indicator is less than the lower one.

2. RESULTS

$$F_{3n}/2 = \sum_{j=1}^{n} F_{3j-2}. \tag{2.1}$$

$$F_{4n}/3 = rF_2 + \sum_{j=1}^{\lfloor n/2 \rfloor}(F_{4(2j+r)-5} + F_{4(2j+r)-3}). \tag{2.2}$$

$$F_{6n}/4 = \sum_{j=1}^{n} F_{6j-3}. \tag{2.3}$$

$$F_{5n}/5 = x(n) + \sum_{j=1}^{\lfloor n/4 \rfloor}\left(F_{5(4j+s)-17} + F_{5(4j+s)-14} + F_{5(4j+s)-4} + \sum_{i=1}^{3} F_{5(4j+s-1)-2i}\right), \tag{2.4}$$

where

$$x(n) = \begin{cases} 0, & \text{if } s = 0, \\ F_2, & \text{if } s = 1, \\ F_4 + F_6, & \text{if } s = 2, \\ F_2 + F_4 + F_6 + F_8 + F_{11}, & \text{if } s = 3. \end{cases} \tag{2.4'}$$

$$F_{12n}/6 = r(F_4 + F_8) + \sum_{j=1}^{\lfloor n/2 \rfloor}\left(F_{12(2j+r)-21} + F_{12(2j+r)-18} + F_{12(2j+r)-16}\right.$$
$$\left. + F_{12(2j+r)-4} + \sum_{i=1}^{3} F_{12(2j+r)-2i-7}\right). \tag{2.5}$$

$$F_{8n}/7 = rF_4 + \sum_{j=1}^{\lfloor n/2 \rfloor} \sum_{i=1}^{4} F_{8(2j+r)-2i-3}. \tag{2.6}$$

$$F_{6n}/8 = rF_2 + \sum_{j=1}^{\lfloor n/2 \rfloor}(F_{6(2j+r)-7} + F_{6(2j+r)-5}). \tag{2.7}$$

$$F_{12n}/9 = r(F_4 + F_7) + \sum_{j=1}^{\lfloor n/2 \rfloor}\left(F_{12(2j+r)-19} + F_{12(2j+r)-5} + \sum_{i=1}^{4} F_{12(2j+r)-2i-7}\right). \tag{2.8}$$

The complete ZD of $F_{15n}/10$ is extremely cumbersome and unpleasant. We confine ourselves to showing it only for $s = 0$.

$$F_{15n}/10 = \sum_{j=1}^{n/4}\left(F_{60j-55} + F_{60j-51} + F_{60j-48} + F_{60j-42} + F_{60j-40} + F_{60j-37} + F_{60j-23}\right.$$
$$\left. + F_{60j-21} + F_{60j-16} + F_{60j-14} + F_{60j-10} + F_{60j-5} + \sum_{i=1}^{4} F_{60j-2i-25}\right). \tag{2.9}$$

$$F_{10n}/11 = \sum_{j=1}^{n} F_{10j-5}. \tag{2.10}$$

$$F_{7n}/13 = x(n) + \sum_{j=1}^{\lfloor n/4 \rfloor}\left(F_{7(4j+s)-23} + F_{7(4j+s)-20} + F_{7(4j+s)-6} + \sum_{i=1}^{5} F_{7(4j+s-1)-2i}\right), \tag{2.11}$$

where

$$x(n) = \begin{cases} 0, & \text{if } s = 0, \\ F_2, & \text{if } s = 1, \\ F_6 + F_8, & \text{if } s = 2, \\ F_2 + F_4 + F_6 + F_8 + F_{10} + F_{12} + F_{15}, & \text{if } s = 3. \end{cases}$$ (2.11′)

$$F_{9n} / 17 = x(n) + \sum_{j=1}^{\lfloor n/4 \rfloor} \left(F_{9(4j+s)-31} + F_{9(4j+s)-6} + \sum_{i=1}^{3} F_{9(4j+s)-2i-22} + \sum_{i=1}^{5} F_{9(4j+s)-2i-11} \right),$$ (2.12)

where

$$x(n) = \begin{cases} 0, & \text{if } s = 0, \\ F_3, & \text{if } s = 1, \\ F_6 + F_{12}, & \text{if } s = 2, \\ F_4 + F_6 + F_8 + F_{10} + F_{12} + F_{14} + F_{21}, & \text{if } s = 3. \end{cases}$$ (2.12′)

$$F_{18n} / 19 = \sum_{j=1}^{n} \sum_{i=1}^{3} F_{18j-2i-5}.$$ (2.13)

$$F_{24n} / 23 = r(F_6 + F_9 + F_{14} + F_{17}) + \sum_{j=1}^{\lfloor n/2 \rfloor} \left(F_{24(2j+r)-41} + F_{24(2j+r)-39} + F_{24(2j+r)-36} \right.$$

$$\left. + F_{24(2j+r)-34} + F_{24(2j+r)-15} + F_{24(2j+r)-10} + F_{24(2j+r)-7} + \sum_{i=1}^{6} F_{24(2j+r)-2i-17} \right).$$ (2.14)

3. A PROOF

Proof of (2.4) (for $s = 1$)

Use (1.4)-(1.6) of [3] and (2.4′) to rewrite the right-hand side of (2.4) as

$$F_2 + \sum_{j=1}^{(n-1)/4} F_{20j-12} + \sum_{j=1}^{(n-1)/4} (F_{20j-9} + F_{20j+1}) + \sum_{j=1}^{(n-1)/4} \sum_{i=1}^{3} F_{20j-2i}$$

$$= F_2 + \sum_{j=1}^{(n-1)/4} F_{20j-12} + 5 \sum_{j=1}^{(n-1)/4} L_{20j-4} + 4 \sum_{j=1}^{(n-1)/4} F_{20j-4}$$

$$= F_2 + \frac{F_{5n+3} - F_{5n-17} - 165}{L_{20} - 2} + 5 \frac{L_{5n+11} - L_{5n-9} - 2200}{L_{20} - 2} + 4 \frac{F_{5n+11} - F_{5n-9} - 990}{L_{20} - 2}$$

$$= F_2 + \frac{55 L_{5n-7} - 165}{15125} + 5 \frac{275 F_{5n+1} - 2200}{15125} + 4 \frac{55 L_{5n+1} - 990}{15125}$$

$$= \frac{L_{5n-7} + 25 F_{5n+1} + 4 L_{5n+1}}{275} = \frac{L_{5n-7} + 5 L_{5n} + 5 L_{5n+2} + 4 L_{5n+1}}{275} \quad \text{(from } I_9 \text{ of } [5])$$

$$= \frac{L_{5n-7} + 10 L_{5n+2} - L_{5n+1}}{275} = \frac{10 L_{5n+2} - 15 F_{5n-3}}{275} = \frac{2 L_{5n+2} - 3 F_{5n-3}}{55}$$

$$= \frac{2F_{5n+3}+2F_{5n+1}-3F_{5n-3}}{55} = \frac{2(F_{5n+3}-F_{5n-3})+2F_{5n+1}-F_{5n-3}}{55}$$

$$= \frac{8F_{5n}+2F_{5n+1}-F_{5n-3}}{55} = \frac{8F_{5n}+3F_{5n}}{55} = \frac{F_{5n}}{5}. \quad \text{Q.E.D.}$$

We do not exclude the possibility that a shorter proof can be given.

4. RESULTS FOR SOME SPECIAL VALUES OF k

From the results presented in Section 2, we noted that the ZD of $F_{Z(k)n}/k$ is independent of the residue of n modulo 2 or 4 whenever

$$Z(k) = P(k), \tag{4.1}$$

so that its expression assumes a rather simple form [cf. (2.1), (2.3), (2.10), and (2.13)]. We were not able to find conditions for (4.1) to hold for k arbitrary. On the other hand, it is known (see (19) of [2]) that, if p is a prime, then

$$Z(p) = P(p) \quad \text{iff} \quad F_{Z(p)-1} \equiv 1 \pmod{p}. \tag{4.2}$$

A list of the first few values of k satisfying (4.1) and their $Z(k)$ is available in [1, p. 33]. In the following, we show the ZD of $F_{Z(k)n}/k$ for all such k greater than 19 [cf. (2.13)] and not exceeding 71:

$$F_{30n}/22 = \sum_{j=1}^{n}\left[F_{30j-23}+\sum_{i=1}^{3}(F_{30j-3i-12}+F_{30j-3i-4})\right]; \tag{4.3}$$

$$F_{14n}/29 = \sum_{j=1}^{n}F_{14j-7}; \tag{4.4}$$

$$F_{30n}/31 = \sum_{j=1}^{n}\left(F_{30j-15}+F_{30j-10}+\sum_{i=1}^{4}F_{30j-5i-3}\right); \tag{4.5}$$

$$F_{18n}/38 = \sum_{j=1}^{n}(F_{18j-11}+F_{18j-8}); \tag{4.6}$$

$$F_{30n}/44 = \sum_{j=1}^{n}(F_{30j-23}+F_{30j-20}+F_{30j-18}+F_{30j-14}+F_{30j-8}); \tag{4.7}$$

$$F_{42n}/58 = \sum_{j=1}^{n}\left(F_{42j-33}+F_{42j-20}+\sum_{i=1}^{3}F_{42j-3i-22}+\sum_{i=1}^{4}F_{42j-3i-6}\right); \tag{4.8}$$

$$F_{58n}/59 = \sum_{j=1}^{n}\left(F_{58j-47}+F_{58j-44}+F_{58j-26}+F_{58j-16}+F_{58j-12}+F_{58j-9}\right.$$
$$\left.+\sum_{i=1}^{3}F_{58j-14i-7}+\sum_{i=1}^{5}F_{58j-5i-13}\right); \tag{4.9}$$

$$F_{30n}/62 = \sum_{j=1}^{n}\left(F_{30j-21}+F_{30j-9}+\sum_{i=1}^{3}F_{30j-2i-11}\right); \tag{4.10}$$

$$F_{70n} / 71 = \sum_{j=1}^{n} \left[F_{70j-61} + F_{70j-48} + F_{70j-36} + F_{70j-23} + F_{70j-15} + F_{70j-9} \right.$$

$$\left. + \sum_{i=1}^{3} (F_{70j-12i-17} + F_{70j-5i-40}) + \sum_{i=1}^{4} F_{70j-7i-11} \right].$$

(4.11)

5. RESULTS FOR *VERY* SPECIAL VALUES OF k

Inspection of the results established in Sections 2 and 4 shows that, for $k = 2, 4, 11$, and 29, the ZD of $F_{Z(k)n} / k$ is constituted by exactly n addends. If we disregard the value 2, it is quite natural to conjecture that the ZD of $F_{Z(L_{2h+1})n} / L_{2h+1}$ $(h = 1, 2, 3, ...)$ has n addends.

Question 1. What is the value of $Z(L_{2h+1})$?

Theorem IV of [5, p. 40], which is credited to L. Carlitz, immediately gives the answer:

$$Z(L_{2h+1}) = 4h + 2.$$

(5.1)

In fact, we state the following proposition.

Proposition 1: For $h = 1, 2, 3, ...$, we have

$$\frac{F_{Z(L_{2h+1})n}}{L_{2h+1}} = \frac{F_{(4h+2)n}}{L_{2h+1}} = \sum_{j=1}^{n} F_{(4h+2)j-2h-1}.$$

(5.2)

The proof of Proposition 1 can be obtained simply by letting $n = 2h + 1$ and $k = n$ in (2.2) of [3].

Question 2. Apart from the particular case $k = 2$ [see (2.1)], do there exist other values of k, that are not odd-subscripted Lucas numbers, for which the ZD of $F_{Z(k)n} / k$ is constituted by exactly n addends?

A computer search showed that none of them exists for $k \le 1000$. This search has not been completely useless as it allowed us to discover a misprint in Brousseau's table of entry points [1, pp. 33-41] where it is reported that $Z(961) = 839$; the correct value is 930.

The ZD of $F_{18n} / 38$ has exactly $2n$ addends [see (4.6)]. The only further value of $k \le 1000$ for which such a decomposition occurs is $k = 682$. Namely, we have

$$F_{30n} / 682 = \sum_{j=1}^{n} (F_{30j-17} + F_{30j-14}).$$

(5.3)

The decompositions (4.6) and (5.3) led us to discover the following result, the proof of which is appended below.

Proposition 2: For $h = 1, 2, 3, ...$, we have

$$\frac{F_{Z(L_{6h+3}/2)n}}{L_{6h+3} / 2} = \frac{F_{(12h+6)n}}{L_{6h+3} / 2} = \sum_{j=1}^{n} (F_{(12h+6)j-6h-5} + F_{(12h+6)j-6h-2}).$$

(5.4)

Remark: Conjecturing this result has been very laborious but, once the conjecture has been made, its proof is quite easy. Observe that expression (5.4) works for $h = 0$ as well, but $12 \cdot 0 + 6 = 6$ is not the entry point of $L_{6 \cdot 0+3} / 2 = 2$.

Proof of Proposition 2: First, observe that, for $h \geq 1$,

$$L_{6h+3} = 4v \quad \text{with} \quad v = 1 + \sum_{r=1}^{h} L_{6r} \quad (v > 1 \text{ odd}), \tag{5.5}$$

and use (5.5) and the basic properties of $Z(k)$ [2, p. 9] to establish that

$$Z(L_{6h+3}/2) = Z(L_{6h+3}) \quad (h \geq 1). \tag{5.6}$$

Then, use Theorem IV of [5, p. 40] to write

$$Z(L_{6h+3}) = 12h + 6. \tag{5.7}$$

Finally, rewrite the right-hand side of (5.4) as

$$2 \sum_{j=1}^{n} F_{(12h+6)j-6h-3} = \frac{2[F_{(12h+6)n+6h+3} - F_{(12h+6)n-6h-3}]}{L_{12h+6} - 2} \quad \text{(from (1.4) of [3])}$$

$$= \frac{2F_{(12h+6)n}L_{6h+3}}{L_{12h+6} - 2} = \frac{2F_{(12h+6)n}L_{6h+3}}{L_{6h+3}^2} \quad \text{(from (1.5) of [3] and } I_{18} \text{ of [5])}$$

$$= \frac{F_{(12h+6)n}}{L_{6h+3}/2} = \frac{F_{Z(L_{6h+3}/2)n}}{L_{6h+3}/2} \quad \text{[from (5.7) and (5.6)]. Q.E.D.}$$

6. CONCLUDING REMARKS

The characterization of classes of integers k for which the ZD of $F_{Z(k)n}/k$ is constituted by n q-tuples of Fibonacci numbers whose subscripts are in arithmetical progression seems to be an interesting subject of study, and might be the aim of a future investigation. Propositions 1 and 2 give a solution to this problem for $q = 1$ and 2, respectively. For $q = 3$, we state the following proposition [cf. (2.13)].

Proposition 3: For $h = 1, 2, 3, \ldots$, we have

$$\frac{F_{Z(L_{6h+3}/4)n}}{L_{6h+3}/4} = \frac{F_{(12h+6)n}}{L_{6h+3}/4} = \sum_{j=1}^{n} \sum_{i=1}^{3} F_{(12h+6)j-2i-6h+1}. \tag{6.1}$$

The decompositions

$$\frac{F_{30n}}{L_{10}+1} = \sum_{j=1}^{n} \sum_{i=1}^{5} F_{30j-2i-9} \quad \text{and} \quad \frac{F_{42n}}{L_{14}+1} = \sum_{j=1}^{n} \sum_{i=1}^{7} F_{42j-2i-13} \tag{6.2}$$

seem to be a good starting point for the above-mentioned investigation. In fact, they led us to claim that

$$\frac{F_{Z(L_{2q}+1)n}}{L_{2q}+1} = \frac{F_{6qn}}{L_{2q}+1} = \sum_{j=1}^{n} \sum_{i=1}^{q} F_{6qj-2i-2q+1} \quad (q \text{ odd}). \tag{6.3}$$

Observe that both letting $h = 1$ in (6.1) and letting $q = 3$ in (6.3), yield (2.13). The proofs of (6.1) and (6.3) are left as an exercise for the interested reader. Note that the proof of (6.3) involves the use of the identity $L_{3q}/L_q = L_{2q} + 1$ (see (2.5) of [3]).

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THE GIRARD-WARING POWER SUM FORMULAS FOR SYMMETRIC FUNCTIONS AND FIBONACCI SEQUENCES

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Dedicated to Professor Leonard Carlitz on his 90th birthday.

The very widely-known identity

$$\sum_{0 \le k \le n/2} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k} (xy)^k = x^n + y^n,$$ (1)

which appears frequently in papers about Fibonacci numbers, and the formula of Carlitz [4], [5],

$$\sum_{i+2j+3k=n} (-1)^j \frac{n}{i+j+k} \frac{(i+j+k)!}{i!\,j!\,k!} (x+y+z)^{n-3k} (xy+yz+zx)^k = x^n + y^n + z^n,$$ (2)

summed over all $0 \le i,\ j, k \le n$, $n > 0$, and where $xyz = 1$, as well as the formula

$$\sum_{0 \le k \le n/3} \frac{n}{n-2k} \binom{n-2k}{k} (x+y+z)^{n-3k} (xyz)^k = x^n + y^n + z^n,$$ (3)

where $xy + yz + zx = 0$, are special cases of an older well-known formula for sums of powers of roots of a polynomial which was evidently first found by Girard [12] in 1629, and later given by Waring in 1762 [29], 1770 and 1782 [30]. These may be derived from formulas due to Sir Isaac Newton.

The formulas of Newton, Girard, and Waring do not seem to be as well known to current writers as they should be, and this is the motivation for our remarks: to make the older results more accessible. Our paper was motivated while refereeing a paper [32] that calls formula (1) the "Kummer formula" (who came into the matter very late) and offers formula (3) as a generalization of (1), but with no account of the extensive history of symmetric functions.

The Girard-Waring formula may be derived from what are called Newton's formulas. These appear in classical books on the "Theory of Equations." For example, see Dickson [9, pp. 69-74], Ferrar [11, pp. 153-80], Turnbull [27, pp. 66-80], or Chrystal [6, pp. 436-38]. The most detailed of these is Dickson's account. But the account by Vahlen [28, pp. 449-79] in the old German Encyclopedia is certainly the best historical record.

Let $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n = 0$ be a polynomial with roots x_1, $x_2, \ldots, x_n$, so that $f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$. Then we define the "elementary" symmetric functions $a_1, a_2, a_3, \ldots$ of the roots as follows:

$$\sum_{1 \le i \le n} x_i = x_1 + x_2 + \cdots + x_n = -a_1,$$

$$\sum_{1 \le i < j \le n} x_i x_j = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + \cdots = a_2,$$

$$\sum_{1 \le i < j < k \le n} x_i x_j x_k = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_2 x_5 + \cdots = -a_3,$$

$$\ldots$$

$$x_1 x_2 x_3 \ldots x_n = (-1)^n a_n.$$

Also, we define the kth power sums of the roots by

$$s_k = \sum_{1 \le i \le n} x_i^k = x_1^k + x_2^k + \cdots + x_n^k.$$ (4)

Then the Newton formulas are:

$$\begin{cases} s_1 + a_1 = 0, \\ s_2 + a_1 s_1 + 2a_2 = 0, \\ s_3 + a_1 s_2 + a_2 s_1 + 2a_3 = 0, \\ \cdots \\ s_n + a_1 s_{n-1} + a_2 s_{n-2} + \cdots + na_n = 0, \\ s_{n+1} + a_1 s_{n-1} + a_2 s_{n-2} + \cdots + s_1 a_n = 0. \end{cases}$$ (5)

These equations may be solved by determinants to express s_n in terms of a_n, or conversely. These determinant formulas are as follows:

$$(-1)^n s_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 2a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ na_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix},$$ (6)

and

$$(-1)^n n! a_n = \begin{vmatrix} s_1 & 1 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & 2 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}.$$ (7)

Turnbull [27, p. 74] gives both these determinants, but not the expanded forms. Father Hagen [16, p. 310] cites Salmon [26] for the determinants. Hagen and Vahlen note that we may write these determinants in expanded form, which is the source of the Girard-Waring formula

$$s_j = j \sum (-1)^{k_1 + k_2 + \cdots + k_n} \frac{(k_1 + k_2 + \cdots + k_n - 1)!}{k_1! k_2! \ldots k_n!} a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n}$$ (8)

summed over all nonnegative integers k_i satisfying the equation $k_1 + 2k_2 + \cdots + nk_n = j$, and its inverse

$$a_j = \sum (-1)^{k_1 + k_2 + \cdots + k_j} \frac{1}{k_1! k_2! \ldots k_j!} \left(\frac{s_1}{1}\right)^{k_1} \left(\frac{s_2}{2}\right)^{k_2} \ldots \left(\frac{s_j}{j}\right)^{k_j}$$ (9)

summed over all nonnegative integers k_i satisfying the equation $k_1 + 2k_2 + \cdots + jk_j = j$. Vahlen [28, p. 451] gives (8) and (9) and a very detailed account of the theory of symmetric functions with historical references.

The determinant (7) and expansion (9), without mention of their use in connection with Girard-Waring power sums, were noted by Thomas Muir [21, pp. 216-17]. Muir cites G. Mola [10] for his second paper on questions of E. Fergola [10] for the expanded form of (7), i.e., (9). Note Problem 67-13 [31] where W. B. Gragg has an incorrect citation, misunderstanding Muir's citation.

Of course, (9) was already known to Waring. A short proof of (9) was given by Richter [23] using generating functions and the multinomial theorem.

With $n = 2$, and suitable change in notation, relation (8) yields

$$x^n + y^n = n \sum_{i+2j=n} (-1)^{i+j} \frac{(i+j+1)!}{i!\,j!} a^i b^j \text{ summed for } 0 \le i,\ j \le n, \tag{10}$$

where $-a = x + y$ and $b = xy$. Clearly, this is the same as formula (1) when we make the dummy variable substitution, $i = n - 2j$.

With $n = 3$, and suitable change in notation, relation (8) yields

$$x^n + y^n + z^n = n \sum_{i+2j+3k=n} (-1)^{i+j+k} \frac{(i+j+k-1)!}{i!\,j!\,k!} a^i b^j c^k, \tag{11}$$

where $-a = x + y + z$, $b = xy + yz + zx$, and $-c = xyz$. When we set $c = 1$ in this, we obtain formula (2) above of Carlitz.

Or, if we specialize (11) by setting $b = 0$, then the only value assumed by j in the summation is $j = 0$, so that we obtain

$$x^n + y^n + z^n = \sum_{i+3k=n} \frac{n}{i+k} \frac{(i+k)!}{i!\,k!} (x+y+z)^i (xyz)^k \tag{12}$$

summed over all $0 \le i,\ k \le n$. Replace i by $n - 3k$ to get

$$x^n + y^n + z^n = \sum_{0 \le k \le n/3} \frac{n}{n-2k} \frac{(n-2k)!}{k!(n-3k)!} (x+y+z)^{n-2k} (xyz)^k, \tag{13}$$

which is formula (3).

For the sum of four n^{th} powers, we obtain

$$x^n + y^n + z^n + w^n = n \sum (-1)^{i+j+k+\ell} \frac{(i+j+k+\ell-1)!}{i!\,j!\,k!\,\ell!} a^i b^j c^k d^\ell \tag{14}$$

summed over all $0 \le i, j, k, \ell \le n$ with $i + 2j + 3k + 4\ell = n$, and where

$$\begin{aligned} -a &= x + y + z + w, & b &= xy + xz + xw + yz + yw + zw, \\ -c &= xyz + xyw + xzw + yzw, & d &= xyzw. \end{aligned}$$

By making special choices for a, b, c, and d, we could obtain simpler formulas somewhat analogous to (1), (2), or (3). For example, let $b = c = 0$, and we get

$$x^n + y^n + z^n + w^n = \sum_{0 \le k \le n/4} (-1)^{n-3k} \frac{n}{n-3k} \binom{n-3k}{k} a^{n-3k} b^k, \tag{15}$$

subject to $xy + xz + xw + yz + yw + zw = xyz + xyw + xzw + yzw = 0$. With similar conditions on the symmetric functions of the roots, we can obtain power sum formulas of the type

$$s_n = \sum_{1 \le i \le r} x_i^n = \sum_{0 \le k \le n/r+1} (-1)^{n-rk} \frac{n}{n-rk} \binom{n-rk}{k} a^{n-rk} b^k. \tag{16}$$

H. A. Rothe [25], a student of Carl F. Hindenburg of the old German school of combinatorial analysis, found the formula

$$\sum_{k=0}^{n} A_k(a,b) A_{n-k}(c,b) = A_n(a+c,b) \tag{17}$$

where

$$A_k(a,b) = \frac{a}{a+bk} \binom{a+bk}{k}, \tag{18}$$

where a, b, and c may be any complex numbers. Formula (17) generalizes the Vandermonde convolution that occurs when $b = 0$.

It is perhaps of interest to note that J. L. Lagrange evidently believed it would be possible to solve the polynomial equation $f(x) = 0$ with a finite closed formula by manipulating the symmetric functions of the roots. Closed formulas using root extractions are, of course, possible for the quadratic, cubic, and biquadratic, but the work of Galois and Abel showed that this is impossible for equations of fifth degree or higher.

Lagrange did succeed in getting infinite series expansions of the roots in his memoir of 1770. For the simple trinomial equation $x^n + px + q = 0$, the series involve binomial coefficient functions of the form (18). More generally, Lagrange developed what is now called the Lagrange-Bürmann inversion formula for series. See any advanced complex analysis text.

Reference [15] contains a list of twelve of my papers since 1956 dealing with the generalized Vandermonde convolution. The first two [13] and [14] deal with Rothe's work. I first became interested in (17) when I found it in Hagen's *Synopsis* [16].

Closely associated with (17)-(18) are the coefficients

$$C_k(a, b) = \binom{a + bk}{k},\tag{19}$$

and, in fact, we easily obtain

$$\sum_{k=0}^{n} C_k(a, b) A_{n-k}(c, b) = C_n(a + c, b).\tag{20}$$

Relating to Fibonacci research is the fact that $C_k(n, -1)$ occurs in a popular formula for the Fibonacci numbers that is easily noted by looking at diagonal sums. This formula is

$$F_{n+1} = \sum_{0 \le k \le n/2} C_k(n, -1) = \sum_{0 \le k \le n/2} \binom{n-k}{k}.\tag{21}$$

This arises because the dual to relation (1) is

$$\sum_{0 \le k \le n/2} (-1)^k \binom{n-k}{k}(x + y)^{n-2k}(xy)^k = \frac{x^{n+1} - y^{n+1}}{x - y}.\tag{22}$$

The classical Fibonacci numbers arise with x and y as the roots of the characteristic equation $z^2 - z - 1 = 0$ that is associated with the Fibonacci recurrence $F_{n+2} - F_{n+1} - F_n = 0$. With the same choices, relation (1) yields the classical Lucas numbers L_n for which, of course, the recurrence relation is $L_{n+2} - L_{n+1} - L_n = 0$.

The A coefficients (18) and the C coefficients (19) may be thought of as relating to generalized Lucas and Fibonacci numbers, respectively, the difference being in the presence of the fraction $a / (a + bk)$. Charles A. Church [7] noted precisely this, defining

$$F_{n+1} = \sum_{0 \le k \le n/b+1} \binom{n - bk}{k} \quad \text{and} \quad L_{n+1} = \sum_{0 \le k \le n/b+1} \frac{n}{n - bk}\binom{n - bk}{k}.$$

More general Fibonacci numbers U_n with

$$U_n = \sum_{0 \le k \le n/r+1} \binom{n - rk}{k} a^{n-rk} b^k\tag{23}$$

were studied in the first volume of *The Fibonacci Quarterly* by J. A. Raab [22]. V. C. Harris and Carolyn Stiles [17] then introduced a generalized Fibonacci sequence that satisfies

$$A_n = \sum_{0 \le k \le n/p+q} \binom{n-pk}{qk} a^{n-rk} b^k \qquad (24)$$

with $a = b = 1$. Verner E. Hoggatt, Jr. [18] wrote about this. Papers too numerous to list (before and since) have studied such sequences. David Dickinson [8] and others have noted that expressions of the form

$$S_m(t) = \sum_k \binom{n+ak}{b+ck} t^{b+ck}, \quad \text{where } m = nc - ab, \ |t| \le 1, \qquad (25)$$

summed over meaningful ranges of k essentially satisfy recursive relations and therefore may be seen as generalized Fibonacci numbers. He finds the associated trinomial equation $x^c - tx^a - 1 = 0$, which has distinct roots α_r, $r = 0, 1, ..., c-1$, and obtains the formula

$$S_m(t) = \sum_{0 \le r \le c-1} \frac{\alpha_r^m}{c - at\alpha_r^{a-c}}. \qquad (26)$$

The late Leon Bernstein [1], [2], [3] examined the zeros of the function

$$f(n) = \sum_{0 \le i \le n/2} (-1)^i \binom{n-2i}{i}, \qquad (27)$$

finding that it has only the zeros 3 and 12. He also found several new combinatorial identities. Carlitz [4], [5] then studied Bernstein's work and in these papers he developed relation (2) above. His techniques are the usual generating function and multinomial theorem approach. Neither Bernstein nor Carlitz mentions or uses the work of Girard and Waring, but (11) with $c = 1$ is found by Carlitz, so that the Girard-Waring formula is implicit there.

We may note that the late John Riordan [24, p. 47] related power sum symmetric functions to the general Bell polynomials he loved to study.

The Girard-Waring formulas offer many extensions of the ordinary Fibonacci-Lucas formulas. These formulas should be more well known.

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A GENERALIZATION OF JACOBSTHAL POLYNOMIALS

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1. INTRODUCTION

In a recent article, André-Jeannin [1] introduced a generalization of the Morgan-Voyce polynomials by defining the sequence of polynomials $\{P_n^{(r)}(x)\}$ by the relation

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x) \quad (n \geq 2),$$ (1a)

with

$$P_0^{(r)}(x) = 1 \quad \text{and} \quad P_1^{(r)}(x) - x+r+1.$$ (1b)

Subsequently, Horadam [2] defined a closely related sequence of polynomials $\{Q_n^{(r)}(x)\}$ by the relation

$$Q_n^{(r)}(x) = (x+2)Q_{n-1}^{(r)}(x) = Q_{n-2}^{(r)}(x) \quad (n \geq 2),$$ (2a)

with

$$Q_0^{(r)}(x) = 1 \quad \text{and} \quad Q_1^{(r)}(x) = x+r+2.$$ (2b)

They also established that

$$P_n^{(0)}(x) = b_n(x),$$ (3a)

$$P_n^{(1)}(x) = B_n(x),$$ (3b)

and

$$Q_n^{(0)}(x) = C_n(x),$$ (3c)

where $b_n(x)$ and $B_n(x)$ are the classical Morgan-Voyce polynomials defined in [6] and $C_n(x) = 2c_n(x)$, where $c_n(x)$ is the polynomial introduced by Swamy and Bhattacharyya [7] in the analysis of ladder networks. It has also been established in [1] and [2] that, if

$$P_n^{(r)}(x) = \sum_{k \geq 0} a_{n,k}^{(r)} x^k$$ (4a)

and

$$Q_n^{(r)}(x) = \sum_{k \geq 0} b_{n,k}^{(r)} x^k,$$ (4b)

then, for any $n \geq 0$, $k \geq 0$,

$$a_{n,k}^{(r)} = \binom{n+k}{2k} + r\binom{n+k}{2k+1}$$ (5a)

and

$$b_{n,k}^{(r)} = \frac{n}{k}\binom{n-1+k}{2k-1} + r\binom{n+k}{2k+1}.$$ (5b)

The purpose of this short article is to introduce two new sequences of polynomials $\{p_n^{(r)}(x)\}$ and $\{q_n^{(r)}(x)\}$, and then relate them to Jacobsthal polynomials (see [3] and [5]).

2. THE NEW POLYNOMIALS $\{p_n^{(r)}(x)\}$ AND $\{q_n^{(r)}(x)\}$

Let us define the following two sequences of polynomials:

$$p_n^{(r)}(x) = p_{n-1}^{(r)}(x) + xp_{n-2}^{(r)}(x) \quad (n \geq 2) \tag{6a}$$

with

$$p_0^{(r)}(x) = 1 \quad \text{and} \quad p_1^{(r)}(x) = r, \tag{6b}$$

and

$$q_n^{(r)}(x) = q_{n-1}^{(r)}(x) + xq_{n-2}^{(r)}(x) \quad (n \geq 2) \tag{7b}$$

with

$$q_0^{(r)}(x) = 2 \quad \text{and} \quad q_1^{(r)}(x) = r + 1, \tag{7b}$$

where r is a real number.

If we now express $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$ by

$$p_n^{(r)}(x) = \sum_{k \geq 0} c_{n,k}^{(r)} x^k \tag{8a}$$

and

$$q_n^{(r)}(x) = \sum_{k \geq 0} d_{n,k}^{(r)} x^k, \tag{8b}$$

we can obtain recurrence relations for $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$, and derive expressions for $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$ using the procedures adopted in [1] and [2]. However, we will use the properties of the sequence $w_n(a, b; p, q)$ defined by Horadam [4] to obtain a direct expression for $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$. From [4], we know that the solution $w_n(a, b; x)$ of the equation

$$w_n(x) = w_{n-1}(x) + xw_{n-2}(x) \quad (n \geq 2) \tag{9a}$$

with

$$w_0(x) = a \quad \text{and} \quad w_1(x) = b \tag{9b}$$

is given by

$$w_n(x) = w_1(x)u_{n-1}(x) + xw_0(x)u_{n-2}(x), \tag{10a}$$

where

$$u_n(x) = w_n(1, 1; x). \tag{10b}$$

Hence, from (6), (7), (9), and (10), we have

$$p_n^{(r)}(x) = ru_{n-1}(x) + xu_{n-2}(x) = u_n(x) + (r-1)u_{n-1}(x) \tag{11a}$$

and

$$q_n^{(r)}(x) = (r+1)u_{n-1}(x) + 2xu_{n-2}(x) = u_n(x) + xu_{n-2}(x) + ru_{n-1}(x). \tag{11b}$$

3. $p_n^{(r)}(x)$, $q_n^{(r)}(x)$ AND JACOBSTHAL POLYNOMIALS

We now observe that

$$u_n(x) = J_{n+1}(x) \tag{12a}$$

and

$$v_n(x) = j_n(x), \tag{12b}$$

where $J_n(x)$ and $j_n(x)$ are the Jacobsthal polynomials (see [3]). It is to be noted that we have used x instead of $(2x)$ in the definitions of $J_n(x)$ and $j_n(x)$; that is, $J_n(x)$ and $j_n(x)$ are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \quad J_o(x) = 0, \; J_o(x) = 1, \tag{13a}$$

and

$$j_n(x) = j_{n-1}(x) + xj_{n-2}(x), \quad j_o(x) = 2, \; j_o(x) = 1. \tag{13b}$$

Hence, from (11), (12), and (13), we get

$$p_n^{(r)}(x) = J_{n+1}(x) + (r-1)J_n(x) \tag{14a}$$

and

$$q_n^{(r)}(x) = j_n(x) + rJ_n(x), \tag{14b}$$

where we have used the relation

$$j_n(x) = J_{n+1}(x) + xJ_{n-1}(x). \tag{15}$$

Using the closed-form expressions for $J_n(x)$ and $j_n(x)$ derived in [3], we can derive polynomial expressions for $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$ and, thus, expressions for $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$. Also, the interrelationship between $c_{n,k}^{(r)}$ and $d_{n,k}^{(r)}$ may be expressed in terms of the following relation, which is a consequence of (14a) and (14b):

$$q_n^{(r)}(x) = p_n^{(r)}(x) + J_n(x) + xJ_{n-1}(x). \tag{16}$$

Of course, we have the following particular cases:

$$p_n^{(0)}(x) = xJ_{n-1}(x), \tag{17a}$$

$$p_n^{(1)}(x) = J_{n+1}(x), \tag{17b}$$

$$q_n^{(0)}(x) = j_n(x), \tag{17c}$$

$$q_n^{(1)}(x) = 2J_{n+1}(x). \tag{17d}$$

We may derive other relations between $p_n^{(r)}(x)$ and $q_n^{(r)}(x)$ by utilizing the properties of $J_n(x)$ and $j_n(x)$. However, we will not pursue them here.

4. THE POLYNOMIALS $P_n^{(r)}(x)$ AND $Q_n^{(r)}(x)$

As was done in Section 2, by using the results of Horadam [4] concerning the generalized Fibonacci sequence, we may show that

$$P_n^{(r)}(x) = U_{n+1}(x) + (r-1)U_n(x) \tag{18a}$$

and

$$Q_n^{(r)}(x) = V_n(x) + rU_n(x), \tag{18b}$$

where

$$U_n(x) = w_n(0, 1; x), \tag{19a}$$

$$V_n(x) = w_n(2, x+2; x), \tag{19b}$$

and $w_n(a, b; x)$ is now the solution of the equation

$$w_n(x) = (x+2)w_{n-1}(x) - w_{n-2}(x) \quad (n \geq 2) \tag{20a}$$

with

$$w_0(x) = a \quad \text{and} \quad w_1(x) = b. \tag{20b}$$

From the properties of the polynomials $B_n(x)$ and $C_n(x)$ given in [6] and [7], we can relate $B_n(x)$ and $C_n(x)$ directly to $U_n(x)$ and $V_n(x)$ by

$$B_n(x) = U_{n+1}(x) \tag{21a}$$

and

$$C_n(x) = V_n(x). \tag{21b}$$

Thus, we have the following relations for $n \geq 1$:

$$P_n^{(r)}(x) = B_n(x) + (r-1)B_{n-1}(x), \tag{22a}$$

$$Q_n^{(r)}(x) = C_n(x) + rB_{n-1}(x), \tag{22b}$$

and

$$Q_n^{(r)}(x) = P_n^{(r)}(x) + b_{n-1}(x), \tag{22c}$$

where we have used the relations (see [7])

$$b_n(x) = B_n(x) - B_{n-1}(x) \tag{23a}$$

and

$$C_n(x) = B_n(x) - B_{n-2}(x). \tag{23b}$$

It can be observed that relations (3a), (3b), and (3c) directly follow from (22a) and (22b).

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EVEN DUCCI-SEQUENCES

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Ducci-sequences are successive iterations of the function

$$D(X) = D(x_1, x_2, \ldots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1|).$$

Note that $D: Z^n \to Z^n$, where Z^n is the set of n-tuples with integer entries. Since the entries of $D(X)$ are less than or equal to those of X, eventually every Ducci-sequence $\{X, D(X), D^2(X), \ldots, D^j(X), \ldots\}$ gives rise to a cycle. That is, there exist integers i and j for which $0 \le i < j$ and $D^j(X) = D^i(X)$. When i and j are as small as possible, we say that the resulting cycle, $\{D^i(X), \ldots, D^{j-1}(X), \ldots\}$, is generated by X and has period $j - i$. If Y is contained in a cycle of period k, then $D^j(Y) = Y$ if and only if $k \mid j$.

Introduced in 1937, Ducci-sequences and their resulting cycles have been studied extensively (see [1]-[7]). It is well known that for a given cycle all the entries in all the tuples are equal to either 0 or a constant C (see [2] and [4]). Since for every λ, $D(\lambda X) = \lambda D(X)$, we can assume without loss of generality that $C = 1$. Thus, when studying cycles of Ducci-sequences, we can restrict our attention to Z_2^n, the set of n-tuples with entries from $\{0, 1\}$. In addition, we can view the operation associated with D as addition modulo 2 since, for $x, y \in \{0, 1\}$, $|x - y| \equiv (x + y)$ (mod 2).

Most of the work on Ducci-sequences has focused on the case when n is odd or a power of 2. Here we consider the case when $n = 2^s \cdot q$, where $s \ge 1$ and q is odd with $q > 1$. We will show that associated with an n-tuple X are 2^s different q-tuples that completely determine the behavior of X. In particular, we will show that an n-tuple X is contained in a cycle if and only if each of the 2^s associated q-tuples is in a cycle. Further, the period of the cycle generated by X is determined by the periods of the cycles generated by the 2^s associated q-tuples.

To motivate the notation that will be introduced shortly, consider the following representations of a 12-tuple X:

$$X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12})$$

$$= (x_1, 0, 0, 0, x_5, 0, 0, 0, x_9, 0, 0, 0) + (0, x_2, 0, 0, 0, x_6, 0, 0, 0, x_{10}, 0, 0)$$

$$+ (0, 0, x_3, 0, 0, 0, x_7, 0, 0, 0, x_{11}, 0) + (0, 0, 0, x_4, 0, 0, 0, x_8, 0, 0, 0, x_{12}).$$

We see that associated with X are the following four 3-tuples:

$$(x_1, x_5, x_9), \quad (x_2, x_6, x_{10}), \quad (x_3, x_7, x_{11}), \quad \text{and} \quad (x_4, x_8, x_{12}).$$

When we form these smaller tuples, we will say that we compress the original tuple. Conversely, we can begin with these four 3-tuples and expand them to a 12-tuple by inserting zeros and adding.

Since we are interested in even tuples, we will often need to work with powers of 2. To simplify notation, we will write 2^s as $2^{\wedge}s$ whenever this expression appears as a superscript or subscript.

Let X be an n-tuple where $n = 2^s \cdot q$ with $s \geq 1$. For $i \in \{1, 2, ..., 2^s\}$, the compression functions $C_{i, 2^\wedge s}: Z_2^n \to Z_2^q$ are defined by $C_{i, 2^\wedge s}(X) = (c_j)$, where

$$c_j = x_{i+(j-1) \cdot 2^\wedge s}.$$

For $i \in \{1, 2, ..., 2^s\}$, the expansion functions $E_{i, 2^\wedge s}: Z_2^q \to Z_2^n$ are defined by $E_{i, 2^\wedge s}(Y) = (e_j)$, where

$$\begin{cases} e_j = y_{\lambda+1} & \text{when } j = i + \lambda \cdot 2^s \text{ for } \lambda = 0, 1, ..., q-1, \\ e_j = 0 & \text{when } j \not\equiv i \pmod{2^s}. \end{cases}$$

The observations below follow immediately from the definitions of the compression and expansion functions:

$$X = \sum_{i=1}^{2^\wedge s} E_{i, 2^\wedge s}(C_{i, 2^\wedge s}(X)) \text{ for } X \in Z_2^n, \text{ where } 2^s | n; \tag{1}$$

$$C_{i, 2}(E_{i, 2}(Y)) = Y \text{ for } Y \in Z_2^q, \text{ where } n = 2 \cdot q; \tag{2}$$

$$C_{i, 2}(E_{j, 2}(Y)) = (0, 0, ..., 0) \text{ for } Y \in Z_2^q, \text{ where } n = 2 \cdot q \text{ and } i \neq j; \tag{3}$$

$$C_{j, 2}(C_{i, 2^\wedge s}(X)) = C_{i+(j-1) \cdot 2^\wedge s, 2^\wedge (s+1)}(X) \text{ for } X \in Z_2^n, \text{ where } 2^{s+1} | n; \tag{4}$$

$$E_{i, 2^\wedge s}(E_{j, 2}(Y)) = E_{i+(j-1) \cdot 2^\wedge s, 2^\wedge (s+1)}(Y) \text{ for } Y \in Z_2^q, \text{ where } n = 2^{s+1} \cdot q. \tag{5}$$

We use these observations to express $D^{2^\wedge s \cdot m}(X)$ in terms of $D^m(C_{i, 2^\wedge s}(X))$.

Theorem 1: Let X be an n-tuple, where $2 | n$. Then

$$D^2(X) = \sum_{i=1}^{2} E_{i, 2}(D(C_{i, 2}(X))).$$

Proof: Let $X = (x_1, x_2, ..., x_n)$. Then

$$D(X) = (x_1 + x_2, \ x_2 + x_3, \ x_3 + x_4, \ x_4 + x_5, ..., x_{n-1} + x_n, \ x_n + x_1),$$

$$D^2(X) = (x_1 + x_3, \ x_2 + x_4, \ x_3 + x_5, \ x_4 + x_6, ..., x_{n-1} + x_1, \ x_n + x_2).$$

On the other hand,

$$C_{1, 2}(X) = (x_1, x_3, x_5, ..., x_{n-1}),$$

$$D(C_{1, 2}(X)) = (x_1 + x_3, \ x_3 + x_5, ..., x_{n-1} + x_1),$$

$$E_{1, 2}(D(C_{1, 2}(X))) = (x_1 + x_3, \ 0, \ x_3 + x_5, \ 0, ..., x_{n-1} + x_1, \ 0).$$

Similarly,

$$E_{2, 2}(D(C_{2, 2}(X))) = (0, \ x_2 + x_4, \ 0, \ x_4 + x_6, ..., 0, \ x_n + x_2).$$

Thus

$$D^2(X) = E_{1, 2}(D(C_{1, 2}(X))) + E_{2, 2}(D(C_{2, 2}(X))). \quad \square$$

Theorem 2: Let X be an n-tuple, where $2 \mid n$. Then

$$D^{2m}(X) = \sum_{i=1}^{2} E_{i,2}(D^m(C_{i,2}(X))).$$

Proof: By Theorem 1, the result holds for $m = 1$. Assume it holds for m and consider $m+1$. Now $D^{2(m+1)}(X) = D^2(D^{2m}(X))$. Thus

$$D^{2(m+1)}(X) = D^2\left(\sum_{i=1}^{2} E_{i,2}(D^m(C_{i,2}(X)))\right) = \sum_{j=1}^{2} E_{j,2}\left(D\left(C_{j,2}\left(\sum_{i=1}^{2} E_{i,2}(D^m(C_{i,2}(X)))\right)\right)\right). \quad (6)$$

Using observations (2) and (3), (6) simplifies to

$$D^{2(m+1)}(X) = E_{1,2}(D(D^m(C_{1,2}(X)))) + E_{2,2}(D(D^m(C_{2,2}(X))))$$

$$= \sum_{i=1}^{2} E_{i,2}(D^{m+1}(C_{i,2}(X))). \quad \square$$

Theorem 3: Let X be an n-tuple, where $2^s \mid n$ with $s \geq 1$. Then

$$D^{2^s \cdot m}(X) = \sum_{i=1}^{2^s} E_{i,2^s}(D^m(C_{i,2^s}(X))).$$

Proof: By Theorem 2, the result holds for $s = 1$. Assume it holds for s and consider $s+1$. Using the induction hypothesis, we have

$$D^{2^{(s+1)} \cdot m}(X) = D^{(2^s) \cdot (2m)}(X) = \sum_{i=1}^{2^s} E_{i,2^s}(D^{2m}(C_{i,2^s}(X)))$$

$$= \sum_{i=1}^{2^s} E_{i,2^s}\left(\sum_{j=1}^{2} E_{j,2}(D^m(C_{j,2}(C_{i,2^s}(X))))\right). \quad (7)$$

The last equality in (7) follows from Theorem 2. Using observations (4) and (5), (7) simplifies to

$$D^{2^{(s+1)} \cdot m}(X) = \sum_{i=1}^{2^s} \sum_{j=1}^{2} E_{i+(j-1) \cdot 2^s, 2^{(s+1)}}(D^m(C_{i+(j-1) \cdot 2^s, 2^{(s+1)}}(X)))$$

$$= \sum_{i=1}^{2^{(s+1)}} E_{i,2^{(s+1)}}(D^m(C_{i,2^{(s+1)}}(X))). \quad \square$$

Corollary 1: Let X be an n-tuple, where $2^s \mid n$;with $s \geq 1$. X is contained in a cycle if and only if $C_{i,2^s}(X)$ is contained in a cycle for $i \in \{1, \ldots, 2^s\}$.

Proof: Suppose X is contained in a cycle of period k; that is, $D^k(X) = X$. Then

$$D^{2^s \cdot k}(X) = X.$$

Using (1) and Theorem 3, we see that

$$C_{i,2^s}(X) = C_{i,2^s}(D^{2^s \cdot k}(X)) = D^k(C_{i,2^s}(X))$$

for $i \in \{1, \ldots, 2^s\}$. Hence for each i, $C_{i,2^s}(X)$ is in a cycle.

Conversely, suppose that, for each i, $C_{i,2^{\wedge}s}(X)$ is in a cycle of period k_i. Let $m = \text{lcm}(k_1, k_2, ..., k_{2^{\wedge}s})$. Since $D^m(C_{i,2^{\wedge}s}(X)) = C_{i,2^{\wedge}s}(X)$, by Theorem 3 and (1), $D^{2^{\wedge}s \cdot m}(X) = X$. Hence, X is in a cycle. $\square$

For odd n, an n-tuple X is contained in a cycle if and only if the sum of the entries of X is congruent to 0 modulo 2 (see [4]). Thus by Corollary 1, for $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$, an n-tuple X is contained in a cycle if and only if for each $i \in \{1, ..., 2^s\}$ the sum of the entries of $C_{i,2^{\wedge}s}(X)$ is congruent to 0 modulo 2. Although the terminology is different, this result appears in [4]. In a moment we will begin to consider how the period of the cycle containing X is related to the periods of the cycles containing $C_{i,2^{\wedge}s}(X)$, $i = 1, ..., 2^s$. First, we prove a rather technical corollary that we will need later.

Corollary 2: Let X be an n-tuple, where $2^s | n$ with $s \geq 1$. Then

$$C_{i,2^{\wedge}s}(D^{2^{\wedge}(s-1)}(X)) = C_{i,2^{\wedge}s}(X) + C_{i+2^{\wedge}(s-1),2^{\wedge}s}(X)$$

for $i = 1, 2, ..., 2^{s-1}$.

Proof: Let $n = 2^s \cdot q = 2^{s-1} \cdot 2q$. By Theorem 3,

$$D^{2^{\wedge}(s-1)}(X) = \sum_{i=1}^{2^{\wedge}(s-1)} E_{i,2^{\wedge}(s-1)}(D(C_{i,2^{\wedge}(s-1)}(X))).$$

For $Z \in Z_2^{2q}$ and $i = 1, 2, ..., 2^{s-1}$,

$$C_{i,2^{\wedge}s}(E_{i,2^{\wedge}(s-1)}(Z)) = C_{1,2}(Z),$$
$$C_{j,2^{\wedge}s}(E_{i,2^{\wedge}(s-1)}(Z)) = (0, 0, ..., 0) \quad \text{when } j \neq i.$$

Hence $C_{i,2^{\wedge}s}(D^{2^{\wedge}(s-1)}(X)) = C_{1,2}(D(C_{i,2^{\wedge}(s-1)}(X)))$. Now

$$C_{i,2^{\wedge}(s-1)}(X) = (x_i, x_{i+2^{\wedge}(s-1)}, x_{i+2 \cdot 2^{\wedge}(s-1)}, x_{i+3 \cdot 2^{\wedge}(s-1)}, ..., x_{i+(2q-1) \cdot 2^{\wedge}(s-1)}),$$

$$D(C_{i,2^{\wedge}(s-1)}(X)) = (x_i + x_{i+2^{\wedge}(s-1)} + x_{i+2 \cdot 2^{\wedge}(s-1)}, x_{i+2 \cdot 2^{\wedge}(s-1)}, ..., x_{i+(2q-1) \cdot 2^{\wedge}(s-1)} + x_i),$$

$$C_{i,2^{\wedge}s}(D^{2^{\wedge}(s-1)}(X)) = C_{1,2}(D(C_{i,2^{\wedge}(s-1)}(X)))$$

$$= (x_i + x_{i+2^{\wedge}(s-1)}, x_{i+2 \cdot 2^{\wedge}(s-1)} + x_{i+3 \cdot 2^{\wedge}(s-1)}, ..., x_{i+(2q-2) \cdot 2^{\wedge}(s-1)} + x_{i+(2q-1) \cdot 2^{\wedge}(s-1)})$$

$$= (x_i, x_{i+2 \cdot 2^{\wedge}(s-1)}, ..., x_{i+(2q-2) \cdot 2^{\wedge}(s-1)}) + (x_{i+2^{\wedge}(s-1)}, x_{i+3 \cdot 2^{\wedge}(s-1)}, ..., x_{i+(2q-1) \cdot 2^{\wedge}(s-1)})$$

$$= C_{i,2^{\wedge}s}(X) + C_{i+2^{\wedge}(s-1),2^{\wedge}s}(X). \quad \square$$

We now begin considering how the period of the cycle containing X is related to the periods of the cycles containing $C_{i,2^{\wedge}s}(X)$, $i = 1, ..., 2^s$.

Theorem 4: Let $n = 2^s \cdot q$, where $s \geq 1$. Suppose X is an n-tuple which is contained in a cycle of period k. Let k_i be the period of the cycle containing the q-tuple $C_{i,2^{\wedge}s}(X)$, $i = 1, ..., 2^s$. Then $k = 2^t \cdot \text{lcm}(k_1, k_2, ..., k_{2^{\wedge}s})$ for some $0 \leq t \leq s$.

Proof: Let $m = \text{lcm}(k_1, k_2, ..., k_{2^{\wedge}s})$. As noted in the proof of Corollary 1, $D^{2^{\wedge}s \cdot m}(X) = X$. Consequently, $k | 2^s \cdot m$.

We now show that $m|k$. Since $D^k(X) = X$, it follows that $D^{2^s \cdot k}(X) = X$. As we showed in the proof of Corollary 1, $D^k(C_{i,2^s}(X)) = C_{i,2^s}(X)$. Since k_i is the period of the cycle containing $C_{i,2^s}(X)$, $k_i|k$ for $i = 1, \ldots, 2^s$. Consequently, $m|k$. Since $m|k$ and $k|2^s \cdot m$, we conclude that $k = 2^t \cdot m$ for some $0 \le t \le s$. $\square$

Theorem 5: Let $n = 2^s \cdot q$, where $s \ge 1$. Suppose X is an n-tuple which is contained in a cycle of period $k = 2^t \cdot m$, where m is odd and $0 \le t < s$. Then

$$C_{i+2^t,\, 2^{\wedge}(t+1)}(X) = C_{i,\, 2^{\wedge}(t+1)}(X) + D^{\frac{m+1}{2}}\left(C_{i,\, 2^{\wedge}(t+1)}(X)\right)$$

for $i = 1, \ldots, 2^t$.

Proof: Since $0 \le t < s$, $1 \le t+1 \le s$, and $2^{t+1}|n$. Thus by Theorem 3,

$$D^{2^{\wedge}t \cdot (m-1)}(X) = D^{2^{\wedge}(t+1) \cdot \frac{m-1}{2}}(X)$$
$$= \sum_{i=1}^{2^{\wedge}(t+1)} E_{i,\, 2^{\wedge}(t+1)}\left(D^{\frac{m-1}{2}}(C_{i,\, 2^{\wedge}(t+1)}(X))\right). \tag{8}$$

By hypothesis, $D^{2^{\wedge}t \cdot m}(X) = X$. Since $X = D^{2^{\wedge}t \cdot m}(X) = D^{2^{\wedge}t}(D^{2^{\wedge}t \cdot (m-1)}(X))$,

$$C_{i,\, 2^{\wedge}(t+1)}(X) = C_{i,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}t}(D^{2^{\wedge}t \cdot (m-1)}(X))\right).$$

By Corollary 2,

$$C_{i,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}t}(D^{2^{\wedge}t \cdot (m-1)}(X))\right) = C_{i,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}t \cdot (m-1)}(X)\right) + C_{i+2^t,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}t \cdot (m-1)}(X)\right)$$

for $i = 1, \ldots, 2^t$. Thus

$$C_{i,\, 2^{\wedge}(t+1)}(X) = C_{i,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}t \cdot (m-1)}(X)\right) + C_{i+2^t,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}t \cdot (m-1)}(X)\right). \tag{9}$$

Using (8) to find the two terms on the right-hand side of (9), we can rewrite (9) as

$$C_{i,\, 2^{\wedge}(t+1)}(X) = D^{\frac{m-1}{2}}\left(C_{i,\, 2^{\wedge}(t+1)}(X)\right) + D^{\frac{m-1}{2}}\left(C_{i+2^t,\, 2^{\wedge}(t+1)}(X)\right). \tag{10}$$

Applying $D^{\frac{m-1}{2}}$ to (10) gives

$$D^{\frac{m-1}{2}}\left(C_{i,\, 2^{\wedge}(t+1)}(X)\right) = D^m\left(C_{i,\, 2^{\wedge}(t+1)}(X)\right) + D^m\left(C_{i+2^t,\, 2^{\wedge}(t+1)}(X)\right). \tag{11}$$

By hypothesis, $D^{2^{\wedge}t \cdot m}(X) = X$. Hence $D^{2^{\wedge}(t+1) \cdot m}(X) = X$. Thus, using Theorem 3 and (1),

$$C_{j,\, 2^{\wedge}(t+1)}(X) = C_{j,\, 2^{\wedge}(t+1)}\left(D^{2^{\wedge}(t+1) \cdot m}(X)\right) = D^m\left(C_{j,\, 2^{\wedge}(t+1)}(X)\right)$$

for $j = 1, \ldots, 2^{t+1}$. Using this to simplify (11) and rearranging terms gives the desired result. $\square$

We now prove the converse of Theorem 5. To do so, we will need the following well-known result: when n is odd, the period of a cycle of n-tuples divides $n \cdot (2^{\phi(n)} - 1)$, where $\phi(n)$ is Euler's phi function [3]. Actually, a great deal more is known about the period, but this is all we require. Specifically, when n is odd, the period of each cycle of n-tuples is odd.

Theorem 6: Let $n = 2^s \cdot q$, where $s \ge 1$ and q is odd with $q > 1$. Suppose X is an n-tuple that is contained in a cycle. Let $m = \mathrm{lcm}(k_1, k_2, \ldots, k_{2^{\wedge}s})$, where k_i is the period of the cycle containing $C_{i,\, 2^{\wedge}s}(X)$ for $i = 1, \ldots, 2^s$. If there exists t, $0 \le t < s$, such that

$$C_{i+2^\wedge t, 2^\wedge(t+1)}(X) = C_{i, 2^\wedge(t+1)}(X) + D^{\frac{m-1}{2}}(C_{i, 2^\wedge(t+1)}(X)) \tag{12}$$

for $i = 1, \ldots, 2^t$, then $D^{2^\wedge t \cdot m}(X) = X$.

Proof: Since q is odd, each k_i is odd and hence m is odd. Further, since $D^{k_i}(C_{i, 2^\wedge s}(X)) = C_{i, 2^\wedge s}(x)$, $D^m(C_{i, 2^\wedge s}(X)) = C_{i, 2^\wedge s}(X)$ for $i = 1, \ldots, 2^s$. Thus, if $t + 1 = s$,

$$D^m(C_{i, 2^\wedge(t+1)}(X)) = C_{i, 2^\wedge(t+1)}(X).$$

On the other hand, if $r = t + 1 < s$, then

$$C_{i, 2^\wedge r}(X) = C_{i, 2^\wedge s}(X) + C_{i+2^\wedge r, 2^\wedge s}(X) + C_{i+2 \cdot 2^\wedge r, 2^\wedge s}(X)$$
$$+ \cdots + C_{i+[2^\wedge(s-r)-1] \cdot 2^\wedge r, 2^\wedge s}(X).$$

This implies $D^m(C_{i, 2^\wedge r}(X)) = C_{i, 2^\wedge r}(X)$; i.e., $D^m(C_{i, 2^\wedge(t+1)}(X)) = C_{i, 2^\wedge(t+1)}(X)$. Hence

$$C_{i, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot m}(X)) = D^m(C_{i, 2^\wedge(t+1)}(X)) = C_{i, 2^\wedge(t+1)}(X),$$

so $D^{2^\wedge(t+1) \cdot m}(X) = X$. We now use this to show that, in fact, $D^{2^\wedge t \cdot m}(X) = X$.

As in the proof of Theorem 5, we consider $D^{2^\wedge t \cdot m}(X)$. Using (8), we have

$$C_{i, 2^\wedge(t+1)}(D^{2^\wedge t \cdot (m-1)}(X)) = D^{\frac{m-1}{2}}(C_{i, 2^\wedge(t+1)}(X)). \tag{13}$$

Likewise, using (8) and (12), we have

$$C_{i+2^\wedge t, 2^\wedge(t+1)}(D^{2^\wedge t \cdot (m-1)}(X)) = D^{\frac{m-1}{2}}(C_{i+2^\wedge t, 2^\wedge(t+1)}(X))$$
$$= D^{\frac{m-1}{2}}(C_{i, 2^\wedge(t+1)}(X)) + D^{\frac{m-1}{2}}(D^{\frac{m+1}{2}}(C_{i, 2^\wedge(t+1)}(X))) \tag{14}$$
$$= D^{\frac{m-1}{2}}(C_{i, 2^\wedge(t+1)}(X)) + C_{i, 2^\wedge(t+1)}(X).$$

Note that (13) and (14) hold for $i = 1, \ldots, 2^t$. Now, by Theorem 3, we have

$$C_{i, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot (m-1)}(X)) = D^{m-1}(C_{i, 2^\wedge(t+1)}(X)). \tag{15}$$

Likewise, using Theorem 3 and (12), we have

$$C_{i+2^\wedge t, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot (m-1)}(X)) = D^{m-1}(C_{i+2^\wedge t, 2^\wedge(t+1)}(X))$$
$$= D^{m-1}(C_{i, 2^\wedge(t+1)}(X)) + D^{\frac{m-1}{2}}(C_{i, 2^\wedge(t+1)}(X)). \tag{16}$$

Note that (15) and (16) hold for $i = 1, \ldots, 2^t$. By Corollary 2,

$$C_{i, 2^\wedge(t+1)}(D^{2^\wedge t}(Z)) = C_{i, 2^\wedge(t+1)}(Z) + C_{i+2^\wedge t, 2^\wedge(t+1)}(Z). \tag{17}$$

We let $Z = D^{2^\wedge(t+1) \cdot (m-1)}(X)$ in (17), note that $2^t + 2^{t+1} \cdot (m-1) = 2^{t+1} \cdot m - 2^t$, and use (15) and (16) to get

$$C_{i, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot m - 2^\wedge t}(X)) = D^{\frac{m-1}{2}}(C_{i, 2^\wedge(t+1)}(X)). \tag{18}$$

Now we let $Z = D^{2^\wedge(t+1) \cdot m - 2^\wedge t}(X)$ in (17). This gives us

$$C_{i, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot m}(X)) = C_{i, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot m - 2^\wedge t}(X))$$
$$+ C_{i+2^\wedge t, 2^\wedge(t+1)}(D^{2^\wedge(t+1) \cdot m - 2^\wedge t}(X)). \tag{19}$$

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We rewrite (19) using (18) and the fact that $C_{i,2^{\wedge}(t+1)}(D^{2^{\wedge}(t+1)\cdot m}(X)) = C_{i,2^{\wedge}(t+1)}(X)$:

$$C_{i,2^{\wedge}(t+1)}(X) = D^{\frac{m-1}{2}}(C_{i,2^{\wedge}(t+1)}(X)) + C_{i,2^{\wedge}(t+1)}(D^{2^{\wedge}(t+1)\cdot m - 2^{\wedge}t}(X))$$

or

$$C_{i+2^{\wedge}t,\,2^{\wedge}(t+1)}(D^{2^{\wedge}(t+1)\cdot m - 2^{\wedge}t}(X)) = D^{\frac{m-1}{2}}(C_{i,2^{\wedge}(t+1)}(X)) + C_{i,2^{\wedge}(t+1)}(X). \tag{20}$$

Comparing (13) to (18), we see that

$$C_{i,2^{\wedge}(t+1)}(D^{2^{\wedge}t\cdot m - 2^{\wedge}t}(X)) = C_{i,2^{\wedge}(t+1)}(D^{2^{\wedge}(t+1)\cdot m - 2^{\wedge}t}(X))$$

for $i = 1, \ldots, 2^t$, and comparing (14) to (20), we see that

$$C_{i+2^{\wedge}t,\,2^{\wedge}(t+1)}(D^{2^{\wedge}t\cdot m - 2^{\wedge}t}(X)) = C_{i+2^{\wedge}t,\,2^{\wedge}(t+1)}(D^{2^{\wedge}(t+1)\cdot m - 2^{\wedge}t}(X))$$

for $i = 1, \ldots, 2^t$. Hence $D^{2^{\wedge}t\cdot m - 2^{\wedge}t}(X) = D^{2^{\wedge}(t+1)\cdot m - 2^{\wedge}t}(X)$. This, in turn, implies that $D^{2^{\wedge}t\cdot m}(X) = D^{2^{\wedge}(t+1)\cdot m}(X) = X$. $\square$

Thus we have completely characterized the period of a cycle of n-tuples. We summarize the results of the last three theorems in the following corollary.

Corollary 3: Let $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$. Suppose X is an n-tuple which is contained in a cycle of period k. Let $m = \text{lcm}(k_1, k_2, \ldots, k_{2^{\wedge}s})$, where k_i is the period of the cycle containing $C_{i,2^{\wedge}s}(X)$. Then $k = 2^t \cdot \text{lcm}(k_1, k_2, \ldots, k_{2^{\wedge}s})$ for some $0 \leq t < s$ if and only if

$$C_{i+2^{\wedge}t,\,2^{\wedge}(t+1)}(X) = C_{i,2^{\wedge}(t+1)}(X) + D^{\frac{m+1}{2}}(C_{i,2^{\wedge}(t+1)}(X))$$

for $i = 1, \ldots, 2^t$, where t is as small as possible. If no such t exists, then $k = 2^s \cdot m$. $\square$

We now show that there is a cycle for each possible period. Although there are many ways to do this, we will continue to use the compression functions.

Theorem 7: Let $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$. Suppose there is a cycle of q-tuples of period m. Then, for $0 \leq t \leq s$, there exists a cycle of n-tuples of period $2^t \cdot m$.

Proof: For $0 \leq r \leq s-1$, suppose there is a $(2^{s-1} \cdot q)$-tuple A that is contained in a cycle of period $2^r \cdot m$. By hypothesis, this holds for $s = 1$. Consider the $(2^s \cdot q)$-tuple $X = E_{1,2}(A)$. Now $C_{1,2}(X) = A$ and $C_{2,2}(X) = (0, 0, \ldots, 0)$. By Corollary 1, X is in a cycle. By Theorem 4, the period of the cycle containing X is either $2^r \cdot m$ or $2 \cdot (2^r \cdot m)$. Assume the period is $2^r \cdot m$. For $r > 0$,

$$\sum_{i=1}^{2} E_{i,2}(C_{i,2}(X)) = X = D^{2^{\wedge}r\cdot m}(X) = \sum_{i=1}^{2} E_{i,2}(D^{2^{\wedge}(r-1)\cdot m}(C_{i,2}(X))).$$

Thus, $D^{2^{\wedge}(r-1)\cdot m}(C_{1,2}(X)) = C_{1,2}(X)$; i.e., $D^{2^{\wedge}(r-1)\cdot m}(A) = A$. This implies that A is in a cycle with period less than or equal to $2^{r-1} \cdot m$. This contradiction shows that the period of the cycle containing X is $2 \cdot (2^r \cdot m) = 2^{r+1} \cdot m$ when $r > 0$. On the other hand, if $r = 0$, then

$$C_{1,2}(D^{m-1}(X)) = D^{\frac{m-1}{2}}(C_{1,2}(X)) = D^{\frac{m-1}{2}}(A)$$

and

$$C_{2,2}(D^{m-1}(X)) = D^{\frac{m-1}{2}}(C_{2,2}(X)) = (0,0,\dots,0).$$

Since

$$C_{1,2}(D^m(X)) = C_{1,2}(D^{m-1}(X)) + C_{2,2}(D^{m-1}(X)) = D^{\frac{m-1}{2}}(A) \neq A = C_{1,2}(X),$$

we see that $D^m(X) \neq X$. Hence the period of the cycle containing X is $2 \cdot m$ when $r = 0$. Therefore there are cycles of $(2^s \cdot q)$-tuples with period $2^t \cdot m$ for $1 \leq t \leq s$.

We now show that there is a cycle of $(2^s \cdot q)$-tuples with period m. Suppose there is a $(2^{s-1} \cdot q)$-tuple B that is contained in a cycle of period m and for which each $C_{i, 2^{\wedge}(s-1)}(B)$, $i = 1, \dots, 2^{s-1}$, is also contained in a cycle of period m. By hypothesis, this holds for $s = 1$. Consider the $(2^s \cdot q)$-tuple

$$Y = E_{1,2}(B) + E_{2,2}(B + D^{\frac{m+1}{2}}(B)). \tag{21}$$

Now $C_{1,2}(Y) = B$ and $C_{2,2}(Y) = B + D^{\frac{m+1}{2}}(B)$; $C_{2,2}(Y)$ is also in a cycle of period m. Thus Y is in a cycle. We want to use Corollary 3 to show that the period of the cycle containing Y is m. Note that

$$\begin{cases} C_{i, 2^{\wedge}s}(Y) = C_{\frac{i+1}{2}, 2^{\wedge}(s-1)}(B) & \text{when } i \text{ is odd}, \\ C_{i, 2^{\wedge}s}(Y) = C_{\frac{i}{2}, 2^{\wedge}(s-1)}(B + D^{\frac{m+1}{2}}(B)) & \text{when } i \text{ is even}. \end{cases}$$

By assumption, when i is odd, the period of the cycle containing $C_{i, 2^{\wedge}s}(Y)$ is m. To show that this is also the case when i is even, it suffices to show that the period of the cycle containing $C_{j, 2^{\wedge}(s-1)}(B + D^{\frac{m+1}{2}}(B))$ is m for $j = 1, \dots, 2^{s-1}$. Since $\gcd(m, 2^{s-1}) = 1$, there exist integers g and h for which

$$g \cdot m + h \cdot 2^{s-1} = \frac{m+1}{2}.$$

Either g or h is positive, but not both. Suppose $g > 0$ and $h < 0$. Then

$$B = D^{g \cdot m}(B) = D^{-h \cdot 2^{\wedge}(s-1)}(D^{\frac{m+1}{2}}(B)),$$

which implies

$$C_{j, 2^{\wedge}(s-1)}(B) = D^{-h}(C_{j, 2^{\wedge}(s-1)}(D^{\frac{m+1}{2}}(B))).$$

Hence, $C_{j, 2^{\wedge}(s-1)}(D^{\frac{m+1}{2}}(B))$ is in the same cycle as $C_{j, 2^{\wedge}(s-1)}(B)$. Since this cycle has period m, the cycle containing $C_{j, 2^{\wedge}(s-1)}(D^{\frac{m+1}{2}}(B))$ also has period m. In a similar manner, it can be shown that this is also the case when $g < 0$ and $h > 0$. Since the cycle containing $C_{i, 2^{\wedge}s}(Y)$, $i = 1, \dots, 2^s$, has period m and since (21) holds, Corollary 3 implies that the cycle containing Y has period m. $\square$

For a given n, the maximal period of cycles of Ducci-sequences is denoted by $P(n)$. By Corollary 3, if $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$, then $P(n)$ divides $2^s \cdot P(q)$. We now show that, in fact, $P(n) = 2^s \cdot P(q)$. This result appears in [2]; the proof there uses matrices and the fact that the cycle which has maximum period is generated by the n-tuple $(1, 0, \dots, 0, 0)$. We offer a new proof here based on the compression functions. The result follows immediately from the proof of Theorem 7.

Theorem 8: Let $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$. Then $P(n) = 2^s \cdot P(q)$.

Proof: Let A be a q-tuple that is contained in a cycle of period $P(q)$. Then the proof of Theorem 7 shows that the $(2^s \cdot q)$-tuple $X = E_{1, 2^\wedge s}(A) = E_{1, 2}(E_{1, 2}(\ldots E_{1, 2}(X)))$ is in a cycle of period $2^s \cdot P(q)$. $\square$

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EULERIAN NUMBERS: INVERSION FORMULAS AND CONGRUENCES MODULO A PRIME

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1. INTRODUCTION

The identity

$$n^m = \sum_{k=1}^{m} C_k^m \binom{n+k-1}{m},$$

(1)

where the C_k^m's are *Eulerian Numbers*, was known by Worpitsky [16]. Eulerian numbers satisfy the recursive relation

$$C_k^m = (m+1-k)C_{k-1}^{m-1} + kC_k^{m-1}$$

(2)

for $2 \le k \le m-1$ and $C_1^m = C_m^m = 1$. This can be proved by induction, see Stanley [13], and is also proved by Carlitz [4], [5] and by Krishnapriyan [11] in different ways.

n^m can also be expressed as a linear combination of binomial coefficients in the form

$$n^m = \sum_{k=1}^{m} D_k^m \binom{n}{k},$$

(3)

where the D_k^m coefficients satisfy the recursive relation

$$D_k^m = k(D_{k-1}^{m-1} + D_k^{m-1})$$

(4)

for $2 \le k \le m-1$; also $D_1^m = 1$ and $D_m^m = m!$, which is not difficult to check using induction. The uniqueness of the C and D in formulas (1) and (3) is also easily proved.

The C numbers form the following triangular array:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1 \\
 & & & & \cdots & & & &
\end{array}
$$

where the *Pascal-like* formation rule is given by formula (2). The D numbers form the array:

$$
\begin{array}{cccc}
1 & & & \\
1 & 2 & & \\
1 & 6 & 6 & \\
1 & 14 & 36 & 24 \\
1 & 30 & 150 & 240 & 120 \\
 & & \cdots & &
\end{array}
$$

where the formation rule for this table is given by formula (4). The D numbers are related to the *Stirling Numbers of the Second Kind* S_k^m by the expression $D_k^m = k!\,S_k^m$, which is easy to prove by considering the recursive relation

$$S_k^m = kS_k^{m-1} + S_{k-1}^{m-1}.$$

Thus, we can deduce

$$n^m = \sum_{k=1}^{m} k! \, S_k^m \binom{n}{k},$$

which is studied in several combinatorics textbooks, (see, e.g., Aigner [1] or Stanley [13]).

We can use expressions (1) and (3) to obtain formulas for the sum of powers $S_m(n) = 1^m + 2^m + \cdots + n^m$. Adding terms, and using the identity

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1},$$

we deduce

$$S_m(n) = \sum_{k=1}^{m} C_k^m \binom{n+k}{m+1} \tag{5}$$

and

$$S_m(n) = \sum_{k=1}^{m} D_k^m \binom{n+1}{k+1}, \tag{6}$$

where the C and D coefficients are defined by (2) and (4). Many papers have been written concerning formulas for $S_m(n)$. Perhaps the best known formulas express this sum as a polynomial in n of degree $m+1$ with coefficients involving Bernoulli numbers. See, for example, the papers by Christiano [6] and by de Bruyn and de Villiers [7]. Burrows and Talbot [2] treat this sum as a polynomial in $(n+1/2)$, and Edwards [8] expresses the sums $S_m(n)$ as polynomials in $\sum k$ and $\sum k^2$. Formulas (5) and (6) express this sum as linear combinations of binomial coefficients. Formula (5) is also discussed by Graham et al. [9]; Shanks [12] deduces (5) by considering sums of powers of binomial coefficients. Hsu [10] obtains formula (6) by studying sums of the form $\sum_{k=0}^{n} F(n, k)k^p$ for different functions $F(n, k)$ and expresses these sums as linear combinations of the D_k^m coefficients.

The combinatorial significance of Eulerian numbers is known. In Section 2 we discuss a combinatorial meaning of D_k^m and deduce some nonrecursive formulas for both C and D numbers by combinatorial means.

In Section 3 we show that the C and D numbers satisfy the inversion formulas

$$D_k^m = \sum_{i=0}^{k-1} \binom{m-k+i}{i} C_{k-i}^m$$

and

$$C_k^m = \sum_{i=0}^{k-1} (-1)^i \binom{m-k+i}{i} D_{k-i}^m.$$

We then use these to obtain a number-theoretical result analogous to the well-known fact that

$$p \, \bigg| \binom{p}{k}$$

whenever p is a prime and $1 \le k \le p-1$.

2. COMBINATORIAL MEANING

The combinatorial significance of the Eulerian numbers is known. C_k^m is the number of permutations $p_1 p_2 \ldots p_m$ of $\{1, 2, \ldots, m\}$ that have $k - 1$ *ascents* [9, pp. 253-58], that is, $k - 1$ places where $p_j < p_{j+1}$.

A combinatorial meaning of the D numbers is given by the following proposition.

Proposition 2.1: D_k^m is the number of surjective functions from the set $\{1, 2, \ldots, m\}$ onto the set $\{1, 2, \ldots, k\}$.

Proof: Consider the number of m-tuples $(a_1, a_2, \ldots, a_m)$, where $1 \le a_i \le n$, $i = 1, 2, \ldots, m$. We have a total of n^m different m-tuples.

Now, the total number of different m-tuples is equal to the number of m-tuples whose elements are equal plus the number of m-tuples whose elements are two different numbers, and so on.

Since the number of subsets of k elements of a set of n elements is given by $\binom{n}{k}$, the number of m-tuples whose elements are k different numbers is $E_k^m \binom{n}{k}$, where E_k^m is the number of m-tuples with k different numbers, which is equal to the number of surjective functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, k\}$. Hence,

$$n^m = \sum_{k=1}^{m} E_k^m \binom{n}{k}.$$

By unicity of the D_k^m, we conclude that $D_k^m = E_k^m$.

We shall now deduce a formula for D_k^m.

Proposition 2.2: The number D_k^m is given by

$$D_k^m = \sum \frac{m!}{x_1! x_2! \ldots x_k!}, \tag{7}$$

where the sum is taken over all the positive integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = m. \tag{8}$$

Proof: By Proposition 2.1, D_k^m is the number of surjective functions from $\{1, 2, \ldots, m\}$ onto $\{1, 2, \ldots, k\}$, so we count the m-tuples formed "using" all the numbers $1, 2, \ldots, k$.

To form an m-tuple with the numbers $1, 2, \ldots, k$, we use the number 1 x_1 times, the number 2 x_2 times, and so on up to the number k x_k times, so that $x_1 + x_2 + \cdots + x_k = m$ and $x_i \ge 1$ for $i = 1, 2, \ldots, k$. For each solution to this equation, we have

$$\frac{m!}{x_1! x_2! \ldots x_k!}$$

ways of ordering the numbers $1, 2, \ldots, k$ in the m-tuple. Therefore,

$$D_k^m = \sum \frac{m!}{x_1! x_2! \ldots x_k!},$$

where the sum is taken over all positive integer solutions of equation (8).

Note that the expression

$$\frac{m!}{x_1! \, x_2! \dots x_k!}$$

is equal to the multinomial coefficient

$$\binom{m}{x_1, x_2, \dots, x_k},$$

which is the coefficient of $a_1^{x_1} a_2^{x_2} \dots a_k^{x_k}$ in the expansion of $(a_1 + a_2 + \cdots + a_k)^m$. For a discussion of multinomial expansions, see Tomescu [14, p. 17]. Thus, we have the expression

$$D_k^m = \sum_{x_i \geq 1} \binom{m}{x_1, x_2, \dots, x_k}.$$

Other formulas for both D_k^m and C_k^m can be obtained. The expression

$$D_k^m = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (k-j)^m$$

can also be obtained by counting functions from $\{1, 2, \dots, m\}$ onto $\{1, 2, \dots, k\}$, see [14, pp. 41-48]. The expression

$$C_k^m = \sum_{j=0}^{k-1} (-1)^j \binom{m+1}{j} (k-j)^m$$

for the Eulerian numbers appears in papers by Carlitz [4], [5], and by Velleman and Call [15].

We are particularly interested in formula (7) because, from it, we can easily deduce that if p is a prime, then

$$D_k^p \equiv \begin{cases} 1 \pmod{p} & \text{if } k = 1, \\ 0 \pmod{p} & \text{if } 2 \leq k \leq p. \end{cases} \tag{9}$$

We will use this result in Section 4.

3. INVERSION FORMULAS

In this section we discuss inversion formulas between the C and D numbers discussed above. For the purposes of this section, let us extend the definition of the C and D numbers by

$$C_k^m = \begin{cases} (m+1-k)C_{k-1}^{m-1} + kC_k^{m-1} & \text{if } 1 \leq k \leq m, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

and

$$D_k^m = \begin{cases} k(D_{k-1}^{m-1} + D_k^{m-1}) & \text{if } 1 \leq k \leq m, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

with $C_1^1 = D_1^1 = 1$. Clearly, these formulas are extensions of (2) and (4) above. For the proof of the next theorem, we will use the following identities:

$$(m+1)\binom{m+1-k+i}{i} - (i+1)\binom{m+1-k+i}{i+1} = k\binom{m+1-k+i}{i}; \tag{12}$$

$$(i+1)\binom{m+1-k+i}{i+1}+\binom{m+1-k+i}{i}=(m-k+2)\binom{m+1-k+i}{i};\tag{13}$$

$$\sum_{i=0}^{k}(-1)^i\binom{n-i}{k-i}\binom{n}{i}=\begin{cases}1 & \text{if } k=0,\\0 & \text{if } 1\le k\le n.\end{cases}\tag{14}$$

Expressions (12) and (13) are easy to verify using the definition of the binomial coefficient, while expression (14) is proven by induction (see [14, p. 20, prob. 2.15]).

Now we state the following theorem.

Theorem 3.1: The numbers C_k^m and D_k^m are related by the inversion formulas

$$D_k^m=\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_{k-i}^m\tag{15}$$

and

$$C_k^m=\sum_{i=0}^{k-1}(-1)^i\binom{m-k+i}{i}D_{k-i}^m.\tag{16}$$

Proof: We first prove (15). This will be done by induction on m. It is easy to verify that formula (15) is true for $m=1$, so assume it is true for some m. We need to show that

$$\sum_{i=0}^{k-1}\binom{m+1-k+i}{i}C_{k-i}^{m+1}=D_k^{m+1}.$$

In order to simplify the notation, let

$$\mathcal{C}=\sum_{i=0}^{k-1}\binom{m+1-k+i}{i}C_{k-i}^{m+1},\quad C_0=C_{k-i}^m,\quad\text{and}\quad C_1=C_{k-i-1}^m.$$

By (10), properties of sums, and Pascal's identity,

$$\mathcal{C}=\sum_{i=0}^{k-1}\binom{m+1-k+i}{i}[(k-i)C_0+(m+2-k+i)C_1]$$

$$=\sum_{i=0}^{k-1}(k-i)\binom{m+1-k+i}{i}C_0+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1$$

$$=\sum_{i=0}^{k-1}(k-i)\left[\binom{m-k+i}{i}+\binom{m-k+i}{i-1}\right]C_0+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1$$

$$=k\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_0-\sum_{i=1}^{k-1}i\binom{m-k+i}{i}C_0$$

$$+\sum_{i=1}^{k-1}(k-i)\binom{m-k+i}{i-1}C_0+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1$$

$$=k\sum_{i=0}^{k-1}\binom{m-k+i}{i}C_0-\sum_{i=0}^{k-2}(i+1)\binom{m-k+i+1}{i}C_1$$

$$+\sum_{i=0}^{k-2}(k-i-1)\binom{m-k+i+1}{i}C_1+\sum_{i=0}^{k-2}(m+2-k+i)\binom{m+1-k+i}{i}C_1$$

$$= k \sum_{i=0}^{k-1} \binom{m-k+i}{i} C_0 + \sum_{i=0}^{k-2} \left[(m+1) \binom{m+1-k+i}{i} - (i+1) \binom{m+1-k+i}{i+1} \right] C_1.$$

Using identity (12),

$$\mathscr{C} = k \sum_{i=0}^{k-1} \binom{m-k+i}{i} C_0 + k \sum_{i=0}^{k-2} \binom{m+1-k+i}{i} C_1.$$

Finally, by our induction hypothesis and (11),

$$\mathscr{C} = k(D_k^m + D_{k-1}^m) = D_k^{m+1}.$$

Similar inductive reasoning, using identity (13), proves formula (16). Another way to prove (16) is by expressing relations (15) and (16) in matrix form, $\mathbf{Cc} = \mathbf{d}$ and $\mathbf{Dd} = \mathbf{c}$, where

$$\mathbf{c} = \begin{pmatrix} C_1^m \\ C_2^m \\ \vdots \\ C_m^m \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} D_1^m \\ D_2^m \\ \vdots \\ D_m^m \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} \binom{m-1}{0} & 0 & \cdots & 0 \\ \binom{m-1}{1} & \binom{m-2}{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m-1}{m-1} & \binom{m-2}{m-2} & \cdots & \binom{0}{0} \end{pmatrix},$$

and

$$\mathbf{D} = \begin{pmatrix} \binom{m-1}{0} & 0 & \cdots & 0 \\ -\binom{m-1}{1} & \binom{m-2}{0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1}\binom{m-1}{m-1} & (-1)^{m-2}\binom{m-2}{m-2} & \cdots & \binom{0}{0} \end{pmatrix},$$

and verifying that $\mathbf{CD} = \mathbf{I}_m$, where $\mathbf{I}_m$ is the $m \times m$ identity matrix.

The i,j^{th} term of $\mathbf{C}$ is given by

$$c_{ij} = \binom{m-j}{i-j},$$

while the i,j^{th} term of $\mathbf{D}$ is given by

$$d_{ij} = (-1)^{i-j} \binom{m-j}{i-j}.$$

Hence, the i,j^{th} term of $\mathbf{CD}$ is given by

$$\sum_{k=1}^{m} c_{ik} d_{kj} = \sum_{k=1}^{m} (-1)^{k-j} \binom{m-k}{i-k} \binom{m-j}{k-j}.$$

If we substitute r for $k - j$, the right-hand side becomes

$$\sum_{r=1-j}^{m}(-1)^r\binom{(m-j)-r}{(i-j)-r}\binom{m-j}{r}. \tag{17}$$

Now, if $r < 0$, then $\binom{m-j}{r} = 0$, and if $r > i - j$, then $\binom{(m-j)-r}{(i-j)-r} = 0$. Hence, expression (17) becomes

$$\sum_{r=0}^{i-j}(-1)^r\binom{(m-j)-r}{(i-j)-r}\binom{m-j}{r},$$

which, by (14), is equal to 0 for $j+1 \le i \le m$, and equal to 1 if $i = j$. It is understood that this sum is 0 in the case $j > i$. Therefore, $\mathbf{CD} = \mathbf{I}_m$.

4. CONGRUENCES MODULO A PRIME

Now let us go back to our result (9), which stated that, if p is a prime, then

$$D_k^p \equiv \begin{cases} 1 \pmod{p} & \text{if } k = 1, \\ 0 \pmod{p} & \text{if } 2 \le k \le p. \end{cases} \tag{18}$$

For instance, the fifth row of the table formed by the D numbers is 1, 30, 150, 240, 120, and we see that all these numbers, except for the first one, which is equal to 1, are multiples of 5. This is analogous to the well-known fact that, if p is a prime number, then

$$\binom{p}{k} \equiv \begin{cases} 1 \pmod{p} & \text{if } k = 0 \text{ or } k = p, \\ 0 \pmod{p} & \text{if } 1 \le k \le p-1. \end{cases} \tag{19}$$

We will prove that statement (18) is equivalent to the statement

$$C_k^p \equiv 1 \pmod{p} \tag{20}$$

whenever p is a prime and $1 \le k \le p$. For instance, in the fifth row of the table formed by the Eulerian numbers, 1, 26, 66, 26, 1, all the numbers are congruent to 1 modulo 5.

For the proof of the equivalence of these two statements, we will use the following identity,

$$\sum_{i=0}^{k-1}\binom{n-k+i}{i} = \binom{n}{k-1}, \tag{21}$$

which is not difficult to verify by induction on n, together with the statement,

$$(-1)^{k-1}\binom{p-1}{k-1} \equiv 1 \pmod{p}, \tag{22}$$

which is easy to show using (19) and *Pascal's identity* (see [3, p. 96. prob. 12]).

Now we prove the equivalence of statements (18) and (20), which we state as a theorem.

Theorem 4.1: If p is a prime, then statements (18) and (20) are equivalent.

Proof: Assume that (20) is true. For $k = 1$, $D_k^p = 1 \equiv 1 \pmod{p}$. By Theorem 3.1, we have

$$D_k^p = \sum_{i=0}^{k-1}\binom{p-k+i}{i}C_{k-i}^p.$$

Then, using (20) and identity (21), for $2 \le k \le p$,

$$D_k^p \equiv \sum_{i=0}^{k-1} \binom{p-k+i}{i} = \binom{p}{k-1} \equiv 0 \pmod{p}.$$

Conversely, assume statement (18) is true. By Theorem 3.1 and (22),

$$C_k^p = \sum_{i=0}^{k-1} (-1)^i \binom{p-k+i}{i} D_{k-i}^p \equiv (-1)^{k-1} \binom{p-1}{k-1} \equiv 1 \pmod{p}.$$

Theorem 4.1 and the validity of (18) imply the validity of (20). We see that this theorem, together with Theorem 3.1, shows a strong relationship between the two sets of numbers C_k^m and D_k^m. We expect this relationship to have a combinatorial significance as well.

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ALGORITHMIC MANIPULATION OF SECOND-ORDER LINEAR RECURRENCES

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1. INTRODUCTION

Consider the second-order linear recurrences defined by

$$u_{n+2} = Pu_{n+1} - Qu_n, \qquad u_0 = 0, \ u_1 = 1, \tag{1}$$

$$v_{n+2} = Pv_{n+1} - Qv_n, \qquad v_0 = 2, \ v_1 = P, \tag{2}$$

and

$$w_{n+2} = Pw_{n+1} - Qw_n, \qquad w_0, \ w_1 \text{ arbitrary.} \tag{3}$$

The sequences $\langle u_n \rangle$ and $\langle v_n \rangle$ were studied extensively by Lucas [17], and the sequence $\langle w_n \rangle$ was popularized by Horadam [10], [11], [12], and was also studied by Zeitlin [23], [26], [27]. The sequence $\langle u_n \rangle$ is known as the fundamental Lucas sequence and the sequence $\langle v_n \rangle$ is known as the primordial Lucas sequence.

The relationship between w_n and the pair of sequences u_n and v_n is well known. Horadam [10] gives several formulas for w_n:

$$w_n = \frac{(2w_1 - Pw_0)u_n + w_0 v_n}{2}, \tag{4}$$

$$w_n = (w_1 - Pw_0)u_n + w_0 u_{n+1}, \tag{5}$$

$$w_n = w_1 u_n - Qw_0 u_{n-1}. \tag{6}$$

In [19], it was shown that Algorithm `LucasSimplify` could be used to prove any polynomial identity involving expressions of the form u_{an+b} and v_{an+b}. Since w_n can be expressed in terms of u_n and v_n, this means that we can algorithmically prove any polynomial identity involving expressions of the form w_{an+b} using Algorithm `LucasSimplify`.

However, Algorithm `LucasSimplify`, when applied to an expression involving w's will return a simplified expression involving u's and v's. Since it may be of interest to get results in terms of w's, we will now develop new algorithms that can be used to transform expressions involving w's from one form to another.

For example, Melham and Shannon [18] found an "addition formula" for simplifying w_{m+n}:

$$w_{m+n} = \frac{(2w_{m+1} - Pw_m)u_n + w_m v_n}{2}.$$

Unfortunately, this formula involves the sequences $\langle u_n \rangle$ and $\langle v_n \rangle$. We call an identity *impure* if it contains terms involving u's or v's. Otherwise, if the identity only involves w's, we call it *pure*. It is our goal to find a pure formula for w_{n+m} and related expressions.

2. OVERVIEW

Two expressions occur frequently enough that we shall give them names:

$$D = P^2 - 4Q \quad \text{and} \quad e = w_0 w_2 - w_1^2. \tag{7}$$

Since $w_2 = Pw_1 - Qw_0$, an equivalent formula for e is

$$e = Pw_0 w_1 - Qw_0^2 - w_1^2. \tag{8}$$

The quantity D is the discriminant of the characteristic equation for the recurrence, and the quantity e is known as the characteristic number of the sequence [2], [1]. Throughout this paper, we shall assume that

$$Q \neq 0, \quad D \neq 0, \quad \text{and} \quad e \neq 0. \tag{9}$$

In Section 3 we develop the Purification Theorem, which shows how to transform impure identities into pure identities. In subsequent sections, we then find the pure analogs (for w_n) of all the classic identities known for u_n and v_n, either by giving a reference to the literature where the pure identity was discovered, or by deriving the pure identity ourselves. If a simpler proof of the result can be given without using the Purification Theorem, then we present the simpler proof. We then give algorithms that allow pure expressions to be transformed from one form to another.

3. THE PURIFICATION THEOREM

To achieve our goal of finding pure identities, we need only express u_n and v_n in terms of members of the sequence $\langle w_n \rangle$.

Theorem 1 (The Purification Theorem): Any identity involving u's, v's, and w's can be transformed into a pure identity (involving only w's). In particular,

$$u_n = \frac{w_0 w_{n+1} - w_1 w_n}{e},$$

$$v_n = \frac{(Pw_0 - 2w_1)w_{n+1} - (2Qw_0 - Pw_1)w_n}{e}. \tag{10}$$

Proof: Algorithm `LucasSimplify` allows us to express both w_n and w_{n+1} in terms of u_n and v_n. Solving these two equations for u_n and v_n gives us formula (10). Thus, any expression involving u's and v's can be transformed into expressions involving w's. $\square$

4. THE ADDITION FORMULA

The addition formulas for u_n and v_n are well known:

$$u_{m+n} = \frac{u_m v_n + u_n v_m}{2},$$

$$v_{m+n} = \frac{v_m v_n + D u_m u_n}{2}. \tag{11}$$

We would like to find a similar formula for w_{m+n}. Horadam [10] gives several such formulas:

$$w_{n+m} = u_m w_{n+1} - Q u_{m-1} w_n; \tag{12}$$

$$w_{n+m} = (u_{m+1} - P u_m) w_n + u_m w_{n+1};$$

$$w_{n+m} = w_m u_{n+1} - Q w_{m-1} u_n;$$

$$w_{n+m} = w_n u_{m+1} - Q w_{n-1} u_m;$$

$$w_{n+m} = w_{m-j} u_{n+j+1} - Q w_{m-j-1} u_{n+j};$$

$$w_{n+m} = w_{n+j} u_{m-j+1} - Q w_{n+j-1} u_{m-j};$$

however, these are all impure.

Applying LucasSimplify to u_{m-1} gives $u_{m-1} = (P u_m - v_m)/2Q$. Substituting this value of u_{m-1} into Horadam's addition formula (12) and then applying the purification theorem gives us:

$$w_{n+m} = \frac{(w_0 w_{m+1} - w_1 w_m) w_{n+1} - (w_1 w_{m+1} - w_2 w_m) w_n}{e}.$$

We state this in another form in the following theorem.

Theorem 2 (The Addition Formula for w): For all integers n and m,

$$w_{n+m} = -\frac{1}{e} \begin{vmatrix} w_0 & w_1 & w_m \\ w_1 & w_2 & w_{m+1} \\ w_n & w_{n+1} & 0 \end{vmatrix}. \tag{13}$$

5. THE NEGATION FORMULA

Having found the addition formula entirely in terms of w's, we now proceed to express all the other standard formulas in the same manner.

Horadam [10] expressed the negation formula in the following ways:

$$w_{-n} = Q^{-n}(w_0 u_{n+1} - w_1 u_n);$$

$$w_{-n} = Q^{-n}(w_0 v_n - w_n).$$

He also found the interesting formula: $w_n w_{-n} = w_0^2 + e Q^{-n} u_n^2$.

Unfortunately, these formulas are all impure. We can use the purification theorem to remove the u's and v's to arrive at a pure negation formula.

Theorem 3 (The Negation Formula for w): For all integers n,

$$w_{-n} = \frac{(w_1^2 - Q w_0^2) w_n + w_0 (P w_0 - 2 w_1) w_{n+1}}{e Q^n}$$

$$= -\frac{1}{e Q^n} \begin{vmatrix} w_{-1} & w_0 & w_1 \\ w_0 & w_1 & Q w_0 \\ w_n & w_{n+1} & 0 \end{vmatrix}. \tag{14}$$

Solving equation (14) for w_{n+1} gives us a useful formula that allows one to express w_{n+1} in terms of w_n and w_{-n}.

Theorem 4 (The Symmetrization Formula): For all integers n,

$$w_{n+1} = \frac{(w_1^2 - Qw_0^2)w_n - eQ^n w_{-n}}{w_0(2w_1 - Pw_0)},$$

(15)

provided that the denominator is not 0.

6. THE SUBTRACTION FORMULA

Melham and Shannon [18] expressed the subtraction formula in the following form:

$$w_{m-n} = \frac{w_m u_{n+1} - w_{m+1} u_n}{Q^n}.$$

Again, this is an impure formula. We can now combine the negation formula with the addition formula to get a pure subtraction formula.

Theorem 5 (The Subtraction Formula for w): For all integers n and m,

$$w_{n-m} = -\frac{1}{eQ^m} \begin{vmatrix} w_{-1} & w_0 & w_{n+1} \\ w_0 & w_1 & Qw_n \\ w_m & w_{m+1} & 0 \end{vmatrix}.$$

(16)

Proof: Horadam [10] found $w_{n+m} + Q^m w_{n-m} = w_n v_m$. Solve for w_{n-m} and then expand w_{n+m} by the addition formula and express v_m in terms of w_m and w_{m+1} by the purification theorem. Upon simplifying, we get the desired result. □

7. THE BINET FORM

The Binet form (see [10]) for w_n is given by the following theorem.

Theorem 6 (The Binet Form): If r_1 and r_2 are the roots of the characteristic equation

$$x^2 - Px + Q = 0,$$

then

$$w_n = Ar_1^n + Br_2^n$$

(17)

where

$$A = \frac{w_1 - w_0 r_2}{r_1 - r_2} \quad \text{and} \quad B = \frac{w_0 r_1 - w_1}{r_1 - r_2}.$$

(18)

This generalizes the result for Fibonacci numbers found by Binet [3].

Note that $r_1 \neq r_2$ since $P^2 - 4Q \neq 0$. One should also note that

$$AB = \frac{e}{D} \quad \text{and} \quad A + B = w_0.$$

(19)

We also have

$$r_1 - r_2 = \sqrt{D}.$$

(20)

Since $w_{n+1} = (Ar_1)r_1^n + (Br_2)r_2^n$, we can solve the system consisting of this equation and equation (17) for r_1^n and r_2^n. We get the following:

$$r_1^n = \frac{w_{n+1} - r_2 w_n}{w_1 - r_2 w_0} \quad \text{and} \quad r_2^n = \frac{w_{n+1} - r_1 w_n}{w_1 - r_1 w_0}. \tag{21}$$

These formulas may be used to replace powers of r_1 and r_2 (with variable exponents) by simpler expressions involving r_1 and r_2.

If we let $x_n = w_{n+1} - Q w_{n-1}$, then x_n may be considered to be a companion sequence to w_n, in the same way that v_n is the companion of u_n. A little computation shows that

$$r_1^n = \frac{1}{2}(x_n + w_n(r_1 - r_2))/(w_1 - r_2 w_0).$$

This gives us the following theorem, since $(r_1^n)^k = r_1^{kn}$.

Theorem 7 (De Moivre's Formula for w_n): If $x_n = w_{n+1} - Q w_{n-1}$ and $c = w_1 - r_2 w_0 \neq 0$, then for all integers $k > 0$,

$$\left(\frac{x_n + w_n\sqrt{D}}{2c}\right)^k = \frac{x_{kn} + w_{kn}\sqrt{D}}{2c}. \tag{22}$$

This theorem is so named because of its resemblance to de Moivre's trigonometry formula. If $\langle w_n \rangle = \langle u_n \rangle$, we have

$$\left(\frac{v_n + u_n\sqrt{D}}{2}\right)^k = \frac{v_{kn} + u_{kn}\sqrt{D}}{2}.$$

8. SIMSON'S FORMULA

In 1753, Robert Simson [21] found the formula

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

The analog for the sequence $\langle w_n \rangle$ was found by Horadam [10].

Theorem 8 (Simson's Formula for w): For all integers n,

$$w_{n-1}w_{n+1} - w_n^2 = Q^{n-1}e. \tag{23}$$

Theorem 8 can also be expressed in the following manner:

$$w_n w_{n+2} - w_{n+1}^2 = Q^n e. \tag{24}$$

Horadam [10] also found the following generalization of Simson's formula.

Theorem 9 (Catalan's Identity for w): For all integers n and r,

$$w_{n+r}w_{n-r} - w_n^2 = eQ^{n-r}u_r^2. \tag{25}$$

This generalizes a result found by Catalan for Fibonacci numbers in 1886 [4].

The determinant form of Simson's theorem is

$$\begin{vmatrix} w_{n-1} & w_n \\ w_n & w_{n+1} \end{vmatrix} = Q^n e. \tag{26}$$

Horadam [10] generalized this to

$$\begin{vmatrix} w_{n-r} & w_{n+t} \\ w_n & w_{n+r+t} \end{vmatrix} = Q^{n-r} e u_r u_{r+t},$$ (27)

which extends a result for generalized Fibonacci numbers found by Tagiuri [22] in 1901. Horadam and Shannon [13] expressed this as

$$w_n^2 = \begin{vmatrix} w_{n+r+s} & w_{n+s} \\ w_{n+r} & w_n \end{vmatrix} = e Q^n u_r u_s.$$ (28)

In this form, it generalizes a 1960 result for Fibonacci numbers [7]. The special case of identity (27) when $r = 1, n = a + 1$, and $t = b - a - 1$ is of interest.

Theorem 10 (D'Ocagne's Identity for w): For all integers a and b,

$$\begin{vmatrix} w_a & w_b \\ w_{a+1} & w_{b+1} \end{vmatrix} = Q^a e u_{b-a}.$$ (29)

This generalizes a result found by d'Ocagne for Fibonacci Numbers in 1885 [6]. The special case of formula (27) when $n = a + r$ and $t = b - a - r$ is also of interest:

$$\begin{vmatrix} w_a & w_b \\ w_{a+r} & w_{b+r} \end{vmatrix} = Q^a e u_r u_{b-a}.$$ (30)

This formulation (with $a = n$ and $b = n + s$) comes from [13]. Catalan's identity can be expressed as $w_{n+r} w_{n-r} = w_n^2 + e Q^{n-r} u_r^2$.

Letting $r = 1$ and $r = 2$ in this formula and multiplying the results together yields a polynomial with w_n^4 and w_n^2 terms. The w_n^2 term can be made to vanish in the case in which $Q = -P^2$. This gives the following result.

Theorem 11: If $P^2 + Q = 0$, then

$$w_n^4 - w_{n-2} w_{n-1} w_{n+1} w_{n+2} = (e Q^{n-1})^2.$$ (31)

This generalizes the identity $F_n^4 - F_{n-2} F_{n-1} F_{n+1} F_{n+2} = 1$ that was stated by Gelin in 1880 and proved by Cesàro [5]. For another generalization of the Gelin-Cesàro identity, see [13].

Letting $r = n$ in formula (25) gives another interesting case.

Theorem 12: For all integers n,

$$w_0 w_{2n} - w_n^2 = e u_n^2.$$ (32)

Gilbert [8] found an interesting pure formula in the form of a 3×3 determinant.

Theorem 13: For all integers a, b, c, x, y, and z,

$$\begin{vmatrix} w_{a+x} & w_{a+y} & w_{a+z} \\ w_{b+x} & w_{b+y} & w_{b+z} \\ w_{c+x} & w_{c+y} & w_{c+z} \end{vmatrix} = 0.$$ (33)

9. CHANGE OF BASIS

We often wish to change an expression involving w_n and w_{n+1} into one involving w_{n+a} and w_{n+b} for two distinct integers a and b.

Theorem 14: For all integers n,

$$\binom{w_n}{w_{n+1}} = \frac{1}{w_{a+1}w_b - w_a w_{b+1}}\begin{pmatrix} w_1 & -w_0 \\ w_2 & -w_1 \end{pmatrix}\begin{pmatrix} w_b & -w_a \\ w_{b+1} & -w_{a+1} \end{pmatrix}\binom{w_{n+a}}{w_{n+b}}. \tag{34}$$

Proof: Use the Addition Formula to express w_{a+1} and w_{b+1} in terms of w_n and w_{n+1}. This gives two equations in two variables w_n and w_{n+1}. We can thus solve for these variables. Putting the result in matrix form gives the above formula. $\square$

Note that the basis change is not always possible. The denominator can be written in the form $-Q^a e u_{b-a}$ by formula (29). Thus, the change of basis is possible if and only if $u_{b-a} \neq 0$.

10. THE FUNDAMENTAL IDENTITY

Theorem 15 (The Fundamental Identity): The fundamental identity connecting w_n and w_{n+1} is

$$Pw_n w_{n+1} - Qw_n^2 - w_{n+1}^2 = eQ^n. \tag{35}$$

Proof: This follows immediately from formula (24) after replacing w_{n+2} by the value given in equation (3). $\square$

This result is not new; it is equivalent to Simson's theorem. If a is a constant, then the fundamental identity connecting w_n and w_{n+a} is

$$v_a w_n w_{n+a} - Q^a w_n^2 - w_{n+a}^2 = eQ^n u_a^2. \tag{36}$$

This was obtained by using formula (34) on the fundamental identity, changing the basis from $\{w_n, w_{n+1}\}$ to $\{w_n, w_{n+a}\}$. Changing n to x and $n + a$ to y gives the fundamental identity connecting w_x and w_y:

$$v_{y-x} w_x w_y - Q^{y-x} w_x^2 - w_y^2 = eQ^x u_{y-x}^2. \tag{37}$$

11. REMOVAL OF P AND Q

It is occasionally useful to be able to remove the quantity P from an expression. If the expression is a polynomial in the variables P and w_{c_i} where the c_i are constants and if P always occurs in a product with one of the w_{c_i}, then we can use the following results to accomplish our goal.

Theorem 16: If k is a positive integer, then

$$P^k w_0 = \sum_{j=0}^{k}\binom{k}{j}Q^j w_{k-2j}. \tag{38}$$

Applying the translation theorem (see Section 17) yields

Theorem 17: If k is a positive integer and r is an integer, then

$$P^k w_r = \sum_{j=0}^{k} \binom{k}{j} Q^j w_{k+r-2j}. \tag{39}$$

We also have

Theorem 18: If k is a positive integer and r is an integer, then

$$Q^k w_r = (-1)^k \sum_{j=0}^{k} \binom{k}{j} (-P)^j w_{2k+r-j} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P^{k-j} w_{k+r+j}. \tag{40}$$

12. THE DOUBLE ARGUMENT FORMULA

Horadam [10] found the double argument formula in the following form:

$$w_{2n} = (-Q)^n \sum_{j=0}^{n} \binom{n}{j} \left(-\frac{P}{Q}\right)^{n-j} w_{n-j}. \tag{41}$$

However, this is not a closed form.

Horadam also found a closed form (for $w_0 \neq 0$): $w_{2n} = (w_n^2 + e u_n^2)/w_0$. Shannon and Horadam [20] found the double argument formula in the following form: $w_{2n} = v_n w_n - w_0 Q^n$.

Unfortunately, both these formulas are impure. To get a pure formula, let $m = n$ in the addition formula. We obtain the following result.

Theorem 19 (The Double Argument Formula for w): For all integers n,

$$w_{2n} = \frac{w_2 w_n^2 - 2 w_1 w_n w_{n+1} + w_0 w_{n+1}^2}{e} = \frac{-\begin{vmatrix} w_0 & w_1 & w_n \\ w_1 & w_2 & w_{n+1} \\ w_n & w_{n+1} & 0 \end{vmatrix}}{\begin{vmatrix} w_0 & w_1 \\ w_1 & w_2 \end{vmatrix}}. \tag{42}$$

13. FORMULAS FOR w_{kn}

To find expressions for w_{kn} where k is a positive integer constant, you can use the recurrence found by Zeitlin [24]:

$$w_{kn} = v_n w_{(k-1)n} - Q^n w_{(k-2)n}, \quad k \geq 2. \tag{43}$$

Lee [16] found a more direct formula for multiple argument reduction. For $k > 1$,

$$w_{kn} = w_n S(k) - w_0 Q^n S(k-1), \tag{44}$$

where

$$S(k) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{k-j-1}{j} (-Q^n)^j v_n^{k-2j-1}. \tag{45}$$

Jarden [14] found the following interesting formula:

$$w_{kn+s} = \sum_{i=0}^{k} \binom{k}{i} u_n^i (-Q u_{n-1})^{k-i} w_{i+s}. \tag{46}$$

Zeitlin has found many related formulas. For example, Zeitlin ([27], eq. 1.14, with $m = 0$ and $n = 0$) found the following interesting formula:

$$w_{kn} = \frac{1}{2^k} \sum_{j=0}^{k} c_j \binom{k}{j} D^{\lfloor j/2 \rfloor} u_n^j v_n^{k-j}, \text{ where } c_j = \begin{cases} w_0, & \text{if } j \text{ is even,} \\ 2w_1 - Pw_0, & \text{if } j \text{ is odd.} \end{cases} \tag{47}$$

Formula (46) can be converted into a pure formula for w_{kn} by letting $s = 0$ and substituting $u_n = (w_0 w_{n+1} - w_1 w_n)/e$ and $u_{n-1} = (w_1 w_{n+1} - w_2 w_n)/(eQ)$. We get the following.

Theorem 20: If $k \geq 0$, then

$$w_{kn} = \frac{1}{e^k} \sum_{i=0}^{k} \binom{k}{i} (w_0 w_{n+1} - w_1 w_n)^i (w_2 w_n - w_1 w_{n+1})^{k-i} w_i. \tag{48}$$

This can be expanded out as a polynomial in w_n and w_{n+1}. Computer experiments suggest the following result.

Conjecture 21 (The Multiple Argument Formula for w): If k is an integer larger than 1, then

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_i \binom{k}{i} (-1)^{k-i} w_n^i w_{n+1}^{k-i}, \tag{49}$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}. \tag{50}$$

14. EXPANSION OF PRODUCTS

Horadam [10] found

$$w_n v_m = w_{n+m} + Q^m w_{n-m}. \tag{51}$$

But we would really like to express $w_n w_m$ as a sum of w's. To do that, we can proceed as follows. Changing m to $m+1$ in equation (51) gives

$$w_n v_{m+1} = w_{n+m+1} + Q^m w_{n-m-1}. \tag{52}$$

But it is easy to show that

$$w_m = \frac{D_1}{D} v_m + \frac{D_2}{D} v_{m+1}, \tag{53}$$

where

$$D = P^2 - 4Q, \quad D_1 = P^2 w_0 - 2Qw_0 - Pw_1, \quad \text{and} \quad D_2 = 2w_1 - Pw_0. \tag{54}$$

Multiplying (51) by D_1/D, multiplying (52) by D_2/D, and adding the results, gives us the following theorem.

Theorem 22 (The Product Formula for w): For all integers n and m,

$$w_m w_n = \frac{1}{D}[Q^{m+1} D_2 w_{n-(m+1)} + Q^m D_1 w_{n-m} + D_1 w_{n+m} + D_2 w_{n+m+1}], \tag{55}$$

where $D, D_1,$ and D_2 are as given in (54).

Applying the symmetrization formula and expressing $w_{n-(m+1)}$ in terms of w_{n-m} and w_{n-m+1} permits us to obtain another variation of the product formula.

Theorem 23 (Symmetric Product Formula for w): If $w_0 \neq 0$, then, for all integers n and m,

$$w_m w_n = \frac{1}{Dw_0}[eQ^n w_{m-n} - eQ^{m+n} w_{-n-m} + eQ^m w_{n-m} + (Dw_0^2 - e)w_{m+n}]. \tag{56}$$

If $m = n$ in formula (55), we get

$$w_n^2 = \frac{1}{D}[2eQ^n + D_1 w_{2n} + D_2 w_{2n+1}], \tag{57}$$

which can be used to turn squares into sums. Using formula (56), this can also be written as

$$Dw_0 w_n^2 = (Dw_0^2 - e)w_{2n} + 2ew_0 Q^n - ew_{-2n} Q^{2n}. \tag{58}$$

Theorem 24: If k is a positive integer, then w_n^k can be expressed in the form

$$(Dw_0)^{k-1} w_n^k = \sum_{i=0}^{k} c_{k,i} Q^{in} w_{(k-2i)n}, \tag{59}$$

where $c_{k,i}$ is a polynomial in d, e, and w_0, with integer coefficients, where $d = Dw_0^2$.

Proof: The proof is by induction. The case $k = 2$ is given above in formula (58). Assuming it is true for w_n^k, take the formula for w_n^k and multiply it by $(Dw_0)w_n$. The symmetric product formula then gives the answer in the desired form. $\square$

15. THE POWER EXPANSION FORMULA

In 1878, Lucas ([17], §XII) found an explicit formula for w_n in terms of w_0, w_1, P, and Q (see also [25], [12], and [16]).

Theorem 25: For all $n > 0$,

$$w_n = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} P^{n-2k}(-Q)^{k-1}\left[\binom{n-k}{k-1}w_1 P - \binom{n-k-1}{k-1}w_0 Q\right]. \tag{60}$$

16. THE UNIVERSAL RECURRENCE

We can solve the system of equations

$$w_{n+2} = Pw_{n+1} - Qw_n,$$

$$w_{n+3} = Pw_{n+2} - Qw_{n+1},$$

for P and Q. Thus, any four consecutive terms of the sequence $\langle w_n \rangle$ are enough to determine P and Q. The result is

$$P = \frac{w_n w_{n+3} - w_{n+1} w_{n+2}}{w_n w_{n+2} - w_{n+1}^2} \quad \text{and} \quad Q = \frac{w_{n+1} w_{n+3} - w_{n+2}^2}{w_n w_{n+2} - w_{n+1}^2}. \tag{61}$$

We can substitute these values of P and Q into the identity $w_{n+4} = Pw_{n+3} - Qw_{n+2}$ to arrive at a recurrence for $\langle w_n \rangle$ that does not involve P, Q, w_0, or w_1. The result is the following.

Theorem 26 (The Universal Recurrence): Any second-order linear recurrence $\langle w_n \rangle$ with constant coefficients satisfies the recurrence

$$w_{n+4} = \frac{w_{n+2}^3 - 2w_{n+1}w_{n+2}w_{n+3} + w_n w_{n+3}^2}{w_n w_{n+2} - w_{n+1}^2}. \tag{62}$$

We call this the "universal recurrence" since it is satisfied by any second-order linear recurrence no matter what the coefficients or initial conditions, subject only to the restriction that the denominator should not be 0. [This is equivalent to the condition that $e \neq 0$ and $Q \neq 0$ by formula (24).]

The universal recurrence can be written in the form

$$\begin{vmatrix} w_{n+4} & w_{n+3} & w_{n+2} \\ w_{n+3} & w_{n+2} & w_{n+1} \\ w_{n+2} & w_{n+1} & w_n \end{vmatrix} = 0. \tag{63}$$

In this form, the result is due to Casorati.

17. THE RECURRENCE FOR MULTIPLES

Zeitlin [24] found the recurrence satisfied by the sequence $\langle w_{kn} \rangle$, where k is a fixed positive integer:

$$w_{kn} = v_k w_{k(n-1)} - Q^k w_{k(n-2)}. \tag{64}$$

This recurrence can be made pure by substituting the value for v_k given by formula (10).

Theorem 27 (The Translation Theorem): Let a be a nonzero integer. Given an identity involving w_n, u_n, and v_n, we can create another valid identity by replacing all occurrences of w_x by w_{x+a}. This operation is called a translation by a.

Proof: Since the original identity is true for a completely arbitrary second-order linear recurrence $\langle w_n \rangle$ it must be true for the particular second-order linear recurrence $\langle w_{n+a} \rangle$. □

Theorem 28 (The Dilation Theorem): Let k be a positive integer. Given an identity involving w_n, u_n, and v_n, we can create another valid identity by replacing all occurrences of w_x by w_{kx}, provided that we also replace u_x by u_{kx}/u_k, v_x by v_{kx}, Q by Q^k, P by v_k, and e by eu_k^2. This operation is called "a dilation by k."

Proof: The sequence $\langle w_{kn} \rangle$ satisfies the second-order linear recurrence given by equation (64). Since the original identity is true for a completely arbitrary second-order linear recurrence $\langle w_n \rangle$ it must be true for the particular second-order linear recurrence $\langle w_{kn} \rangle$. However, this new recurrence has different parameters; namely, $P' = v_k$ and $Q' = Q^k$. If $W_n = w_{kn}$, then the fundamental Lucas sequence $\langle U_n \rangle$ that corresponds to $\langle W_n \rangle$ would satisfy the recurrence $U_n = v_k U_{n-1} - Q^k U_{n-2}$ with initial conditions $U_0 = 0$ and $U_1 = 1$. But the sequence u_{kn} satisfies this recurrence by (64). To meet the initial conditions, we need only scale it down by a factor of u_k. Thus, $U_k = u_{kn}/u_k$. A similar remark holds for the corresponding primordial Lucas sequence $\langle V_n \rangle$.

Thus, if we convert to these new parameters, we should obtain a valid identity. Note that $e = w_0 w_2 - w_1^2$, when converted, becomes $w_0 w_{2k} - w_k^2$, which is equal to eu_k^2 by Theorem 12. □

18. THE RECURRENCE FOR POWERS

Jarden [15] found the recurrence satisfied by the sequence $\langle w_n^t \rangle$, where t is a fixed positive integer:

$$\sum_{j=0}^{t+1} (-1)^j Q^{j(j-1)/2} \begin{bmatrix} t+1 \\ j \end{bmatrix}_u w_{n-j}^t = 0, \tag{65}$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{u_m u_{m-1} \cdots u_{m-r+1}}{u_1 u_2 \cdots u_r}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1. \tag{66}$$

See also [9] and [10] for some related identities. Zeitlin [23], [26], has found many other identities involving powers of w's.

19. THE ALGORITHMS

We now summarize the algorithms found earlier in this paper. For the reader's convenience, we repeat some of the earlier formulas (leaving their original formula numbers). All of the algorithms listed here have been implemented in Mathematica™, and are available from the author via e-mail.

Algorithm `ConvertToUV`–to convert an expression involving w's into one involving u's and v's:

Apply the substitution

$$w_n = \frac{(2w_1 - Pw_0)u_n + w_0 v_n}{2}. \tag{4}$$

Algorithm `ConvertToW`–to convert expressions involving u's and v's into expressions involving w's:

Apply the identities

$$u_n = \frac{w_0 w_{n+1} - w_1 w_n}{e},$$

$$v_n = \frac{(Pw_0 - 2w_1)w_{n+1} - (2Qw_0 - Pw_1)w_n}{e}. \tag{10}$$

Algorithm `WReduce`–to remove sums in subscripts:

Repeatedly apply the addition formula

$$w_{n+m} = -\frac{1}{e} \begin{vmatrix} w_0 & w_1 & w_m \\ w_1 & w_2 & w_{m+1} \\ w_n & w_{n+1} & 0 \end{vmatrix}. \tag{13}$$

Algorithm `WNegate`–to negate subscripts:

Use the identity

$$w_{-n} = \frac{(w_1^2 - Qw_0^2)w_n + w_0(Pw_0 - 2w_1)w_{n+1}}{eQ^n}. \tag{14}$$

Algorithm WShift–to change basis:

To convert an expression involving w_n and w_{n+1} into one involving w_{n+a} and w_{n+b}, apply the substitutions

$$\begin{pmatrix} w_n \\ w_{n+1} \end{pmatrix} = \frac{1}{w_{a+1}w_b - w_a w_{b+1}} \begin{pmatrix} w_1 & -w_0 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} w_b & -w_a \\ w_{b+1} & -w_{a+1} \end{pmatrix} \begin{pmatrix} w_{n+a} \\ w_{n+b} \end{pmatrix}. \tag{34}$$

Algorithm WExpand–to turn products into sums:

Repeatedly apply the substitution

$$w_m w_n = \frac{1}{D}[Q^{m+1}D_2 w_{n-(m+1)} + Q^m D_1 w_{n-m} + D_1 w_{n+m} + D_2 w_{n+m+1}], \tag{55}$$

where $D = P^2 - 4Q$, $D_1 = P^2 w_0 - 2Q w_0 - P w_1$, and $D_2 = 2w_1 - P w_0$.

Algorithm WRemoveP–to remove P in coefficients with terms involving w_c:

If c is a constant, use the identity

$$P^k w_c = \sum_{j=0}^{k} \binom{k}{j} Q^j w_{k+c-2j}. \tag{39}$$

Algorithm WRemoveQ–to remove Q in coefficients with terms involving w_c:

If c is a constant, use the identity

$$Q^k w_c = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P^{k-j} w_{k+c+j}. \tag{40}$$

Algorithm RemovePowersOfWPlus1–to remove powers of w_{n+1}:

Use the identity

$$w_{n+1}^2 = P w_n w_{n+1} - Q w_n^2 - eQ^n \tag{67}$$

repeatedly until no w_{n+1} term has an exponent larger than 1. This identity comes from formula (35).

Algorithm RemovePowersOfW–to remove powers of w_n:

Use the identity

$$w_n^2 = \frac{P w_n w_{n+1} - w_{n+1}^2 - eQ^n}{Q} \tag{68}$$

repeatedly until no w_n term has an exponent larger than 1 This identity comes from formula (35).

Algorithm RemovePowersOfQ–to remove variable powers of Q:

To remove any expressions of the form Q^{an+b} from an expression, where n is a variable and a and b are independent of n with $a \neq 0$, write Q^{an+b} as $Q^b(Q^n)^a$ if $a > 0$ and as $Q^b(Q^{-n})^{-a}$ if $a < 0$. Then replace $Q^{\pm n}$ by the substitution

$$Q^{\pm n} = \frac{P w_{\pm n} w_{\pm n+1} - Q w_{\pm n}^2 - w_{\pm n+1}^2}{e}, \tag{69}$$

which comes from formula (35). If $a < 0$, we cannot in general replace Q^{an} by any polynomial in the w's with subscripts consisting only of positive multiples of n. However, if Q happens to be a root of unity, then simplification is possible. The cases $Q = -1$ and $Q = 1$ frequently occur and are of this form. Let m be the smallest positive integer such that $Q^m = 1$. Write Q^{an+b} as $Q^b Q^{an}$. Let b be the residue of a modulo m, i.e., the positive integer such that $0 \le b < m$ and $b \equiv a \pmod{m}$. Then $Q^a = Q^b$, so we can replace Q^{an} by Q^{bn} with $b \ge 0$. If $b > 0$, we proceed as in the previous case.

Definition: A w-polynomial is any polynomial $f(x_1, x_2, ..., x_r)$ with constant coefficients, where each x_i is of the form w_x, u_x, v_x, or Q^x, with each x of the form $a_1 n_1 + a_2 n_2 + \cdots + a_k n_k + b$, where b and the a_i are integer constants and the n_i are variables. For purposes of this definition, the quantities P, Q, w_0, and w_1 are to be considered constants.

Algorithm `WSimplify`–to convert an expression to canonical form:

INPUT: A w-polynomial.

OUTPUT: Its "canonical form". Two expressions that are identical will have the same canonical form. In particular, an expression is identically 0 if and only if its canonical form is 0.

Step 1. [Convert to w.] If any expression of the form u_x or v_x occurs, apply Algorithm `ConvertToW` to remove it.

Step 2. [Remove variable sums in subscripts.] If any expression of the form $a_1 n_1 + a_2 n_2$ occurs in a subscript, apply Algorithm `WReduce` to remove such sums. Treat $a_1 n_1 - a_2 n_2$ as $a_1 n_1 + (-a_2)n_2$.

Step 3. [Make multipliers positive.] All subscripts are now of the form $an + b$, where a and b are integers and n is a variable. For any term in which the multiplier a is negative, apply Algorithm `WNegate`.

Step 4. [Remove multipliers.] All subscripts are now of the form $an + b$, where a is a non-negative integer, b is an integer, and n is a variable. If $a > 1$, write $an + b$ as $n + n + \cdots + n + b$ with a copies of n and then apply Algorithm `WReduce` repeatedly until all these subscripts are of the form $n + c$, where c is an integer.

Step 5. [Remove constants in subscripts.] If any expression of the form $n + b$ with $b \ne 0$ and $b \ne 1$ occurs in a subscript, apply Algorithm `WReduce` to remove such sums.

Step 6. [Remove powers of w_{n+1}.] If any term involves an expression of the form w_{n+1}^k, where $k > 1$ and n is a variable, apply Algorithm `RemovePowersOfWPlus1` to leave only linear terms in w_{n+1}.

Step 7. [Evaluate constants.] If any term involves an expression of the form w_c^k, where c is an integer constant, replace w_c by its numerical equivalent. If the symbols D or e occur, replace them by their equivalent values from formula (7).

Step 8. [Simplify powers of Q.] If Q is a primitive m^{th} root of unity, then replace all constants appearing in an exponent with base Q by their residues modulo m.

The canonical form is a polynomial $f(x_1, x_2, ..., x_r)$ with constant coefficients, where each x_i is of the form w_{n_i}, w_{n_i+1}, or $Q^{\pm n_i}$, where the n_i are variables, and the degree of each w_{n_i+1} term is 0 or 1. If Q is a root of unity, then no exponent with base Q is negative.

Alternatively, to prove an identity, you can apply Algorithm `LucasSimplify` and show that the resulting canonical form is 0.

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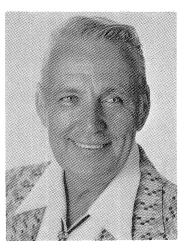
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AMS Classification Numbers: 11Y16, 11B37

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In Memoriam

Richard Spain Vine (1913-1999)

Richard Vine, retired subscription manager of *The Fibonacci Quarterly* (17 years), retiree from Lockheed, active participant in professional and community theater, avid tennis player, took his final bow and left this stage of life on January 25, 1999, after a long fight with bone cancer. Richard sends the following message to his friends in the Fibonacci Association: "It was a wonderful life; please think of me kindly and with love as I did you: *La commedia è finita!*"

THE GOLDEN SECTION AND NEWTON APPROXIMATION

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(Submitted July 1997-Final Revision October 1997)

In this note we combine number theory (continued fraction convergents (see [1], ch. X) to the golden section) and calculus (Newton approximants to zeros (see [3], ch. 4)).

The golden section $g := \frac{\sqrt{5}-1}{2}$ satisfies $g^2 + g = 1$; for $G := g^{-1} = g + 1$, we have $G^2 = G + 1$. The even (continued fraction) convergents to g are

$$g_n := \frac{F_{2n}}{F_{2n+1}} \quad (n = 0, 1, 2, 3, \ldots).$$

The arbitrary function $H : [0, g] \to \mathbb{R}$ of class C^2 may satisfy $H(0) = 1$, $H(g) = 0$, and $H'(x) < 0$, $H''(x) > 0$ $(0 \le x < g)$. Let

$$N(x) := x - \frac{H(x)}{H'(x)};$$

then Newton approximation applies with

$$x_0 := 0, \quad x_{n+1} := N(x_n) > x_n \quad (n = 0, 1, 2, \ldots), \quad \lim_{n \to \infty} x_n = g.$$

In this note we give H explicitly such that $x_n = g_n$ $(n = 0, 1, 2, \ldots)$. For this, we look at

$$D(x) := \frac{1 - x - x^2}{2 + x} = \frac{(g - x)(G + x)}{2 + x} = \frac{(1 - Gx)(1 + gx)}{2 + x};$$

$y = D(x)$ is a hyperbola with the asymptotes $x = -2$ and $x + y = 1$. Thus, we have

$$D(-G) = D(g) = 0, \quad D(-1) = 1, \quad D'(-1) = 0, \quad D(0) = \frac{1}{2}, \quad D(x) > 0 \quad (-G < x < g).$$

By $G^3 + g^3 = 2\sqrt{5}$, $G^2 - g^2 = \sqrt{5}$, we have

$$\frac{\sqrt{5}}{D(x)} = \frac{G^3}{1 - Gx} + \frac{g^3}{1 + gx}.$$

To be specific, we choose

$$H(x) := \exp\left(-\int_0^x \frac{dt}{D(t)}\right) \quad (0 \le x \le g).$$

Using log and differentiation, we find that

$$H(x) = (1 - Gx)^{G^2/\sqrt{5}} (1 + gx)^{-g^2/\sqrt{5}} \quad (0 \le x \le g)$$

and also that

$$\frac{H'(x)}{H(x)} = -\frac{1}{D(x)} \quad (0 \le x < g).$$

We observe the following:

$$H(x) > 0, \quad H'(x) < 0 \ \ (0 \le x < g), \quad H(g) = 0, \quad H'(g) = 0,$$

$$N(x) = x + D(x) = \frac{x+1}{x+2} = 1 - \frac{1}{x+2}, \quad N(g) = g, \quad N'(x) = \frac{1}{(x+2)^2};$$

$y = N(x)$ is a hyperbola with the asymptotes $x = -2$, $y = 1$. Thus, we have

$$N(-1) = 0, \quad N(0) = \frac{1}{2}.$$

From $D(x)H'(x) + H(x) = 0$, $D(x)H''(x) + N'(x)H'(x) = 0$, we deduce $H''(x) > 0 \ \ (0 \le x < g)$. We also note that

$$x_0 := 0, \quad x_{n+1} := \frac{x_n + 1}{x_n + 2} \quad (n = 0, 1, 2, \ldots).$$

Theorem: We have $x_n = g_n \ \ (n = 0, 1, 2, \ldots)$.

Proof: We know that $x_0 = g_0 = 0$. It remains to show that

$$\frac{F_{2n+2}}{F_{2n+3}} = \frac{\frac{F_{2n}}{F_{2n+1}} + 1}{\frac{F_{2n}}{F_{2n+1}} + 2} \quad \text{or} \quad \frac{F_{2n+2}}{F_{2n+3}} = \frac{F_{2n} + F_{2n+1}}{F_{2n} + 2F_{2n+1}} \quad (n = 0, 1, 2, \ldots);$$

but the numerators are equal and also the denominators.

For integers $a, b > 0$, $c, d > 0$, let $bc - ad = 1$, then $(a, b) = (c, d) = 1$, and

$$\frac{a}{b} < \frac{a+c}{b+d} \ \ (\text{"mediant"}) \ < \frac{c}{d}.$$

Let $a' := a + b$, $b' := a + 2b > 0$, $c' := c + d$, $d' := c + 2d > 0$, then

$$(a', b') = (a+b, a+2b) = (a+b, b) = (a, b) = 1, \quad (c', d') = \cdots = 1,$$

$$N\left(\frac{a}{b}\right) = \frac{a'}{b'}, \quad N\left(\frac{a+c}{b+d}\right) = \frac{(a+c)+(b+d)}{(a+c)+2(b+d)} = \frac{a'+c'}{b'+d'}, \quad N\left(\frac{c}{d}\right) = \frac{c'}{d'};$$

hence, N respects mediants.

I treated this topic during my visit to Johannesburg in 1985 (see [2]). I am grateful to the referee for a careful reading of the manuscript.

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AMS Classification Numbers: 11B39, 41A05

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to stanley@tiac.net on the Internet. All correspondence will be acknowledged.

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-872 *Proposed by Murray S. Klamkin, University of Alberta, Canada*

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for r_n^2.

B-873 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that 3 is the only positive integer that is both a prime number and of the form $L_{3n} + (-1)^n L_n$.

B-874 *Proposed by David M. Bloom, Brooklyn College, NY*

Prove that 3 is the only positive integer that is both a Fibonacci number and a Mersenne number. [A Mersenne number is a number of the form $2^a - 1$.]

B-875 *Proposed by Richard André-Jeannin, Cosnes et Romain, France*

Prove that 3 is the only positive integer that is both a triangular number and a Fermat number. [A triangular number is a number of the form $n(n+1)/2$. A Fermat number is a number of the form $2^a + 1$.]

B-876 *Proposed by N. Gauthier, Royal Military College of Canada*

Evaluate

$$\sum_{k=1}^{n} \sin\left(\frac{\pi F_{k-1}}{F_k F_{k+1}}\right) \sin\left(\frac{\pi F_{k+2}}{F_k F_{k+1}}\right).$$

B-877 *Proposed by Indulis Strazdins, Riga Technical University, Latvia*
Evaluate

$$\begin{vmatrix} F_n F_{n+1} & F_{n+1} F_{n+2} & F_{n+2} F_{n+3} & F_{n+3} F_{n+4} \\ F_{n+4} F_{n+5} & F_{n+5} F_{n+6} & F_{n+6} F_{n+7} & F_{n+7} F_{n+8} \\ F_{n+8} F_{n+9} & F_{n+9} F_{n+10} & F_{n+10} F_{n+11} & F_{n+11} F_{n+12} \\ F_{n+12} F_{n+13} & F_{n+13} F_{n+14} & F_{n+14} F_{n+15} & F_{n+15} F_{n+16} \end{vmatrix}.$$

SOLUTIONS

The Right Angle to Success

B-854 *Proposed by Paul S. Bruckman, Edmonds, WA*
(Vol. 36, no. 3, August 1998)
Simplify

$$3\arctan(\alpha^{-1}) - \arctan(\alpha^{-5}).$$

Solution by L. A. G. Dresel, Reading, England

Let $\theta = \arctan(\alpha^{-1})$, so that $\tan\theta = \alpha^{-1}$. Using the formula

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

we find that

$$\tan 2\theta = 2\alpha^{-1} / (1 - \alpha^{-2}) = 2\alpha / (\alpha^2 - 1) = 2\alpha / \alpha = 2,$$

and

$$\tan 3\theta = (2 + \alpha^{-1}) / (1 - 2\alpha^{-1}) = (2 - \beta) / (1 + 2\beta) = (1 + \alpha) / (\beta^2 + \beta) = \alpha^2 / \beta^3 = -\alpha^5.$$

Hence, $3\arctan(\alpha^{-1}) = \pi - \arctan(\alpha^5)$, and since $\arctan(\alpha^{-5}) + \arctan(\alpha^5) = \pi / 2$, we have

$$3\arctan(\alpha^{-1}) - \arctan(\alpha^{-5}) = \pi / 2.$$

Solutions also received by Richard André-Jeannin, Charles K. Cook, Steve Edwards, Russell Jay Hendel, Walther Janous, Murray S. Klamkin, Angel Plaza & Miguel A. Padrón, Maitland A. Rose, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Recurrence for a Ratio

B-855 *Proposed by the editor*
(Vol. 36, no. 3, August 1998)
Let $r_n = F_{n+1} / F_n$ for $n > 0$. Find a recurrence for r_n.

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

$$r_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{r_{n-1}} \quad \text{for } n > 1.$$

Generalization by Murray S. Klamkin, University of Alberta, Canada: More generally, we determine a recurrence for $r_n = G_{n+1} / G_n$, where $G_{n+1} = aG_n + bG_{n-1}$ by simply dividing the latter recurrence by G_n to give

$$r_n = a + b / r_{n-1}.$$

Klamkin gave generalizations to third-order recurrences as well as several other generalizations, one of which we present to the readers as problem B-872 in this issue.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Charles K. Cook, Mario DeNobili, Leonard A. G. Dresel, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Walther Janous, Daina Krigens, Angel Plaza & Miguel A. Padrón, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Weak Inequality

B-856   *Proposed by Zdravko F. Starc, Vršac, Yugoslavia*
   (Vol. 36, no. 3, August 1998)
   If n is a positive integer, prove that

$$L_1\sqrt{F_1} + L_2\sqrt{F_2} + L_3\sqrt{F_3} + \cdots + L_n\sqrt{F_n} < 8F_n^2 + 4F_n.$$

Solution 1 by Richard André-Jeannin, Cosnes et Romain, France
   We see that

$$L_1\sqrt{F_1} + L_2\sqrt{F_2} + L_3\sqrt{F_3} + \cdots + L_n\sqrt{F_n} \le L_1F_1 + L_2F_2 + \cdots + L_nF_n$$

$$= F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1 < F_{2n+1}$$

$$= F_n^2 + F_{n+1}^2 < F_n^2 + (2F_n)^2 = 5F_n^2 < 8F_n^2 + 4F_n.$$

Solution 2 by L. A. G. Dresel, Reading, England
   We shall prove the much stronger result

$$L_1\sqrt{F_1} + L_2\sqrt{F_2} + L_3\sqrt{F_3} + \cdots + L_n\sqrt{F_n} < 4.35F_n^{3/2}.$$

Let $\gamma = \beta/\alpha = -\alpha^2$ and $\delta = \sqrt{5}$. Then $L_k = \alpha^k(1+\gamma^k)$, $F_k = \alpha^k(1-\gamma^k)/\delta$, $\sqrt{F_k} < \alpha^{k/2}(1-\gamma^k/2)/\sqrt{\delta}$, and $L_k\sqrt{F_k} < \alpha^{3k/2}(1+\gamma^k/2)/\sqrt{\delta}$. Summing for $1 \le k \le n$, we have two geometric progressions, giving

$$\sum L_k\sqrt{F_k} < (\alpha^{3(n+1)/2} - \alpha^{3/2})/(\alpha^{3/2} - 1)\sqrt{\delta} - \tfrac{1}{2}(\alpha^{-1/2} - (-1)^n\alpha^{-(n+1)/2})/(1+\alpha^{-1/2})\sqrt{\delta}$$

$$< (\alpha^{3n/2} - 1)/(1 - \alpha^{-3/2})\sqrt{\delta}.$$

Now

$$F_n^{3/2} = \alpha^{3n/2}(1-\gamma^n)^{3/2}/\delta^{3/2} > \alpha^{3n/2}(1-3\gamma^n/2)/\delta^{3/2}$$

$$> (\alpha^{3n/2} - 3/2\alpha)/\delta^{3/2} > (\alpha^{3n/2} - 1)/\delta^{3/2}.$$

Hence,

$$\sum L_k\sqrt{F_k} < cF_n^{3/2},$$

where $c = \sqrt{5}/(1 - \alpha^{-3/2}) = 4.34921... < 4.35$.

All solvers strengthened the proposed equality. Upper bounds found were:

Jaroslav Seibert:	$7F_n^2 - 2F_n$
H.-J. Seiffert:	$5F_n^2$
Walther Janous:	$8F_n^{3/2}$
Paul S. Bruckman:	$2.078\sqrt{F_{3n}}$

Linear Number of Digits

B-857 *Proposed by the editor*
   (Vol. 36, no. 3, August 1998)

   Find a sequence of integers $\langle w_n \rangle$ satisfying a recurrence of the form $w_{n+2} = Pw_{n+1} - Qw_n$ for $n \geq 0$ such that, for all $n > 0$, w_n has precisely n digits (in base 10).

Solution by Richard André-Jeannin, Cosnes et Romain, France

   The sequence $w_n = 10^n - 1$ has n digits in base 10 and satisfies the recurrence:

$$w_n = 11w_{n-1} - 10w_{n-2}.$$

Solutions also received by Paul S. Bruckman, Aloysius Dorp, Leonard A. G. Dresel, Gerald A. Heuer, Walther Janous, H.-J. Seiffert, and the proposer.

Calculating Convolutions

B-858 *Proposed by Wolfdieter Lang, Universität Karlsruhe, Germany*
   (Vol. 36, no. 3, August 1998)

   (a) Find an explicit formula for $\sum_{k=0}^{n} kF_{n-k}$ which is the convolution of the sequence $\langle n \rangle$ and the sequence $\langle F_n \rangle$.

   (b) Find explicit formulas for other interesting convolutions.

   (The convolution of the sequence $\langle a_n \rangle$ and $\langle b_n \rangle$ is the sum $\sum_{k=0}^{n} a_k b_{n-k}$.)

Solution to (a) by Steve Edwards, Southern Polytechnic State Univ., Marietta, GA

   We show that

$$\sum_{k=0}^{n} kG_{n-k} = G_{n+3} - [(n+2)G_1 + G_0]$$

for any generalized Fibonacci sequence $\langle G_n \rangle$, and this gives as a special case the sum in (a), which sums to $F_{n+3} - (n+3)$.

   Proof by induction: For $n = 0$, $0G_0 = 0 = G_3 - (G_2 + G_1) = G_3 - (2G_1 + G_0)$. For $n = m+1$,

$$\sum_{k=0}^{m+1} kG_{(m+1)-k} = \sum_{j=0}^{m} (j+1)G_{m-j} = \sum_{j=0}^{m} jG_{m-j} + \sum_{j=0}^{m} G_{m-j}$$

$$= G_{m+3} - [(m+2)G_1 + G_0] + [G_{m+2} - G_1] \quad \text{(by a variation of (33) in [1])}$$

$$= G_{m+4} - [(m+3)G_1 + G_0].$$

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Solution to (b) by H.-J. Seiffert, Berlin, Germany

   Let $F_n(x)$ denote the Fibonacci polynomial, defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Then we have

$$\sum_{k=0}^{n} F_k(x)F_{n-k}(y) = \frac{F_n(x) - F_n(y)}{x - y}.$$

Several solvers found the convolution of $\langle n^2 \rangle$ and $\langle F_n \rangle$ to be $F_{n+6} - (n^2 + 4n + 8)$. Dresel found the convolution of $\langle n \rangle$ and $\langle L_n \rangle$ to be $L_{n+3} - (n+4)$.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, Pentti Haukkanen, Walther Janous, Hans Kappus, Murray S. Klamkin, Carl Libis, Jaroslav Seibert, Indulis Strazdins, and the proposer.

Fun Determinant

B-859   *Proposed by Kenneth B. Davenport, Pittsburgh, PA*
           (Vol. 36, no. 3, August 1998)

     Simplify

$$\begin{vmatrix} F_n F_{n+1} & F_{n+1} F_{n+2} & F_{n+2} F_{n+3} \\ F_{n+3} F_{n+4} & F_{n+4} F_{n+5} & F_{n+5} F_{n+6} \\ F_{n+6} F_{n+7} & F_{n+7} F_{n+8} & F_{n+8} F_{n+9} \end{vmatrix}.$$

Solution by Russell Hendel, Philadelphia, PA

     The determinant's value is $32(-1)^n$.

     It is easy to verify this for the seven values $n = -3, -2, -1, 0, 1, 2, 3$. The result now follows for all n by Dresel's Verification Theorem [1], since the determinant is a homogeneous algebraic form of degree 6.

Reference

1.   L. A. G. Dresel. "Transformations of Fibonacci-Lucas Identities." In *Applications of Fibonacci Numbers* **5**:169-84. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1996.

Seiffert found that

$$\begin{vmatrix} F_n F_{n+1} & F_{n+1} F_{n+2} & F_{n+2} F_{n+3} \\ F_{n+p} F_{n+p+1} & F_{n+p+1} F_{n+p+2} & F_{n+p+2} F_{n+p+3} \\ F_{n+q} F_{n+q+1} & F_{n+q+1} F_{n+q+2} & F_{n+q+2} F_{n+q+3} \end{vmatrix} = (-1)^{n+p-1} F_p F_q F_{q-p}.$$

For a related problem, see problem B-877 in this issue.

Solutions also received by Richard André-Jeannin, Paul S. Bruckman, Leonard A. G. Dresel, Walther Janous, Carl Libis, Stanley Rabinowitz, Jaroslav Seibert, H.-J. Seiffert, Indulis Strazdins, and the proposer.

Addenda. We wish to belatedly acknowledge solutions from the following solvers:

     Murray S. Klamkin—B-848, 849, 850, 851
     Harris Kwong—B-831, 832
     A. J. Stam—B-853

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-550 *Proposed by Paul S. Bruckman, Highwood, IL*

Suppose n is an odd integer, p an odd prime $\neq 5$. Prove that $L_n \equiv 1 \pmod{p}$ if and only if either (i) $\alpha^n \equiv \alpha$, $\beta^n \equiv \beta \pmod{p}$, or (ii) $\alpha^n \equiv \beta$, $\beta^n \equiv \alpha \pmod{p}$.

H-551 *Proposed by N. Gauthier, Royal Military College of Canada*

Let k be a nonnegative integer and define the following restricted double-sum,

$$S_k := \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} (br+as)^k,$$

where a, b are relatively prime positive integers.

a. Show that

$$S_{k-1} = \frac{1}{kb}\left[\sum_{r=0}^{b-1} ((ab+r)^k - a^k r^k) - \sum_{m=2}^{k} \binom{k}{m} b^m S_{k-m} \right]$$

for $k \geq 1$. The convention that $\binom{k}{m} = 0$ if $m > k$ is adopted.

b. Show that

$$S_2 = \frac{ab}{12}[3a^2b^2 + 2a^2b + 2ab^2 - a^2 - b^2 - 9ab + a + b + 2].$$

H-552 *Proposed by Paul S. Bruckman, Highwood, IL*

Given $m \geq 2$, let $\{U_n\}_{n=0}^{\infty}$ denote a sequence of the following form:

$$U_n = \sum_{i=1}^{m} a_i (\theta_i)^n,$$

where the a_i's and θ_i's are constants, with the θ_i's distinct, and the U_n's satisfy the initial conditions: $U_n = 0, n = 0, 1, \ldots, m-2;\ U_{m-1} = 1$.

Part A. Prove the following formula for the U_n's:

$$U_n = \sum_{S(n-m+1, m)} (\theta_1)^{i_1} (\theta_2)^{i_2} \ldots (\theta_m)^{i_m}, \tag{a}$$

where

$$S(N,m) = \{(i_1, i_2, \ldots, i_m) : i_1 + i_2 + \cdots + i_m = N, \ 0 \le i_j \le N, \ j = 1, 2, \ldots, m\}. \tag{b}$$

Part B. Prove the following determinant formula for the U_n's:

$$U_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^2 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^n & (\theta_2)^n & (\theta_3)^n & \cdots & (\theta_m)^n \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^2 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^{m-1} & (\theta_2)^{m-1} & (\theta_3)^{m-1} & \cdots & (\theta_m)^{m-1} \end{vmatrix}.$$

SOLUTIONS

Sum Problem

H-535 *Proposed by Piero Filipponi & Adina Di Porto, Rome, Italy*
(Vol. 35, no. 4, November 1997)

For given positive integers n and m, find a closed form expression for $\sum_{k=1}^{n} k^m F_k$.

Conjecture by the proposers:

$$\Sigma_{m,n} = \sum_{k=1}^{n} k^m F_k = p_1^{(m)}(n) F_{n+1} + p_2^{(m)}(n) F_n + C_m, \tag{1}$$

where $p_1^{(m)}(n)$ and $p_2^{(m)}(n)$ are polynomials in n of degree m,

$$p_1^{(m)}(n) = \sum_{i=0}^{m} (-1)^i a_{m-i}^{(m)} n^{m-i}, \quad p_2^{(m)}(n) = \sum_{i=0}^{m} (-1)^i b_{m-i}^{(m)} n^{m-i}, \tag{2}$$

the coefficients $a_k^{(m)}$ and $b_k^{(m)}$ ($k = 0, 1, \ldots, m$) are positive integers and C_m is an integer.

On the basis of the well-known identity

$$\Sigma_{1,n} = (n-2) F_{n+1} + (n-1) F_n + 2, \tag{3}$$

which is an alternate form of Hoggatt's identity I_{40}, the above quantities can be found recursively by means of the following algorithm:

1. $$p_1^{(m+1)}(n) = (m+1) \int p_1^{(m)}(n)\, dn + (-1)^{m+1} a_0^{(m+1)},$$
   $$p_2^{(m+1)}(n) = (m+1) \int p_2^{(m)}(n)\, dn + (-1)^{m+1} b_0^{(m+1)}.$$

2. $$a_0^{(m+1)} = \sum_{i=1}^{m+1} (a_i^{(m+1)} + b_i^{(m+1)}).$$

3. $$b_0^{(m+1)} = \sum_{i=1}^{m+1} a_i^{(m+1)}.$$

4. $$C_{m+1} = (-1)^m a_0^{(m+1)}.$$

Example: The following results were obtained using the above algorithm.

$$\Sigma_{2,n} = (n^2 - 4n + 8)F_{n+1} + (n^2 - 2n + 5)F_n - 8,$$

$$\Sigma_{3,n} = (n^3 - 6n^2 + 24n - 50)F_{n+1} + (n^3 - 3n^2 + 15n - 31)F_n + 50,$$

$$\Sigma_{4,n} = (n^4 - 8n^3 + 48n^2 - 200n + 416)F_{n+1} + (n^4 - 4n^3 + 30n^2 - 124n + 257)F_n - 416,$$

$$\Sigma_{5,n} = (n^5 - 10n^4 + 80n^3 - 500n^2 + 2080n - 4322)F_{n+1}$$
$$+ (n^5 - 5n^4 + 50n^3 - 310n^2 + 1285n - 2671)F_n + 4322.$$

Remarks:

(i) Obviously, these results can be proved by induction on n.

(ii) It can be noted that, using the same algorithm, $\Sigma_{1,n}$ can be obtained by the identity

$$\Sigma_{0,n} = F_{n+1} + F_n - 1.$$

(iii) It appears that

$$a_k^{(m+k)} / b_k^{(m+k)} = \text{const.} = a_0^{(m)} / b_0^{(m)} \quad (k = 1, 2, \ldots)$$

and

$$\lim_{m \to \infty} a_0^{(m)} / b_0^{(m)} = \alpha.$$

Solution by Paul S. Bruckman, Highwood, IL

We begin by defining certain polynomials of degree n, as follows:

$$F_{m,n}(x) = \sum_{k=1}^{n} k^m x^k . \tag{1}$$

We then see that

$$\Sigma_{m,n} = 5^{-1/2} \{ F_{m,n}(\alpha) - F_{m,n}(\beta) \}. \tag{2}$$

Now let U denote the operator $x d / dx$. It is easily seen that

$$U(F_{m,n}(x)) = F_{m+1,n}(x). \tag{3}$$

Note that

$$F_{0,n}(x) = (x^{n+1} - x) / (x - 1). \tag{4}$$

Repeated application of the recurrence in (3) yields the following:

$$F_{1,n}(x) = \{ n x^{n+2} - (n+1)x^{n+1} + x \} / (x - 1)^2; \tag{5}$$

$$F_{2,n}(x) = \{ n^2 x^{n+3} - (2n^2 + 2n - 1)x^{n+2} + (n+1)^2 x^{n+1} - x^2 - x \} / (x - 1)^3; \tag{6}$$

More generally (by induction or otherwise),

$$F_{m,n}(x) = \{ x^{n+1} P_{m,n}(x) - P_{m,-1}(x) \} / (x - 1)^{m+1}, \quad m > 0, \tag{7}$$

where $P_{m,n}(x)$ is a polynomial in x of degree m. We may suppose

$$P_{m,n}(x) = \sum_{r=0}^{m} A_r(m, n) x^r . \tag{8}$$

Note that $P_{0,n}(x) = 1$, $P_{1,n}(x) = nx - (n+1)$, $P_{2,n}(x) = n^2x^2 - (2n^2 + 2n - 1)x + (n+1)^2$, etc. Repeated application of (3), using (7) and (8), yields the following general formula for $A_r(m, n)$:

$$A_r(m, n) = \sum_{s=r}^{m} (-1)^{s-r} {}_{m+1}C_{s-r}(n - m + s)^m. \tag{9}$$

Substituting the last expression into (8) yields the following development in the "umbral" calculus, involving the finite difference operators E and Δ (with operand z^m at indicated values of z):

$$P_{m,n}(x) = \sum_{r=0}^{m} x^{m-r} \sum_{s=0}^{r} (-1)^s {}_{m+1}C_s (n - r + s)^m = x^m \left\{ \sum_{s=0}^{m} (-E)^s {}_{m+1}C_s \sum_{r=s}^{m} (1/Ex)^r \right\} (z^m) \Big|_{z=n}$$

$$= x^m \left\{ \sum_{s=0}^{m+1} (-E)^s {}_{m+1}C_s ((Ex)^{-m-1} - (Ex)^{-s}) / ((Ex)^{-1} - 1) \right\} (z^m) \Big|_{z=n}$$

$$= \left\{ E^{-m} / (1 - Ex) \cdot (1 - E)^{m+1} - E / (1 - Ex) \cdot (x - 1)^{m+1} \right\} (z^m) \Big|_{z=n}$$

Note that $(E - 1)^{m+1}(z^m) = \Delta^{m+1}(z^m) = 0$. Thus, since $E = 1 + \Delta$,

$$P_{m,n}(x) = -(x - 1)^{m+1} \{ (1 - xE)^{-1} \} (z^m) \Big|_{z=n+1}$$

$$= (x - 1)^m \{ (1 + \Delta x / (x - 1))^{-1} \} (z^m) \Big|_{z=n+1}.$$

In particular,

$$P_{m,n}(\alpha) = \alpha^{-m} \{ (1 + \alpha^2 \Delta)^{-1} \} (z^m) \Big|_{z=n+1} = \alpha^{-m} \sum_{s=0}^{m} (-1)^s \alpha^{2s} \Delta^s (z)^m \Big|_{z=n+1},$$

and likewise,

$$P_{m,n}(\beta) = \beta^{-m} \sum_{s=0}^{m} (-1)^s \beta^{2s} \Delta^s (z^m) \Big|_{z=n+1}.$$

Now, substituting these last results into the formula in (7), we obtain

$$F_{m,n}(\alpha) = \alpha^{m+1} \left\{ \alpha^{n+1-m} \sum_{s=0}^{m} (-1)^s \alpha^{2s} \Delta^s (n + 1)^m - \alpha^{-m} \sum_{s=0}^{m} (-1)^s \alpha^{2s} \Delta^s (0)^m \right\}$$

$$= \sum_{s=0}^{m} (-1)^s \left\{ \alpha^{n+2s+2} \Delta^s (n + 1)^m - \alpha^{2s+1} \Delta^s (0)^m \right\},$$

where, for brevity, we write $\Delta^s(a)^m$ for the more precise expression $\Delta^s(z)^m \big|_{z=a}$. Similarly, we obtain the following expression:

$$F_{m,n}(\beta) = \sum_{s=0}^{m} (-1)^s \left\{ \beta^{n+2s+2} \Delta^s (n + 1)^m - \beta^{2s+1} \Delta^s (0)^m \right\}.$$

We may now substitute these last expressions into (2) and obtain

$$\Sigma_{m,n} = \sum_{s=0}^{m} (-1)^s \left\{ F_{n+2s+2} \Delta^s (n + 1)^m - F_{2s+1} \Delta^s (0)^m \right\}. \tag{10}$$

This is still not in the form that is envisioned by the proposers, but it only takes a bit more effort to put it into such form; we resort once more to the umbral calculus. Returning to our earlier notation, we proceed as follows:

$$\alpha^m P_{m,n}(\alpha) = \sum_{s=0}^{m} (-1)^s \alpha^{2s} \Delta^s (n+1)^m$$

$$= \{(1-(-\alpha^2\Delta)^{m+1})/(1+\alpha^2\Delta)\}(n+1)^m$$

$$= \{1/(1+\alpha^2\Delta)\}(n+1)^m.$$

Then,

$$5^{-1/2}\{\alpha^{m+n+2}P_{m,n}(\alpha) - \beta^{m+n+2}P_{m,n}(\beta)\}$$

$$= 5^{-1/2}\{\alpha^{n+2}/(1+\alpha^2\Delta) - \beta^{n+2}/(1+\beta^2\Delta)\}(n+1)^m$$

$$= \{(F_{n+2} + F_n\Delta)/(1+3\Delta+\Delta^2)\}(n+1)^m$$

$$= \{(F_{n+1} + (1+\Delta)F_n)/(1+3\Delta+\Delta^2)\}(n+1)^m;$$

finally, we may recast (10) as follows:

$$\Sigma_{m,n} = F_{n+1}G(\Delta)(n+1)^m + F_n G(\Delta)(n+2)^m - G(\Delta)(1)^m, \tag{11}$$

where

$$G(\Delta) = 1/(1+3\Delta+\Delta^2). \tag{12}$$

Comparing this with the desired format, we have

$$p_1^{(m)}(n) = G(\Delta)(n+1)^m, \quad p_2^{(m)}(n) = p_1^{(m)}(n+1), \quad C_m = -p_1^{(m)}(0). \tag{13}$$

Therefore, we may express $p_1^{(m)}$ and $p_2^{(m)}$, as well as C_m, in terms of essentially only one polynomial; thus,

$$\Sigma_{m,n} = F_{n+1}p^{(m)}(n) + F_n p^{(m)}(n+1) - p^{(m)}(0), \tag{14}$$

where $p^{(m)}(n) \equiv p_1^{(m)}(n)$, as given in (12) and (13).

This is a somewhat stronger statement than the initial conjecture, since it gives a more specific expression, relating the three quantities $p_1^{(m)}(n)$, $p_2^{(m)}(n)$, and C_m. Now we need to determine the coefficients of these polynomials. Let

$$p^{(m)}(n) = \sum_{i=0}^{m} (-1)^{m-i} a_i^{(m)} n^i;$$

note that $p^{(m)}(0) = (-1)^m a_0^{(m)} = -C_m$, which gives, essentially, Part 4 of the problem. Also, from (13), $p^{(m)}(n) = G(\Delta)(n+1)^m$. Now

$$G(\Delta) = 1/(1+3\Delta+\Delta^2) = 5^{-1/2}\{\alpha^2/(1+\alpha^2\Delta) - \beta^2/(1+\beta^2\Delta)\}$$

$$= \sum_{s=0}^{m}(-1)^s F_{2s+2}\Delta^s = \sum_{s=0}^{m}(-1)^s F_{2s+2}(E-1)^s = \sum_{s=0}^{m}(-1)^s F_{2s+2}\sum_{r=0}^{s}(-1)^r {}_sC_r E^{s-r};$$

therefore,

$$p^{(m)}(n) = \sum_{s=0}^{m}(-1)^s F_{2s+2}\sum_{r=0}^{s}(-1)^r {}_sC_r (n+1+s-r)^m$$

$$= \sum_{s=0}^{m} (-1)^s F_{2s+2} \sum_{r=0}^{s} (-1)^r \, {}_sC_r \sum_{i=0}^{m} mC_i n^i (s+1-r)^{m-i}$$

$$= \sum_{i=0}^{m} n^i \, {}_mC_i \sum_{s=0}^{m} (-1)^s F_{2s+2} \sum_{r=0}^{s} (-1)^r \, {}_sC_r \, E^{s-r} (1)^{m-i}$$

$$= \sum_{i=0}^{m} n^i \, {}_mC_i \sum_{s=0}^{m} (-1)^s F_{2s+2} \Delta^s (1)^{m-i} = \sum_{i=0}^{m} n^i \, {}_mC_i \, G(\Delta)(1)^{m-i}.$$

Comparison of coefficients in the two expressions for $p^{(m)}(n)$ yields

$$a_i^{(m)} = (-1)^{m-i} \, {}_mC_i \, G(\Delta)(1)^{m-i}. \tag{15}$$

In similar fashion, we may obtain the following expression:

$$b_i^{(m)} = (-1)^{m-i} \, {}_mC_i \, G(\Delta)(2)^{m-i}. \tag{16}$$

We then see from (15) and (16) that $a_i^{(m+i)} / b_i^{(m+i)} = G(\Delta)(1)^m / G(\Delta)(2)^m$. However, we see that such result is independent of i, so we express such ratio as $\rho(m)$. Therefore, we have

$$a_i^{(m+i)} / b_i^{(m+i)} = a_0^{(m)} / b_0^{(m)} = \rho(m). \tag{17}$$

We now return to the problem of determining $\rho = \lim_{m \to \infty} \rho(m)$. First, however, we deduce some additional properties of the coefficients $a_i^{(m)}$ and $b_i^{(m)}$. From the expressions in (15) and (16), it readily follows that

$$a_i^{(m)} = m / i \, a_{i-1}^{(m-1)}, \ b_i^{(m)} = m / i \, b_{i-1}^{(m-1)}, \ i = 1, 2, \ldots, m. \tag{18}$$

Then

$$p^{(m)}(n) = \sum_{i=0}^{m} (-1)^{m-i} a_i^{(m)} n^i = (-1)^m a_0^{(m)} + \sum_{i=1}^{m} (-1)^{m-i} m / i \, a_{i-1}^{(m-1)} n^i$$

$$= (-1)^m a_0^{(m)} + m \sum_{i=0}^{m-1} (-1)^{m-1-i} a_i^{(m-1)} (n^{i+1} / i + 1).$$

We see then that

$$p^{(m)}(n) = (-1)^m a_o^{(m)} + m \int_0^n p^{(m-1)}(t) \, dt, \tag{19}$$

which is essentially Part 1 of the problem (stated somewhat more precisely).

We now return to the expression in (13), namely,

$$p^{(m)}(n) = G(\Delta)(n+1)^m. \tag{20}$$

Note that we may allow $n = 0$ and $n = -1$ in this last expression, but that if $n \leq -2$, extraneous and unintended terms arise that make the expression incorrect.

Note that $p^{(m)}(n+1) = (1+\Delta)p^{(m)}(n)$, while $p^{(m)}(n+2) = (1+\Delta)^2 p^{(m)}(n)$; then $p^{(m)}(n+2) + p^{(m)}(n+1) = \{1 + \Delta + 1 + 2\Delta + \Delta^2\} p^{(m)}(n) = \{1 + 1 + 3\Delta + \Delta^2\} G(\Delta)(n+1)^m = \{1 + G(\Delta)\}(n+1)^m$, or

$$p^{(m)}(n+2) + p^{(m)}(n+1) = (n+1)^m + p^{(m)}(n). \tag{21}$$

In particular, setting $n = -1$ (and assuming $m > 0$),

$$p^{(m)}(1) + p^{(m)}(0) = p^{(m)}(-1). \tag{22}$$

Now observe that

$$p^{(m)}(-1) = (-1)^m \sum_{i=0}^{m} a_i^{(m)},$$

$$p^{(m)}(0) = (-1)^m a_0^{(m)} = p_2^{(m)}(-1) = (-1)^m \sum_{i=0}^{m} b_i^{(m)},$$

and

$$p^{(m)}(1) = p_2^{(m)}(0) = (-1)^m b_0^{(m)}.$$

From (22), we then deduce the following:

$$a_0^{(m)} + b_0^{(m)} = \sum_{i=0}^{m} a_i^{(m)} \quad \text{or} \quad b_0^{(m)} = \sum_{i=1}^{m} a_i^{(m)}, \quad \text{valid for } m > 0;$$

this is essentially Part 3 of the problem. Also

$$a_0^{(m)} = b_0^{(m)} + \sum_{i=1}^{m} b_i^{(m)} = \sum_{i=1}^{m} (a_i^{(m)} + b_i^{(m)}), \quad \text{valid for } m > 0;$$

this is essentially Part 2 of the problem.

That the $a_i^{(m)}$ and $b_i^{(m)}$ are integers is obvious from the formulas in (15) and (16). That they are positive requires more effort. An easy induction on (18) yields the following relations:

$$a_i^{(m)} = {}_m C_i a_0^{(m-i)}, \quad b_i^{(m)} = {}_m C_i b_0^{(m-i)}, \quad i = 0, 1, \dots, m. \tag{23}$$

Thus, $a_i^{(m)}(b_i^{(m)}) > 0$ if and only if $a_0^{(m)}(b_0^{(m)}) > 0$, $m = 0, 1, \dots$.

From (23) and Parts 2 and 3 of the problem, we see that, if $a_0^{(m)} > 0$, $m = 0, 1, 2, \dots$, then $a_i^{(m)} > 0 \Rightarrow b_0^{(m)} > 0 \Rightarrow b_i^{(m)} > 0$ ($i = 0, 1, \dots, m$). Thus, it suffices to prove that $a_0^{(m)} > 0$, $m = 0, 1, \dots$.

Our proof of this assertion is by induction (on m). The inductive step depends on the following recursive formula:

$$a_0^{(m+1)} = \sum_{i=0}^{[m/2]} a_{2i}^{(m)} / (2i+1), \quad m = 0, 1, 2, \dots. \tag{24}$$

If (24) is valid, our inductive hypothesis is that $a_0^{(m)} > 0$ for some $m \geq 0$. From our foregoing discussion, this implies that $a_{2i}^{(m)} > 0$, $i = 0, 1, \dots, [m/2]$. Then (24) implies that $a_0^{(m+1)} > 0$. Since $a_0^{(0)} = 1$, this proves the hypothesis. Our task is thus reduced to proving (24).

We return to the results in (19) and (22). Expressing the integral recurrence in terms of the coefficients (and replacing m by $m+1$), we obtain

$$(-1)^m a_0^{(m+1)} = (m+1) \int_0^1 \sum_{i=0}^{m} (-1)^{m-i} \{1 + (-1)^i\} a_i^{(m)} t^i \, dt;$$

then

$$a_0^{(m+1)} = (m+1) \sum_{i=0}^{m} \{1 + (-1)^i\} a_i^{(m)} / (i+1),$$

which is equivalent to (24). $\square$

Note: As noted, we allowed the values $n = 0$ and $n = -1$ although, in the original statement of the problem, n was required to be positive. It may be observed that if we substitute the values $n = 0$ or $n = -1$ in the formula given by (14), we find that the expression for $\Sigma_{m,n}$ dutifully vanishes, as we should expect from its original definition.

The only remaining task is to establish that $\rho = \alpha$, as conjectured by the proposers.

We write a_m for $a_0^{(m)}$ and b_m for $b_0^{(m)}$. From (18), we easily deduce the following recursive formulas:

$$a_m = \sum_{i=0}^{m-1} {}_mC_i(a_i + b_i), \quad b_m = \sum_{i=0}^{m-1} {}_mC_i\,a_i, \quad m = 1, 2, \ldots. \tag{25}$$

We may also express a_m and b_m in the umbral calculus as follows:

$$a_m = (-1)^m G(\Delta)(1)^m, \quad b_m = (-1)^{m+1}\{\Delta G(\Delta)\}(0)^m. \tag{26}$$

It is apparent that a_m and b_m are unbounded (as $m \to \infty$). Also note that $0 < b_m < a_m$ for all $m \geq 1$. Then (25) implies that $1 < \rho(m) \leq 2$ for all $m \geq 1$. From the expressions for a_m and b_m in (26), it is not difficult to show (by expansion of the operators) that ρ exists.

We may express (25) in the following form:

$$a_m = \sum_{i=0}^{m-1} {}_mC_i(\rho(i) + 1)b_i, \quad b_m = \sum_{i=0}^{m-1} {}_mC_i\,\rho(i)b_i, \quad m = 1, 2, \ldots. \tag{27}$$

Hence, $\rho(m) = 1 + 1/\rho(m-1)$, where $\rho(m-1)$ is a "weighted average" ρ, and the "weights" are the quantities ${}_mC_i\,b_i$ in the sums. Letting $m \to \infty$, we may therefore deduce that $1 \leq \rho \leq 2$ and $\rho = 1 + \rho^{-1}$ This, in turn, implies that $\rho = \alpha$.

For additional confirmation, the simple continued fraction (s.c.f.) expansion for $\rho(m)$ approaches the infinite s.c.f. $[1, 1, 1, \ldots]$, which is known to be the s.c.f. for α. $\square$

Also solved by I. Strazdins.

❖ ❖ ❖

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THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965. $18.00

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. $23.00

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

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