GENERALIZED BASES FOR THE REAL NUMBERS

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Throughout this paper $\{r_i\}_1^\infty$ will denote a non-increasing real number sequence with limit zero; each of $\{k_i\}_1^\infty$ and $\{m_i\}_1^\infty$ denotes a non-negative integer sequence

$$S = \sum_1^\infty k_i r_i \quad \text{and} \quad S^* = \sum_1^\infty m_i r_i$$

(finite or infinite). We shall consider the possibility of expressing each number x in the interval $(-S^*, S)$ in the form

$$x = \sum_1^\infty a_i r_i$$

where each a_i is an integer satisfying $-m_i \le a_i \le k_i$.

In the classical n-scale number representation, each x in $[0,1]$ can be expressed in the above form, where $n > 1$, and $r_i = n^{-i}$, $k_i = n - 1$, and $m_i = 0$ for each i. Previous generalizations ([6] and [8]) have considered only the expansion of <u>positive</u> numbers with certain restrictions on the coefficient bounds $\{k_i\}_1^\infty$.

In this note we shall extend the previous work to include negative number representations as well as relaxing the restrictions on the coefficients $\{a_i\}_1^\infty$. We shall also consider the question of uniqueness of such representations and the expansion of real numbers using a base sequence $\{\pm r_i\}_1^\infty$ of both positive and negative terms.

DEFINITION. The sequence $\{r_i\}_1^\infty$ is a $\{k, m\}$-base for the interval $(-S^*, S)$ if for each x in $(-S^*, S)$ there is an integer sequence $\{a_i\}_1^\infty$ such that

(1) $\qquad x = \sum_1^\infty a_i r_i$, and $-m_i \le a_i \le k_i$ for each i.

Our main purpose is to develop an explicit characterization of a $\{k, m\}$-base; to this end we first consider the case where $m_i = 0$ for each i; i.e., a $\{k, 0\}$-base.

LEMMA. The sequence $\{r_i\}_1^\infty$ is a $\{k, 0\}$-base for the interval $(0, S)$ if and only if

$$(2) \qquad r_n \leq \sum_{n+1}^{\infty} k_i r_i \qquad \text{for each } n.$$

Proof. If (2) does not hold and

$$r_n > x > \sum_{n+1}^{\infty} k_i r_i \quad,$$

for some n, it is easily seen that x cannot be expressed in the form (1).

Assume that (2) holds and let x be in $(0, S)$, the conclusion being trivial for $x = 0$. Let $i(1)$ be the least positive integer such that $r_{i(1)} \leq x$, and choose $a_{i(1)}$ to be the greatest integer such that $a_{i(1)} \leq k_{i(1)}$ and $a_{i(1)} r_{i(1)} \leq x$.

If $a_{i(1)} r_{i(1)} < x$, we continue inductively:
Let $i(n)$ be the least positive integer such that

$$(3) \qquad r_{i(n)} \leq x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \qquad \text{and} \qquad i(n) > i(n-1) \quad;$$

Choose $a_{i(n)}$ to be the greatest integer such that $a_{i(n)} \leq k_{i(n)}$ and

$$(4) \qquad a_{i(n)} r_{i(n)} \leq x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \quad.$$

In case equality does not hold in (4) for any n, we assert that

$$(5) \qquad \sum_{p=1}^{\infty} a_{i(p)} r_{i(p)} = x \ .$$

Suppose, to the contrary, that for some positive ϵ

$$\sum_{p=1}^{n} a_{i(p)} r_{i(p)} \leq x - \epsilon \ , \quad \text{for each} \ n \ .$$

If $r_{i(n)} < \epsilon$ it follows that

$$(6) \qquad (a_{i(n)} + 1) r_{i(n)} \leq x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \ .$$

By the choice of $a_{i(n)}$ this implies that $a_{i(n)} = k_{i(n)}$; furthermore, (6) also yields

$$r_{i(n)+1} \leq r_{i(n)} \leq x - \sum_{p=1}^{n} a_{i(p)} r_{i(p)} \ ,$$

so that $i(n + 1) = i(n) + 1$. Hence,

$$(7) \qquad \sum_{p=1}^{\infty} a_{i(p)} r_{i(p)} = \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} + \sum_{p=i(n)}^{\infty} k_p r_p \leq x \ .$$

Applying (2) to (7) we see that

$$(8) \qquad r_{i(n)-1} \leq x - \sum_{p=1}^{n-1} a_{i(p)} r_{i(p)} \ .$$

By the choice of $i(n)$, (8) implies that $i(n) - 1 = i(n - 1)$, so that (8) can be written as

$$(a_{i(n-1)} + 1)r_{i(n-1)} \leq x - \sum_{p=1}^{n-2} a_{i(p)}r_{i(p)} \quad,$$

whence $a_{i(n-1)} = k_{i(n-1)}$. Thus it is readily seen that for every n, $i(n) = n$ and $a_{i(n)} = k_n$, which contradicts $x < S$; this establishes (5) and completes the proof.

REMARK. From this Lemma the following is clear:

If $\{r_i\}_1^\infty$ is a $\{k, 0\}$-base for $[0, S)$ and N is a positive integer, then $\{r_i\}_N^\infty$ is a $\{k, 0\}_N^\infty$-base for the interval

$$\left[0, \sum_N^\infty k_i r_i \right) \quad .$$

Theorem 1. The sequence $\{r_i\}_1^\infty$ is a $\{k, m\}$-base for $(-S^*, S)$ if and only if

(9) $$r_n \leq \sum_{i=n+1}^\infty (k_i + m_i)r_i \quad \text{for each n .}$$

Proof. If (9) does not hold and

$$r_n > x > \sum_{n+1}^\infty (k_i + m_i)r_i \quad,$$

it follows easily from the Lemma that $x - S^*$ is in $(-S^*, S)$ but $x - S^*$ cannot be expressed as in (1).

To show the sufficiency of (9) we first consider the case where S^* is finite. Let x be in $(-S^*, S)$. By the Lemma, (9) guarantees a sequence $\{a_i\}_1^\infty$

such that

$$x + S^* = \sum_{1}^{\infty} a_i r_i , \quad \text{and} \quad 0 \leq a_i \leq k_i + m_i \quad \text{for each } i .$$

Letting $b_i = a_i - m_i$, we have

$$x = \sum_{1}^{\infty} b_i r_i , \quad \text{and} \quad -m_i \leq b_i \leq k_i \quad \text{for each } i .$$

The case in which S is finite is proved similarly. If both S^* and S are infinite it follows immediately from the Lemma that every non-negative x can be expressed as

$$\sum_{1}^{\infty} a_i r_i ,$$

where $0 \leq a_i \leq k_i$, and every negative x can be so expressed with $-m_i \leq a_i \leq 0$.

We now wish to establish conditions under which the representations in the form (1) are unique. Since the common decimal expansion is not unique, and this is the special case where $r_i = 10^{-i}$, $m_i = 0$, and $k_i = 9$, we cannot hope for total uniqueness in any non-trivial case. Therefore we adopt a convention similar to that used in identifying the decimal $.0999\cdots$ with $.1000\cdots$, viz., we disallow a representation in which $a_i = k_i$ for every i greater than some n. Note that in the proof of the Lemma such representations were not necessary. (This is also the reason that we did not consider the closed interval $[0, S]$ even when S was finite.

Theorem 2. The sequence $\{r_i\}_1^{\infty}$ yields exactly one $\{k, m\}$-base representation of each x in $(-S^*, S)$ if and only if

$$(10) \qquad\qquad r_n = \sum_{i=n+1}^{\infty} (k_i + m_i) r_i \quad \text{for each } n .$$

Proof. The sufficiency of (10) is fairly straightforward. Conversely, it is easily seen that for unique representation it is necessary that S^* (and S) be finite. Suppose that S^* is finite and $\{r_i\}_1^\infty$ satisfies (9) but not (10). Then there exists an integer n and a number x such that

$$ r_n < x < \sum_{n+1}^\infty (k_i + m_i) r_i \ . $$

Using the construction in the proof of the Lemma, we get a sequence $\{a_i\}_1^\infty$ satisfying

$$ x = \sum_1^\infty a_i r_i \ , $$

and $0 \le a_i \le k_i + m_i$; moreover, since $r_n < x$, at least one of $a_1, \cdots, a_n$ is non-zero. Taking $b_i = a_i - m_i$, we have

$$ (11) \qquad x - S^* = \sum_1^\infty b_i r_i \ , \qquad \text{where } -m_i \le b_i \le k_i \ , $$

and for some $i \le n$, $b_i \ne -m_i$.

On the other hand $\{r_i\}_{n+1}^\infty$ is a $\{k + m, 0\}_{n+1}^\infty$-base for the interval

$$ \left[0, \sum_{n+1}^\infty (k_i + m_i) r_i \right) \ , $$

by the Remark following the Lemma. This yields a second $\{k, m\}$-base representation: $x - s = \sum_1 d_i r_i$, where $d_i = -m_i$ for <u>all</u> $i \le n$.

COROLLARY. The sequence $\{r_i\}_1^\infty$ yields a unique $\{k, m\}$-base representation of each x in $(-S^*, S)$ if and only if

$$r_n = (S + S^*)\Big/ \Pi_{i=1}^{n} (1 + k_i + m_i) \qquad \text{for each } n.$$

Proof. This is straightforward induction using Theorem 2.

The foregoing theory can be used to consider representations of real numbers in which the base sequence $\{r_i\}_1^\infty$ takes on both positive and negative values. Let A and B be disjoint sets whose union is the set of positive integers, and let C_A and C_B denote their respective characteristic functions. We shall use

$$\left\{ (-i)^{C_B(i)} r_i \right\}_1^\infty$$

as the base sequence.

Theorem 3. If $\{q_i\}_1^\infty$ is a positive integer sequence, then

$$\left\{ (-1)^{C_B(i)} r_i \right\}_1^\infty$$

is a $\{q, 0\}$-base for the interval

$$\left(-\sum_{i \in B} q_i r_i , \sum_{i \in A} q_i r_i \right)$$

if and only if

$$(12) \qquad r_n \le \sum_{n+1}^{\infty} q_i r_i \qquad \text{for each } n.$$

Proof. Let $k_i = C_A(i) q_i$ and $m_i = C_B(i) q_i$, so that $k_i + m_i = q_i$, $\Sigma_{i \in A} q_i r_i = S$, and $\Sigma_{i \in B} q_i r_i = S^*$. Thus by Theorem 1, (12) is equivalent to $\{r_i\}_1^\infty$ being a $\{k, m\}$-base for $(-S^*, S)$. If (12) holds and x is in $(-S^*, S)$, then

$$x = \sum_{1}^{\infty} b_i r_i , \quad \text{where} \quad -C_B(i) q_i \le b_i \le C_A(i) q_i .$$

Taking $a_i = (-1)^{C_B(i)} b_i$, we have

$$(13) \qquad \sum_1^\infty a_i \left[(-1)^{C_B(i)} r_i \right] \quad \text{and} \quad 0 \le a_i \le q_i \ .$$

The converse is proved similarly.

REMARK. It is clear that the representations in (13) are unique if and only if equality holds in (12) for each n.

A related problem is that of expressing a given number x in the form

$$(14) \qquad x = \sum_1^\infty \epsilon_i r_i \ , \quad \text{where} \quad \epsilon_i = 1 \text{ or } -1 \ .$$

The following solution is proved using Theorem 1.

PROPOSITION. If

$$r_n \le \sum_{n+1}^\infty r_i \quad \text{for each } n, \text{ and } |x| \le \sum_1^\infty r_i \ ,$$

then x can be expressed in the form (14).

The special case of Theorem 1 in which $k_i = 1$ and $m_i = 0$, for all i, is apparently an old result first proved by Kakeya [7] (cf. [2]). Generalizations of the n-scale (radix n) representation of positive integers which are analogous to the theory presented here have been developed by Alder [1] and Brown [3-5].

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ON A PARTIAL DIFFERENCE EQUATION OF L. CARLITZ

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SUMMARY

Eine von L. CARLITZ behandelte partielle Differenzengleichung zweiter Ordnung, die mit den FIBONACCI-Zahlen in Beziehung steht, wird mit Hilfe einer algebraisch begründeten, zweidimensionalen Operatorenrechnung gelöst. Die sich hierbei ergebende Lösung ist allgemeiner als diejenige von L. CARLITZ.

In an article [1] by L. Carlitz, a solution of the equation

$$(1) \qquad u_{mn} - u_{m-1,n} - u_{m,n-1} - u_{m-2,n} + 3u_{m-1,n-1} - u_{m,n-2} = 0$$

$$(m,n \geq 2, \text{ integral})$$

was given with the aid of a power series expansion related to the Fibonacci numbers. Although the solution contains only three arbitrary constants (viz., u_{00}, u_{01}, and u_{10}), it is called a "general solution" — a terminology which appears justified only if, besides equation (1), the following secondary conditions, not mentioned in [1], are also imposed:

$$(2) \qquad u_{11} - u_{01} - u_{10} + 3u_{00} = 0 \quad,$$

$$(3) \qquad u_{0n} - u_{0,n-1} - u_{0,n-2} = 0 \quad \text{for } n \geq 2 \,,$$

$$(4) \qquad u_{m0} - u_{m-1,0} - u_{m-2,0} = 0 \quad \text{for } m \geq 2 \,,$$

$$(5) \qquad u_{1n} - u_{0n} - u_{1,n-1} + 3u_{0,n-1} - u_{1,n-2} = 0 \quad \text{for } n \geq 2 \,,$$

$$(6) \qquad u_{m1} - u_{m-1,1} - u_{m0} - u_{m-2,1} + 3u_{m-1,0} = 0 \quad \text{for } m \geq 2 \,.$$

The conclusion (1.4) from [1] is valid only under the assumptions (2) to (6). From (2), u_{11} is fixed, and from (3) to (6) the initial values u_{0n}, u_{1n} and u_{m0}, u_{m1} are uniquely determined for $n,m \geq 2$. The general solution of (3), for instance, is

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(7) $u_{0n} = u_{01} F_n + u_{00} F_{n-1}$ for $n \geq 0$,

where

(8) $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ with $\alpha = \dfrac{1}{2}(1 + \sqrt{5})$, $\beta = \dfrac{1}{2}(1 - \sqrt{5})$ (n integral).

One can easily verify that the solution (5.4) given by L. Carlitz in [1] reduces to equation (7) for $m = 0$.

The general solution of (1) without secondary conditions thus contains two pairs of arbitrary functions of only one of the two integral variables m and n. We now wish to determine this solution with the aid of the "two-dimensional, discrete operational calculus" developed in [2].

According to the fundamental idea of J. Mikusiński (see perhaps [3]), the usual addition and the two dimensional Cauchy product

(9) $a_{mn} \cdot b_{mn} = \displaystyle\sum_{\mu,\nu=0}^{m,n} a_{\mu\nu} b_{m-\mu,n-\nu}$ as multiplication

are introduced in the set of complex-valued functions of two nonnegative integral variables, and the quotient field Q_2 belonging to the integral domain D_2 arising from this means is considered. In order to conform with the relations in [2], we make an index shift in (1) and write

(1') $D(u_{mn}) = u_{m+2,n+2} - u_{m+1,n+2} - u_{m+2,n+1} - u_{m,n+2} + 3 u_{m+1,n+1}$

$$- u_{m+2,n} = 0 \qquad\qquad (m,n \geq 0) .$$

After application of the difference theorem from [2],

$$u_{m+k,n+\ell} = p^k q^\ell u_{mn} - q^\ell \sum_{\kappa=0}^{k-1} p^{k-\kappa} u_{\kappa n} - p^k \sum_{\lambda=0}^{\ell-1} q^{\ell-\lambda} u_{m\lambda} +$$

$$\sum_{\kappa=0}^{k-1} \sum_{\lambda=0}^{\ell-1} p^{k-\kappa} q^{\ell-\lambda} u_{\kappa\lambda}$$

($u_{mn} \in D_2$; $u_{m\lambda}$, $u_{\kappa n}$ initial values; k, ℓ natural numbers; p, q inverses to shift operators in Q_2), one obtains the operator representation

$$(10) \qquad u = \frac{h(p,q,m,n)}{g(p,q)} \quad ,$$

where the numerator is

$$h = \alpha_n h_1 + \gamma_m h_2 + \beta_n h_3 + \delta_m h_4 + \alpha_0 h_5 + \alpha_1 h_6 + \beta_0 h_7 + \beta_1 h_8 \quad ,$$

as one can easily verify with the polynomials

$$h_1(p,q) = p^2 q^2 - pq^2 - p^2 q + 3pq - p^2 \quad , \quad h_2 = h_1(q,p) \quad ,$$
$$h_3(p,q) = pq^2 - pq - p, \quad h_4 = h_3(q,p), \quad h_5 = -p^2 q^2 + pq^2 + p^2 q - 3pq,$$
$$h_6(p,q) = -p^2 q + pq \,, \quad h_7 = h_6(q,p) \,, \quad h_8 = -pq$$

and the coefficients, the given initial values,

$$(11) \qquad \alpha_n = u_{0n} \,, \; \gamma_m = u_{m0} \,, \; \beta_n = u_{1n} \,, \; \delta_m = u_{m1} \qquad \text{with}$$
$$\alpha_0 = \gamma_0 \,, \; \beta_0 = \gamma_1, \; \alpha_1 = \delta_0 \,, \; \beta_1 = \delta_1 \quad .$$

The denominator, a polynomial of degree 4 in p,q is

$$g(p,q) = p^2 q^2 - pq^2 - p^2 q + 3pq - p^2 - q^2 = g_1(p,q)\, g_2(p,q)$$

with

$$g_1 = pq - \alpha p - \beta q \quad \text{and} \quad g_2 = pq - \beta p - \alpha q \quad ,$$

where α and β have the values given in (8). As can be immediately proved,

$$\frac{h_i}{g(p,q)} \; \epsilon\, D_2$$

holds for $i = 1, \cdots, 8$; and these terms are indeed functions of the Fibonacci numbers F_k. If considerations for the operator

$$\frac{p^2 q^2}{g(p,q)}$$

are indicated, the calculation for the remaining members of u_{mn} then follows easily. If one conceives

$$\frac{pq^2}{g(p,q)}$$

as a (proper fractional) rational operator of p alone, then there results, by decomposition into partial fractions,

$$\frac{p^2q^2}{g(p,q)} = \frac{1}{\alpha - \beta}\left(\frac{\alpha pq}{g_2(p,q)} - \frac{\beta pq}{g_1(p,q)}\right)\frac{q}{q - 1} \quad ,$$

and on account of the obvious relations

$$\frac{pq}{g_2(p,q)} = \binom{m+n}{m}\alpha^m\beta^n \quad , \qquad \frac{pq}{g_1(p,q)} = \binom{m+n}{m}\alpha^n\beta^m$$

and of the meaning of $q/(q - 1)$ as a "partial summation operator"

$$\frac{q}{q - 1}b_{mn} = \sum_{\nu=0}^{n}b_{m\nu} \quad ,$$

it follows that

$$\frac{p^2q^2}{g(p,q)} = \frac{1}{\alpha - \beta}\sum_{k=0}^{n}\binom{m+k}{m}\left[\alpha^{m+1}\beta^k - \alpha^k\beta^{m+1}\right]$$

from which, on account of $\alpha^k\beta^k = (-1)^k$ (k integral), of definition (8), of the symmetry of $g(p,q)$, and with the notation G_{mn} for $(p^2q^2)/g$, there finally results

$$(12) \quad \frac{p^2q^2}{g(p,q)} = G_{mn} = G_{nm} = \sum_{k=0}^{n}(-1)^k\binom{m+k}{k}F_{m+1-k} =$$

$$\sum_{k=0}^{m}(-1)^k\binom{n+k}{k}F_{n+1-k} \quad \text{for } m,n \geq 0.$$

With the aid of (12) the operators h_i/g $(i = 1,\cdots,8)$ can now be immediately

represented as functions from D_2. In order to simplify the notation, we define $G_{mn} = 0$ in case an index is negative; according to (12) this is also achieved by stipulating the following:

$$(13) \qquad \sum_{k=0}^{\ell} a_k = 0 \;, \quad \binom{\ell}{k} = 0 \qquad \text{for } \ell < 0 \;.$$

(With this agreement, $(1/p^2)\, G_{mn} = G_{m-2,n}$, for example, holds for all $m,n \geq 0$.)

Therewith we obtain, after easy calculation from (10),

$$(14) \quad u_{mn} = \alpha_n \cdot (1 + G_{m-2,n}) + \gamma_m \cdot (1 + G_{m,n-2})$$
$$+ \beta_n \cdot (G_{m-1,n} - G_{m-1,n-1} - G_{m-1,n-2})$$
$$+ \delta_m \cdot (G_{m,n-1} - G_{m-1,n-1} - G_{m-2,n-1}) - \alpha_0 G_{mn}$$
$$+ (\alpha_0 - \beta_0) G_{m-1,n} + (\alpha_0 - \alpha_1) G_{m,n-1} - (3\alpha_0 - \beta_0 - \alpha_1 + \beta_1) G_{m-1,n-1}$$
$$\text{for all } m,n \geq 0 .$$

(In this the multiplication symbol means multiplication in D_2 and the summand 1 is the identity element of D_2.) If we finally use

$$G_{m-1,n} - G_{m-1,n-1} = (-1)^n \binom{m+n-1}{n} F_{m-n}$$

and correspondingly

$$G_{m,n-1} - G_{m-1,n-1} = (-1)^m \binom{m+n-1}{m} F_{n-m} \quad \text{for } m,n \geq 0 \;,$$

and carry out the multiplication in D_2 then after simple transformations for $m,n \geq 0$ we obtain from (14)

$$(15) \quad u_{mn} = \alpha_n + \gamma_m + \sum_{\nu=0}^{n} G_{m-2,\nu}\,\alpha_{n-\nu} + \sum_{\mu=0}^{m} G_{\mu,n-2}\gamma_{m-\mu}$$

$$+ \sum_{\nu=0}^{n} (-1)^{\nu} \binom{m+\nu-1}{\nu} F_{m-\nu}\beta_{n-\nu} - \sum_{\nu=0}^{m} G_{m-1,\nu-2}\beta_{n-\nu}$$

$$+ \sum_{\mu=0}^{m} (-1)^{\mu} \binom{n+\mu-1}{\mu} F_{n-\mu}\delta_{m-\mu} - \sum_{\mu=0}^{m} G_{\mu-2,n-1}\delta_{m-\mu}$$

$$- \alpha_0(-1)^m \binom{m+n}{m} F_{n-m+1} - \beta_0(-1)^n \binom{m+n-1}{n} F_{m-n}$$

$$+ (\alpha_0 - \alpha_1)(-1)^m \binom{m+n-1}{m} F_{n-m} - (2\alpha_0 + \beta_1) G_{m-1,n-1} \qquad .$$

We verify that (15) satisfies equation (1'), however, we only indicate the calculation: to begin with, $D(\alpha_n) = -\alpha_{n+2}$ holds and $D(\gamma_m) = -\gamma_{m+2}$.

Furthermore,

$$D\left(\sum_{\nu=0}^{n} G_{m-2,\nu}\alpha_{n-\nu}\right) = \sum_{\nu=2}^{n+2} \alpha_{n+2-\nu}D(G_{m-2,\nu-2}) + n - \alpha_{n+2}[G_{m0} - G_{m-1,0} - G_{m-2,0}]$$

$$+ \alpha_{n+1}[G_{m1} - G_{m-1,1} - G_{m0} - G_{m-2,1} + 3G_{m-1,0}]$$

$$= \alpha_{n+2} ;$$

for, it is true that $G_{m0} - G_{m-1,0} - G_{m-2,0} = \begin{cases} 1 & \text{for } m = 0 , \\ 0 & \text{for } m > 0 , \end{cases}$

$$G_{m1} - G_{m-1,1} - G_{m0} - G_{m-2,1} + 3G_{m-1,0} = 0 \quad \text{for all } m \geq 0$$

and

$$D(G_{m-2,\nu-2}) = 0 \quad \text{for } m \geq 0 , \quad \nu \geq 2 ,$$

as one recognizes after some calculation with the aid of (12) and $F_k = (-1)^{k+1}F_{-k}$ (k integral) or as one can read off directly from the fact that G_{mn} in D_2 is inverse to

$$\frac{g}{p^2q^2} = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{q^2} + \frac{3}{pq} - \frac{1}{p^2} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & \cdots \\ -1 & 3 & 0 & 0 & \cdots \\ -1 & 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdots \end{pmatrix}$$

by (9).

Analogously one completes the verification. By appropriate calculation one recognizes that the initial conditions (11) are satisfied by (15). Since $\delta_0 = \alpha_1$, and because of definition (13) and of the validity of the relation (3) for F_n, there results for $m = 0$, $n \geq 0$, for instance,

$$u_{0n} = \alpha_n + {}_0 + G_{0,n-2} {}_0 + \binom{n-1}{0} F_n {}_0 - \alpha_0 F_{n+1} + (\alpha_0 - \alpha_1)\binom{n-1}{0}F_n = \alpha_n .$$

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RECURRING SEQUENCES

Review of Book by Dov Jarden
By Brother Alfred Brousseau

For some time the volume, Recurring Sequences, by Dov Jarden has been unavailable, but now a printing has been made of a revised version. The new book contains articles published by the author on Fibonacci numbers and related matters in Riveon Lematematika and other publications. A number of these articles were originally in Hebrew and hence unavailable to the general reading public. This volume now enables the reader to become acquainted with this extensive material (some thirty articles) in convenient form.

In addition, there is a list of Fibonacci and Lucas numbers as well as their known factorizations up to the 385th number in each case. Many new results in this section are the work of John Brillhart of the University of San Francisco and the University of California.

There is likewise, a Fibonacci bibliography which has been extended to include articles to the year 1962.

This valuable reference for Fibonacci fanciers is now available through the Fibonacci Association for the price of $6.00. All requests for the volume should be sent to Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif. , 94575.

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The Fibonacci Association invites Educational Institutions to apply for Academic Membership in the Association. The minimum subscription fee is $25 annually. (Academic Members will receive two copies of each issue and will have their names listed in the Journal.

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ENUMERATION OF PARTITIONS SUBJECT TO LIMITATIONS ON SIZE OF MEMBERS

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1. INTRODUCTION

In a previous work [1], it was shown that 'the partition enumeration* $P(n|p| \le n + p - 1)$ is given by

$$(1) \qquad P(n|p| \le n + p - 1) = \left[\frac{n - p + 2}{2}\right] + \sum_{i=1} \left[\frac{n - p + 2 - w_i}{2}\right] \qquad p \ne 1,$$

$$(1a) \qquad\qquad P(n|p| \le n + p - 1) = 1 \qquad\qquad p = 1.$$

The w_i are the sums of each partition in the set of partitions described by $PV(\ge 3, \le n - p| \ge 1, \le [(n-p)/3] \ge 3, \le p)$. It was stated in [1] that the summation term of (1) is zero for those values of p and $(n - p)$ for which $PV(\ge 3, \le n-p| \ge 1, \le [(n-p)/3] \ge 3, \le p)$ does not exist. (See the footnote below for a brief description of nomenclature.) For $n - p < 3$ and/or $p = 2$, $w_i = 0$. One raison d'être for (1) is the adaptability of w_i to digital computation.

$P(n|p| \le n - p + 1)$ is a basic enumeration form which is extremely useful in evaluating more restrictive enumerations [2]. $PV(n|p| \le n - p + 1)$ shares this versatility in that sets of many other partition types can be constructed by operations on the members of the partitions of the basic set. When $PV(n|p| \le n - p + 1)$ is under consideration, it is convenient to arrange the p members of a partition so that

$$(2) \qquad\qquad a_p \le a_{p-1} \le a_{p-2} \le \cdots \le a_2 \le a_1 \quad ,$$

$P(n|p| \le q)$ is the enumeration of the partitions of n into exactly p members, no member of which is greater than q. The appended notation $PV(n|p| \le q)$ is the actual set of such partitions. The use of $\ge$ and/or $\le$ symbols with n, p, or q defines lower limits and/or upper limits of the quantity modified. Note that [ ] (except for obvious reference use) is used with real numbers to indicate the greatest integer less than or equal to the number bracketed.

where a_k is an individual partition member. The arrangement of (2) leads to an initial partition of $PV(n|p| \leq n - p + 1)$ as

(3)

$$\begin{array}{c|c|c|c|c|c} a_p & a_{p-1} & a_{p-2} & \cdots & a_2 & a_1 \\ \hline 1 & 1 & 1 & \cdots & 1 & n - p + 1 \end{array}$$

One method [3] of generating successive partitions of $PV(n|p| \leq n - p + 1)$ starts with (3) and successively increases a_2 by 1 and decreases a_1 by 1 until (2) is just barely satisfied. New members, a_p, $a_{p-1}, \cdots, a_2, a_1$ are chosen exhaustively, and the increase a_2—decrease a_1 process is repeated.

   Based on the above brief background, it is possible to consider the following enumeration extensions to (1):

(a) $P(n|p| \geq s)$.   No member less than s, where s is a positive integer such that $s \leq n - p + 1$.

(b) $P(n|p| \leq r)$.   No member greater than r where r is a positive integer such that $r \leq n - p + 1$.

(c) $P(n|p| \geq s, \leq r)$.   No member less than s, or greater than r .

2. ENUMERATION OF $P(n|p| \geq s)$

   There exists one member of a partition in the set $PV(n|p| \geq s)$ which is at least as large as any member of any partition in the set. Let this member be q_s which can readily be found as

(4) $q_s = n - s(p - 1)$.

This implies that for any a_1,

(5) $n - ps + s \geq a_1 \geq s$,

from which a necessary condition of $P(n|p| \geq s)$ is seen to be

(6) $\left(\dfrac{n}{p} \right) \geq s$.

The initial partition of $PV(n|p|\geq s)$ is

$$(7) \qquad\qquad s,\ s,\ s,\ \cdots,s,\ n - s(p - 1)$$

If $s - 1$ is subtracted from each member of (7), the result is a modified initial partition

$$(8) \qquad\qquad 1,\ 1,\ 1,\ \cdots,1,\ n - sp + 1\ .$$

The complete enumeration for a partition set starting with (8) is, according to (1), $P(n'|p|\leq n' - p + 1)$, where

$$(9) \qquad\qquad n' = n - sp + p\ .$$

Because the a_1 and a_2 members of the initial partitions (7) and (8) differ by the same integer $(n - sp)$ and because each a_k of each partition developed from (7) is $(s - 1)$ greater than the corresponding a_k of the corresponding partition developed from (8), there are exactly as many partitions developable from the start of (7) as there are from (8). Hence, $P(n|p|\geq s)$ appears in the form of (1) as

$$(10) \qquad\qquad P(n|p|\geq s) = P(n'|p|\leq n' - p + 1)\ .$$

As a simple example, consider $P(15|6|\geq 2)$. For this case, $n = 15$, and $n' = 9$. It is seen below that $P(15|6|\geq 2) = P(9|6|\leq 4) = 3$.

$PV(15|6|\geq 2)$	$PV(9|6|\leq 4)$
2,2,2,2,2,5	1,1,1,1,1,4
2,2,2,2,3,4	1,1,1,1,2,3
2,2,2,3,3,3	1,1,1,2,2,2

3. ENUMERATION OF $P(n|p|\leq r)$

The partitions of the set $PV(\geq 3,\leq n - p|\geq 1,\leq [(n - p)/3]|\geq 3 \leq p)$ can be arranged in columns according to the number of members in a partition. This is illustrated in Table 1 for $n = 16$, $p = 5$.

i	0	1	2	3
PV($\geq 3,\leq 11 \mid \geq 1,\leq 3 \mid \geq 3,\leq 5$)	0	3 4 5	3,3 3,4 3,5 4,4 4,5 5,5	3,3,3 3,3,4 3,3,5 3,4,4

Table 1

The sum of members of each partition is equal to a w_i for use in (1). The use of the index i can be extended somewhat to allow it to designate the column from which the summed partition was taken. Although w_i might stand for any of several sums, no loss in generality results thereby since all of these sums must eventually be considered. To account for the non-summation term in (1), a zero[th] column with a lone zero entry is added to indicate that an added $w_0 = 0$. Table 2 shows values of w_i for $n = 16$, $p = 5$.

i	0	1	2	3
	0	3 4 5	6 7 8 8 9 10	9 10 11 11

Table 2   Values of w_i

If, as the w_i's are successively selected for enumerating $P(n|p| \leq n - p + 1)$ in (1), a simultaneous generation of the partitions in the set $PV(n|p| \leq n - p + 1)$ is made (by the increase a_2—decrease a_1 method, for example) there would result subsets of $PV(n|p| \leq n - p + 1)$ each having $[(n - p + 2 - w_i)/2]$ partitions of n. For $i = 0$, the subset can easily be constructed. It is seen that the a_2 and a_3 members of the initial partition must necessarily be one. For $i = 1$, the a_2 and a_3 members of the initial partition assume the least

possible value two since $i = 0$ has accounted for the value one. It can be argued in this fashion that the a_2 and a_3 members of an initial partition in a subset must be $(i + 1)$. The a_1 member of the initial partition of the subset would not generally be known in advance. However, this member is certainly not less than any member of any partition in the subset. Set d_i be the a_1 member of the initial partition corresponding to the particular w_i. If b_i are the number of partitions in the subset, the bracketed terms of (1) limit the possibilities of b_i to either

$$(11) \qquad\qquad n - p + 1 - w_i = 2b_i - 1 \quad ,$$

or

$$(11a) \qquad\qquad n - p + 1 - w_i = 2b_i \quad .$$

The arrangement of the subset of b_i partitions is

	a_p	a_{p-1}	$\cdots$	a_3	a_2	a_1	
	x	x	$\cdots$	i+1	i+1	d_i	$\longleftarrow$ (Initial Partition)
	.	.	$\cdots$	.	∘	∘	
(12) b_i partitions	.	.	$\cdots$	.	.	.	
	.	.	$\cdots$	.	.	.	
	x	x	$\cdots$	i+1	i+b_i	d_i-b_i+1	

From (12), it can be deduced that either

$$(13) \qquad\qquad d_i = 2b_i - 1 + i \quad ,$$

or

$$(13a) \qquad\qquad d_i = 2b_i + i \quad .$$

Comparison of (13) with (11) and (13a) with (11a) yields the desired

$$(14) \qquad\qquad d_i = (n - p + 1 - w_i + i) \quad .$$

An illustration is given in Table 3 for construction of $PV(16|5|\leq 12)$, consistent with the w_i from $PV(\geq 3,\leq 11|\geq 1,\leq 3|\geq 3,\leq 5)$ as arranged in Tables 1 and 2.

i	0	1	2	3
PV(16|5|≤12)	1,1,1,1,1,12	1,1,2,2,10	1,1,3,3,8	1,1,4,4,6
	1,1,1,1,2,11	1,1,2,3, 9	1,1,3,4,7	1,1,4,4,5
	1,1,1,1,3,10	1,1,2,4, 8	1,1,3,5,6	
	1,1,1,1,4, 9	1,1,2,5, 7		
	1,1,1,1,5, 8	1,1,2,6, 6		
	1,1,1,1,6, 7			
		1,2,2,2, 9	1,2,3,3,7	1,2,4,4,5
		1,2,2,3, 8	1,2,3,4,6	
		1,2,2,4, 7	1,2,3,5,5	
		1,2,2,5, 6		
		2,2,2,2, 8	1,3,3,3,6	1,3,4,4,4
		2,2,2,3, 7	1,3,3,4,5	
		2,2,2,4, 6		
		2,2,2,5, 5		
			2,2,3,3,6	2,2,4,4,4
			2,2,3,4,5	
			2,3,3,3,5	
			2,3,3,4,4	
			3,3,3,3,4	

Table 3　$PV(16|5|\leq 12)$

Table 4 shows b_i corresponding to w_i of Table 1 for $P(16|5|\leq 12) = \sum_{i=0} b_i$.

i	0	1	2	3
b_i	6	5	3	2
		4	3	1
		4	2	1
			2	1
			2	
			1	

Table 4　$\sum_i b_i = P(16|5|\leq 12) = 37$

Let the b_i for $P(n|p|\leq r)$ be b_{ir}. It follows that for $P(n|p|\leq r)$ each b_{ir} can have no more (and will possibly have less) than b_i partitions. The non-negative integer by which b_{ir} is less than b_r can be observed by comparing r with the entries in the a_1 column of (12). This leads immediately to

$$(15) \quad P(n|p|\leq r) = \left[\frac{n-p+2}{2}\right] - \alpha_0 + \sum_{i=1}\left(\left[\frac{n-p+2-w_i}{2}\right] - \alpha_1\right) = \sum_{i=0} b_{ir} \, ,$$

where

$$(16) \qquad \alpha_1 = \begin{cases} 0 & (r \geq d_i) \, , \\ d_i - r & (d_i > r \geq (d_i - b_i + 1)) \, , \\ \left[\dfrac{n-p+2-w_i}{2}\right] & (r < (d_i - b_i + 1)). \end{cases}$$

Table 5 serves to illustrate (15) for $n = 16$, $p = 5$, $r = 7$.

i	0	1	2	3
	1	2	2	2
		2	3	1
b_{ir}		3	2	1
			2	1
			1	

Table 5 $\sum_i b_{ir} = P(16|5|\leq 7) = 23$

4. ENUMERATION OF $P(n|p|\geq s, \leq r)$

The combination of the previous two methods leads quickly to the desired enumeration. Reference to (10) reveals a $P(n'|p|\leq n' - p + 1)$ for which every member of each partition of $PV(n'|p|\leq n' - p + 1)$ is $(s - 1)$ less than the corresponding member of the appropriate counterpart in $PV(n|p|\geq s)$. If the desired r is depressed to r' where

(17) $$r' = r - (s - 1) \quad ,$$

the enumeration $P(n, |p| \ge s, \le r)$ is equal to $P(n' |p| \le r')$.

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DID YOU KNOW?

Prof. D. E. Knuth of California Institute of Technology is working on a 3-volume book, The Analysis of Algorithms, which has 39 exercises at the end of the section which introduces the Fibonacci Sequence. However, the Fibonacci Sequence occurs in many different places, both as an operational tool, or to serve as examples of good sequences and also bad sequences. He reports that there are at least 12 different algorithms directly or indirectly connected with the Fibonacci Sequence. In the age of computers, the Fibonacci Sequence is coming of age in many ways. This book will be a most welcome addition to the growing list of Fibonacci related books and articles.

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Prof. C. T. Long of Washington State University has written a very nice book, Elementary Introduction to Number Theory, 1965, Heath, Boston. It contains a good discussion of the Fibonacci Numbers in Chapter One and several Fibonacci Problems in Chapters I and II.

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ON THE DIVISIBILITY PROPERTIES OF FIBONACCI NUMBERS

John H. Halton, University of Colorado, Boulder, Colorado

1. INTRODUCTION

The Fibonacci sequence is defined by the recurrence relation

(1)
$$F_{n+2} = F_{n+1} + F_n \quad ,$$

together with the particular values

(2)

$$F_0 = 0, \quad F_1 = 1 \quad ,$$

whence

$F_2 = 1, \; F_3 = 2, \; F_4 = 3, \; F_5 = 5, \; F_6 = 8 = 2^3, \; F_7 = 13$

$F_8 = 21 = 3 \cdot 7, \; F_9 = 34 = 2 \cdot 17, \; F_{10} = 55 = 5 \cdot 11, \cdots \quad ;$

and, in particular,

$F_{12} = 144 = 2^4 \cdot 3^2, \; F_{14} = 377 = 13 \cdot 29, \; F_{15} = 610 = 2 \cdot 5 \cdot 61,$

$F_{18} = 2584 = 2^3 \cdot 17 \cdot 19, \; F_{20} = 6765 = 3 \cdot 5 \cdot 11 \cdot 41,$

$F_{21} = 10946 = 2 \cdot 13 \cdot 421, \; F_{24} = 46368 = 2^5 \cdot 3^2 \cdot 7 \cdot 23 ,$

$F_{25} = 75025 = 5^2 \cdot 3001, \; F_{28} = 317811 = 3 \cdot 13 \cdot 29 \cdot 281 ,$

$F_{30} = 832040 = 2^3 \cdot 5 \cdot 11 \cdot 31 \cdot 61, \; F_{35} = 9227465 = 5 \cdot 13 \cdot 141961 ,$

$F_{36} = 14930352 = 2^4 \cdot 3^3 \cdot 17 \cdot 19 \cdot 107, \; F_{42} = 267914296 = 2^3 \cdot 13 \cdot 29 \cdot 211 \cdot 421$

$F_{70} = 190392490709135 = 5 \cdot 11 \cdot 13 \cdot 29 \cdot 71 \cdot 911 \cdot 141961 \quad .$

In this paper, we shall be concerned with the sub-sequence of Fibonacci numbers which are divisible by powers of a given integer. We shall also be interested in the associated problem of the periodic nature of the sequence of remainders, when the Fibonacci numbers are divided by a given integer.

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The Fibonacci sequence is defined for all integer values of the index n.
However, the well-known identity

$$(3) \qquad\qquad F_{-n} = (-1)^{n+1} F_n$$

shows that negative indices add nothing to the divisibility properties of the
Fibonacci numbers. We shall consequently simplify our discussion, without
loss of generality, by imposing the restriction that $n \geq 0$.

Of the many papers dealing with our problem, perhaps the most useful
are those of Carmichael [1], Robinson [5], Vinson [6], and Wall [7]; and the
reader can find many additional references in these. Most of the other papers
in the field give either less complete results, or give them for more general
sequences.

We shall make use, in what follows, of the well-known identities:*

$$(4) \qquad F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right\} \quad ;$$

$$(5) \qquad F_n = \left(\frac{1}{2} \right)^{n-1} \sum_{s=0}^{[\frac{1}{2}(n-1)]} \binom{n}{2s+1} 5^s, \quad \text{if } n \geq 1 \quad ;$$

$$(6) \qquad F_n^2 - F_{n-1} F_{n+1} = (-1)^{n-1} \quad ;$$

$$(7) \qquad F_{kn+r} = \sum_{h=0}^{k} \binom{k}{n} F_n^h F_{n-1}^{k-h} F_{r+h}, \quad \text{if } k \geq 0 \quad ;$$

and since $F_0 = 0$,

$$(8) \qquad F_{kn} = F_n \sum_{h=1}^{k} \binom{k}{h} F_n^{h-1} F_{n-1}^{k-h} F_h \quad .$$

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*See, for example, equations (6), (3), (5), (67), and (34), in my earlier paper
[3]. Equation (5) above follows from (4) by the binomial theorem.

Also

$$(9) \qquad \binom{k + 1}{h} = \binom{k}{h} + \binom{k}{h - 1} \ ,$$

and

$$(10) \qquad p \ \text{divides} \ \binom{p}{s} \ \text{if} \ p \ \text{is prime and} \ 0 < s < p \ ,$$

and Fermat's theorem, that

$$(11) \qquad m^{p-1} \equiv 1 \ (\text{mod} \ p) \ \text{if} \ p \ \text{is prime and} \ (m,p) = 1$$

As is customary, we use $(A, B, C, \cdots)$ to represent the greatest common factor of integers $A, B, C, \cdots$, and $[A, B, C, \cdots]$ to represent their least common multiple. We have

$$(12) \qquad m^{\frac{1}{2}(p-1)} \equiv (m/p) \ (\text{mod} \ p) \ ,$$

where p is an odd prime and (m/p) denotes the Legendre index, which is ± 1 if $(m,p) = 1$, and 0 otherwise.

Each writer seems to have invented his own notation. I shall adopt the following, which comes closest to that of Robinson in [5].

Definition 1. The least positive index α such that F_α is divisible by m^n (that is, $F_\alpha \equiv 0 \ (\text{mod} \ m^n)$) will be written

$$(13) \qquad \alpha(m, n) = \alpha(m^n, 1) = \alpha(m^n) \ .$$

This is variously called the "rank of apparition" (why not "appearance"?) of m^n, or the "restricted period" of the Fibonacci sequence modulo m^n.

Definition 2. The least positive index μ such that both $F_\mu \equiv 0$ and $F_{\mu+1} \equiv 1 \ (\text{mod} \ m^n)$ will be written

$$(14) \qquad \mu(m, n) = \mu(m^n, 1) = \mu(m^n) \quad .$$

This notation follows Carmichael [2], who named μ the "characteristic number" of the Fibonacci sequence modulo m^n. It is also called the "period" of the sequence modulo m^n.

Definition 3. I shall write

$$(15) \qquad \mu(m, n)/\alpha(m, n) = \beta(m, n) = \beta(m^n, 1) = \beta(m^n) \quad .$$

Definition 4. The greatest integer ν such that $F_{\alpha(m, n)}$ is divisible by m^ν will be written

$$(16) \qquad \nu(m, n) = \nu(m^n, 1) = \nu(m^n) \quad .$$

It is then clear that

$$(17) \quad \alpha(m, n) = \alpha(m, n + 1) = \cdots = \alpha(m, \nu(m, n)) < \alpha(m, \nu(m, n) + 1)$$

or, equivalently,

$$(18) \qquad \nu(m, \nu(m, n)) = \nu(m, n) \quad .$$

Definition 5. I shall call the sequence

$$(19) \qquad F_{\alpha(m, 1)} , \quad F_{\alpha(m, 2)} , \cdots , F_{\alpha(m, n)} , \cdots \quad ,$$

the divisibility sequence of m.

2. PRELIMINARIES

We shall need a number of preliminary results, whose proofs will be outlined for completeness.

Lemma 1. F_n, F_{n+1}, and F_{n+2} are always pairwise prime.
[If f divides two of the numbers, it must divide the third, by (1). Thus, by

induction along the sequence, using (1), we see that f must divide every F_m. Thus, since $F_1 = 1$, $f = 1$.]

Lemma 2. If $n \geq 2$, F_n is a strictly increasing positive function of n. [By (1), if $F_{n-2} \geq 0$ and $F_{n-1} \geq 1$, $F_{n+1} > F_n \geq 1$. By (2), $F_0 = 0$ and $F_1 = 1$, whence the lemma follows by induction.]

Lemma 3. If $n \geq 3$

$$(20) \qquad\qquad \alpha(F_n) = n \ .$$

[By Lemma 2, if $n \geq 3$, the least index m such that $F_m \geq F_n$ is n.]

Lemma 4.

$$(21) \qquad\qquad (F_m, F_n) = F_{(m,n)} \ .$$

[Let $(m,n) = g$ and $(F_m, F_n) = G$. There are integers x and y (not both negative) such that $xm + yn = g$. Suppose $x \geq 0$; then, by (7),

$$F_g = \sum_{h=0}^{x} \binom{x}{h} F_m^h F_{m-1}^{x-h} F_{yn+h} \equiv 0 \pmod{G} \ ,$$

since G divides F_m and F_n, and by (8), F_n divides F_{yn}. Thus F_g is divisible by G. Again, by (8), $F_{kg} \equiv 0 \pmod{F_g}$. Thus, since g divides both m and n, F_g divides both F_m and F_n, and so G is divisible by F_g.]

Lemma 5. F_m is divisible by F_n, if and only if either m is divisible by n, or $n = 2$. [By Lemma 4, $(F_m, F_n) = F_n$ if and only if $F_{(m,n)} = F_n$; that is, $(m,n) = n$ or $n = 2$.]

Definition 6. The remainder when F_n is divided by m will be written $F_n^{(m)}$ and will be called the residue of F_n modulo m. Clearly

$$(22) \qquad\qquad F_n \equiv F_n^{(m)} \pmod{m}, \quad 0 \leq F_n^{(m)} < m \ .$$

Lemma 6. The sequence of residues $F_n^{(m)}$, modulo any integer $m \geq 2$, is periodic with period $\mu(m)$. That is

$$(23) \quad \left\{ \begin{array}{l} F_{n+k\mu(m)}^{(m)} = F_n^{(m)} \\[2mm] \text{or} \\[2mm] F_{n+k\mu(m)} \equiv F_n \pmod{m}. \end{array} \right.$$

[The ordered pair of integers $F_n^{(m)}$, $F_{n+1}^{(m)}$ can take at most m^2 distinct values. Thus the $m^2 + 1$ such consecutive pairs in $F_0^{(m)}$, $F_1^{(m)}$, $\cdots$, $F_{m^2+1}^{(m)}$ must have a duplication. By backward induction on the indices of two equal pairs, using (1), we see that there must be a pair $F_k^{(m)}$, $F_{k+1}^{(m)}$ equal to $F_0^{(m)} = 0$, $F_1^{(m)} = 1$, with $2 \leq k \leq m^2$. By definition, the least such k is $\mu(m)$. The periodicity now follows from (1).]

Lemma 7. For any integer m, we can find an F_n divisible by m.
[For example, $n = k\mu(m)$, for any integer k, by Lemma 6.]

Lemma 8. F_n is divisible by m if and only if n is divisible by $\alpha(m)$.
[Since m is a factor of $F_{\alpha(m)}$; if n is divisible by $\alpha(m)$, F_n is divisible by m, by Lemma 5. Let $n = k\alpha(m) + r$, $0 \leq r < \alpha(m)$, and let m divide F_n. Then, by (7), $F_{\alpha(m)-1}^k F_r \equiv F_n \equiv 0 \pmod{m}$. Thus, since by Lemma 1, $(F_{\alpha(m)}, F_{\alpha(m)-1}) = 1$; $F_r \equiv 0 \pmod{m}$. Since $r < \alpha(m)$, which is minimal, $F_r = 0$; whence $r = 0$ and n is divisible by $\alpha(m)$.]

Lemma 9. For all integers m and $r \geq s > 0$, $\alpha(m,s)$ divides $\alpha(m,r)$.
[$F_{\alpha(m,r)}$ is divisible by m^r and so by m^s. The result follows from Lemma 8.]

Lemma 10. $\mu(m)$ is divisible by $\alpha(m)$. That is, $\beta(m)$ is an integer.
[Since $F_{\mu(m)}^{(m)} = F_0^{(m)} = 0$, $F_{\mu(m)}$ is divisible by m. The lemma follows from Lemma 8.]

Lemma 11. If p is an odd prime, then p divides only one of F_{p-1}, F_p, and F_{p+1}; namely, F_m, where $m = p - (5/p)$.
[(p,2) = 1. Using (5), (10), and (11), we obtain that

$$(24) \quad F_p \equiv 2^{p-1} F_p = \sum_{s=0}^{\frac{1}{2}(p-1)} \binom{p}{2s+1} 5^s \equiv 5^{\frac{1}{2}(p-1)} \pmod{p}$$

Thus p divides F_p if and only if $(5/p) = 0$, by (12); that is, when $p = 5$. By (5), (9), (10), and (11),

$$(25) \quad 2F_{p+1} \equiv 2^p F_{p+1} = \sum_{s=0}^{\frac{1}{2}(p-1)} \left\{ \binom{p}{2s+1} + \binom{p}{2s} \right\} 5^s \equiv 1 + 5^{\frac{1}{2}(p-1)} \pmod{p}$$

and, by (1), (24), and (25),

$$(26) \quad 2F_{p-1} \equiv 1 - 5^{\frac{1}{2}(p-1)} \pmod{p} \qquad .$$

The lemma now follows. We may note that all but the dependence on $(5/p)$ follows directly from (6), which yields that, if $p \neq 5$, by (11) and (24),

$$F_{p-1}F_{p+1} = F_p^2 - 1 \equiv 0 \pmod{p} \quad ;$$

and from (1).]

Lemma 12. $\alpha(p)$ divides $p - (5/p)$, if p is an odd prime; and if $\alpha(p)$ is itself prime and $p \neq 5$, $\alpha(p) < p$.

[The first part follows from Lemmas 8 and 11. Thus $\alpha(p) < p + 1$. By Lemma 11, if $p \neq 5$ and $\alpha(p)$ is prime, since $p \pm 1$ is not prime, $\alpha(p) \leq p - 2$.]

Lemma 13. If

$$(27) \quad m = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_k^{\lambda_k} \quad ,$$

where the p_i are distinct primes and the λ_i are positive integers, then

$$(28) \quad \alpha(m,n) = [\alpha(p_1, n\lambda_1), \alpha(p_2, n\lambda_2), \cdots, \alpha(p_k, n\lambda_k)]$$

and

$$(29) \quad \mu(m,n) = [\mu(p_1, n\lambda_1), \mu(p_2, n\lambda_2), \cdots, \mu(p_k, n\lambda_k)]$$

[By Lemma 8, F_t is divisible by $p_i^{n\lambda_i}$ if and only if t is divisible by $\alpha(p_i, n\lambda_i)$. Thus F_t is divisible by m^n if and only if t is a multiple of all the $\alpha(p_i, n\lambda_i)$. Since $\alpha(m,n)$ is minimal, (28) follows. By Lemma 6, $F_{s+t} \equiv F_s \pmod{p_i^{n\lambda_i}}$ for every s if and only if t is a multiple of $\mu(p_i, n\lambda_i)$. Thus, by the Chinese remainder theorem, $F_{s+t} \equiv F_s \pmod{m^n}$ for every s if and only if t is a common multiple of the $\mu(p_i, n\lambda_i)$. Since $\mu(m,n)$ is the minimal such t, (29) follows.]

Lemma 14. For any integers m and n,

$$(30) \quad \begin{cases} \alpha([m,n]) = [\alpha(m), \alpha(n)] \\[1em] \text{and} \\[1em] \mu([m,n]) = [\mu(m), \mu(n)] \end{cases} .$$

[This follows from Lemma 13, by expanding m and n in prime factors.]

Definition 7. The greatest integer n such that N is divisible by m^n will be written

$$(31) \qquad n = \text{pot}_m N$$

and called the "potency" of N to base m, following H. Gupta. It is then clear that, in particular,

$$(32) \qquad \nu(m,n) = \text{pot}_m F_{\alpha(m,n)} .$$

Lemma 15. $\text{pot}_m F_N = n$ if and only if N is divisible by $\alpha(m,n)$ but not by $\alpha(m, n+1)$.

[This is an immediate consequence of Lemma 8.]

Lemma 16. If k and n are positive integers, then $(F_{kn}/F_n, F_n)$ is a factor of k.

[By (8), $F_{kn}/F_n \equiv kF_{n-1}^{k-1} \pmod{F_n}$. Thus, if $(F_{kn}/F_n, F_n) = g$, g divides kF_{n-1}^{k-1}. By Lemma 1, $(F_{n-1}, F_n) = 1$; so g divides k.]

Lemma 17. If k and n are both integers greater than one, then F_{kn}/F_n is a strictly increasing function of n and of k.

[By (8), $F_{kn}/F_n = \Sigma_{h=1}^{k} \binom{k}{h} F_n^{h-1} F_{n-1}^{k-h} F_h$. Every term in the sum is positive, and increases with F_n, F_{n-1}, and k. The result follows from Lemma 2.]

Of these results, those in Lemmas 1, 2, 4 — 7, 11, and 16 have been known for a long time. Lemmas 8 — 10 and 12 — 15 appear, or are implicit, in the papers of Robinson [5], Vinson [6], and Wall [7]. [My $\alpha(m)$, $\beta(m)$, $\mu(m)$ are written $\alpha(m)$, $\beta(m)$, $\delta(m)$ by Robinson, and $f(m)$, $t(m)$, $s(m)$ by Vinson,, respectively; and Wall writes $d(m)$, $k(m)$ for my $\alpha(m)$, $\mu(m)$.]

3. THE DIVISIBILITY SEQUENCE

Theorem 1. If p is an odd prime and $n \geq \nu(p)$, then

(33) $\alpha(p,n) = p^{n-\nu(p)}\alpha(p)$,

(34) $\nu(p,n) = n$.

If $p \neq 5$, $(p,\alpha(p)) = 1$; while

(35) $\alpha(5,n) = 5^n$.

Further,

(36) $\alpha(2) = 3, \ \alpha(4) = 6 = \alpha(8)$,

and if $n \geq 3$,

(37) $\alpha(2,n) = 2^{n-2}\alpha(2) = 2^{n-2} \cdot 3$.

Proof. By Lemma 9, $\alpha(p,n) = k\alpha(p,n-1)$, for some integer k. Write

$$F_{\alpha(p,n)} = p^n A, \ F_{\alpha(p,n-1)} = p^{n-1}B, \ F_{\alpha(p,n-1)-1} = C.$$

Then, by (8),

$$(38) \qquad pA = \sum_{h=1}^{k} \binom{k}{h} p^{(n-1)(h-1)} B^h C^{k-h} F_h \quad .$$

Thus, if $n > \nu(p) \geq 1$, since $(p, C) = 1$, kB must be divisible by p. Hence, if $\nu(p, n-1) = n-1$, $(p, B) = 1$, whence p divides k. Since $\alpha(p, n)$ and so k, is minimal, $k = p$. Now, by (10), since $k > 2$, (38) yields that $A \equiv BC^{p-1} \pmod{p}$. Since the factors on the right are prime to p, so is A, whence $\nu(p, n) = n$. By (18), $\nu(p, \nu(p)) = \nu(p)$, so that, by induction, if $n \geq \nu(p)$, (34) holds and $\alpha(p, n) = p^{n-\nu(p)} \alpha(p, \nu(p))$. By (17), $\alpha(p, \nu(p)) = \alpha(p)$, yielding (33).

By Lemma 12, $\alpha(p)$ divides $p - (5/p)$. Thus, if $p \neq 5$, $(p, \alpha(p)) = 1$. If $p = 5$, then, by (2), $\alpha(5) = 5$, $\nu(5) = 1$, and, by (33), we get (35).

Finally, if $p = 2$, (38) still holds, and we see, as before, that $k = p = 2$ if $\nu(2, n-1) = n - 1$. Thus $2A = 2^{n-1}B^2 + 2BC$, whence $(2, A) = 1$ and $\nu(2, n) = n$, as before, if $n \geq 3$. By (2), we have (36), whence we obtain (37) like (33).

Theorem 2. If $\operatorname{pot}_p F_m = n \geq 1$, where p is prime and $p^n \neq 2$, and if $r \geq 0$ and $(p, t) = 1$; then $\operatorname{pot}_p F_{p^r tm} = n + r$. If $p^n = 2$, tm is an odd multiple of 3 and F_{tm} is an odd multiple of 2, while, if $r \geq 1$, $\operatorname{pot}_2 F_{2^r tm} = r + 2$.

Proof. We repeatedly use Lemma 15 and Theorem 1. If $n \geq 1$ and $p^n \neq 2$, either p is odd and $n \geq \nu(p)$, or $p = 2$ and $n \geq 3$; whence, by (33) or (37),

$$(39) \qquad \alpha(p, n + r) = p^r \alpha(p, n) \quad .$$

Thus, $m = k\alpha(p, n)$ for some k prime to p. Hence $p^r tm = tk\alpha(p, n + r)$, so that $\operatorname{pot}_p F_{p^r tm} = n + r$. By (36), if $p^n = 2$, m and tm are divisible by 3 but not by 6, so that $\operatorname{pot}_2 F_{tm} = 1$, and similarly by (37), $\operatorname{pot}_2 F_{2^r tm} = r + 2$, if $r \geq 1$.

Theorems 1 and 2 have a fairly long history. Lucas [4] (see pages 209 – 210) proved the simplest formula (39) with $r = 1$, but failed to notice the anomaly

when $p^n = 2$. Carmichael [1] (see pages 40 — 42) proved Theorem 2 in full,* using the theory of cyclotomic polynomials. Both Lucas' and Carmichael's results apply to a more general sequence** than that defined by (1) and (2). Robinson [5] proves Theorem 1, for odd primes only, by a matrix method.

Theorem 3. If $pot_p F_m = n \geq 1$, where p is prime and $p^n \neq 2$, and if $r \geq 0$; then there is a strictly increasing sequence of pairwise prime integers $\ell_s = \ell_s (m,p) (s = 0,1,2,\cdots)$, all prime to p, such that

$$(40) \qquad\qquad F_{p^r m} = p^{n+r} \ell_0 \ell_1 \cdots \ell_r \quad .$$

Proof. When $r = 0$, we define $F_m = p^n \ell_0$, where $(p, \ell_0) = 1$. By Theorem 2, if $r \geq 1$, there are integers A, B, and C, such that

$$F_{p^r m} = p^{n+r} A, \quad F_{p^{r-1} m} = p^{n+r-1} B, \quad F_{p^{r-1} m-1} = C \quad ,$$

and $(p, A) = (p, B) = 1$, while, by Lemma 1, $(pB, C) = 1$. Thus, by (8),

$$(41) \qquad A = B \sum_{h=1}^{p} \binom{p}{h} p^{(n+r-1)(h-1)-1} B^{h-1} C^{p-h} F_h$$

where, as in the proof of Theorem 1, the sum on the right is an integer, since $n \geq 1$. Thus A is divisible by B. If we write $A = \ell_r B$, it is clear that $A = \ell_0 \ell_1 \cdots \ell_r$, yielding (40). Further (41) gives us that

$$(42) \qquad \ell_r = pB \sum_{h=2}^{p} \binom{p}{h} p^{(n+r-1)(h-1)-2} B^{h-2} C^{p-h} F_h + C^{p-1} \quad ,$$

---

* He has a misprint, making the greatest power of 2 too small by one.
**The sequence is $D_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where $\alpha + \beta$ and $\alpha\beta$ are mutually prime integers. For F_n, by (4), $\alpha = (1/2)(1 + \sqrt{5})$ and $\beta = (1/2)(1 - \sqrt{5})$.

where the sum is again an integer, since either $p \geq 3$ and $n \geq 1$, or $p = 2$ and $n \geq 3$. Thus $\ell_r \equiv C^{p-1}$ (mod pB); so that, since C is prime to p, $\ell_0, \ell_1, \cdots, \ell_{r-1}$, so is ℓ_r. Again, since ℓ_r exceeds a positive integer multiple of pB, we have that

$$(43) \qquad \ell_r > p\ell_0\ell_1 \cdots \ell_{r-1} > \ell_{r-1} \qquad .$$

Corollary 1. If $\mathrm{pot}_p F_m = n \geq 1$ and $p^n \neq 2$, and if $r > s \geq 0$, then

$$(44) \qquad \ell_{r-s}(p^s m, \ p) = \ell_r(m, p)$$

and

$$(45) \qquad \ell_0(p^s m, \ p) = \ell_0(m, p)\ell_1(m, p) \cdots \ell_s(m, p) \qquad .$$

Corollary 2. If $\mathrm{pot}_2 F_m = 1$ and $r \geq 1$, then

$$(46) \qquad F_{2^r m} = 2^{r+2}\ell_0(2m, 2)\ell_1(2m, 2) \cdots \ell_{r-1}(2m, 2) \qquad .$$

Theorem 3, with its corollaries, contains a definition of $\ell_s(m, p)$ whenever $\mathrm{pot}_p F_m = n \geq 1$ and $p^n \neq 2$. By analogy with (40), (44), (45) and (46), we shall adopt the following definition for the remaining case.

Definition 8. If $m = 3t$ where t is odd (so that, by Theorem 2, $\mathrm{pot}_2 F_m = 1$), the sequence $\ell_s(m, 2)$ is defined by

$$(47) \qquad \ell_0(m, 2) = \tfrac{1}{2} F_m \qquad ,$$

$$(48) \qquad \ell_1(m, 2) = 2\ell_0(2m, 2)/\ell_0(m, 2) \qquad ,$$

and

$$(49) \qquad \ell_s(m, 2) = \ell_{s-1}(2m, 2) \text{ if } s \geq 2 \quad .$$

<u>Corollary 3.</u> Adopting Definition 8, we obtain equation (40) for every prime p, and every positive integer m such that $\text{pot}_p F_m = n \geq 1$. In every case, the numbers $\ell_s = \ell_s(m,p)(s = 0,1,2,\cdots)$ are integers, all pairwise prime, and all but $\ell_1(m,p)$ are always prime to p. If m is an odd multiple of 3, $\ell_1(m,2)$ is an odd multiple of 2; in every other case, $\ell_1(m,p)$ is prime to p.

<u>Proof.</u> If $p^n \neq 2$, the corollary coincides with Theorem 3. If $p^n = 2$ (that is, m is an odd multiple of 3, by (36)) and $r \geq 1$, Corollary 2 and Definition 8 (equations (46), (48), and (49)) show that equation (40) holds, with $\frac{1}{2}\ell_0(m,2)\ell_1(m,2)$, $\ell_2(m,2)$, $\ell_3(m,2)$, $\cdots$ all pairwise prime odd integers, by Theorem 3. Finally, when $p^n = 2$ and $r = 0$, we get (40) from the definition (47), and, by Theorem 2, $\ell_0(m,2)$ is an odd integer.

Further, by (8), $F_{2m} = F_m(F_m + 2F_{m-1})$, which yields through (40) that $\ell_1(m,2) = \ell_0(m,2) + F_{m-1}$. Since $(\ell_0, F_{m-1}) = (\ell_1, F_{m-1}) = 1$ (by Lemma 4), and both ℓ_0 and F_{m-1} are odd, we see that $\ell_1(m,2)$ is even and prime to $\ell_0(m,2)$. Finally, since $\frac{1}{2}\ell_0\ell_1$ is odd, ℓ_1 must be an odd multiple of 2.

<u>Theorem 4.</u> Let $P = \{p_1, p_2, \cdots, p_k\}$ be a set of k distinct primes. Then P contains all the prime factors of $F_{p_1}, F_{p_2}, \cdots, F_{pk}$ only if

$$
(50) \quad \left\{ \begin{array}{l} k = 1 \text{ and } P = \{2\} \text{ or } \{5\} \ , \\[2mm] k = 2 \text{ and } P = \{2, 3\} \text{ or } \{2, 5\} \ , \\[2mm] \quad \text{or} \\[2mm] k = 3 \text{ and } P = \{2, 3, 5\} \quad . \end{array} \right.
$$

<u>Proof.</u> Let $k = 1$. Then F_{p_1} can have no prime factor other than p_1. By (2), Lemma 2, and Lemma 11, the only possible values of p_1 are 2 and 5.

Let $k \geq 2$, and first suppose that $2 \notin P$. By Lemma 4, if $i \neq j$, $(F_{p_i}, F_{p_j}) = 1$, so that no prime factor is common to two of the F_{p_i}; and by Lemma 2, since every $p_i \geq 3$, every F_{p_i} has at least one prime factor. Thus every F_{p_i} has exactly one prime factor. Let us now renumber the p_i, if necessary, so that p_1 is the least prime in P not equal to 5, and

$$
(51) \quad F_{p_1} = p_2^{r_2}, \ F_{p_2} = p_3^{r_3}, \ \cdots, \ F_{p_{i-1}} = p_i^{r_i}, \ \cdots,
$$

where each $r_i \geq 1$. This can always be done, and, since $p_1 \neq 5$, inductively $p_2, p_3, \cdots \neq 5$, and no $p_{i-1} = p_i$. Finally, by Lemma 8, each $p_{i-1} = \alpha(p_i)$ and so, by Lemma 12, $\alpha(p_i) = p_{i-1} < p_i$. Thus the sequence defined by (51) cannot terminate, and this contradicts the finiteness of P. Therefore $2 \in P$ and we may write $p_1 = 2$. If the F_{p_i} (i = 2, 3, $\cdots$, k) are all odd, the p_i(i = 2, 3, $\cdots$, k) from a set of k - 1 distinct odd primes containing all the prime factors of the corresponding set of F_{p_i}. We have just shown that this can only happen if k - 1 = 1 and $p_2 = 5$. Suppose now that one of the F_{p_i} is even. Then, by (2), we can write $p_2 = 3$, since $F_3 = 2$. If k = 2, this completes the enumeration of possible cases. If $k \geq 3$, then $p_3, p_4, \cdots, p_k$ form a set of k - 2 distinct odd primes containing all the prime factors of the corresponding set of F_{p_i}, because $\alpha(3) = 4$, which is not prime. Again, we know that this can only happen if k - 2 = 1 and $p_3 = 5$. This completes the proof.

Definition 9. If $\mathrm{pot}_p F_N = n$, and if either $n \geq 1$ and p = 5, or n > $\nu(p)$, we shall call p a multiple prime factor (mpf) of F_N. If, on the contrary, $p \neq 5$ and $n = \nu(p)$, then p is a simple prime factor (spf) of F_N.

Lemma 18. p is a multiple prime factor of F_N if and only if it is a prime factor of both F_N and N. A prime factor of F_N which is not multiple is a simple prime factor.
[This follows from Definition 9, Lemma 8, and Theorem 1.]

Lemma 19. If k and n are positive integers and p is a multiple prime factor of F_n, it is also a multiple prime factor of F_{kn}. Conversely, if p is a simple prime factor of F_{kn}, it is also a simple prime factor of F_n.
[This follows from Lemmas 5 and 18.]

Theorem 5. F_N has at least one simple prime factor, unless N = 1, 2, 5, 6, or 12.

Proof. $F_1 = F_2 = 1$, so that these F_N have no prime factors at all, and so no spf, as stated. Let $N \geq 3$, and let $F_N = m$ satisfy (27). By Lemma 2, the set P of prime factors of F_N is not empty. If F_N has only mpfs, by Lemma 16, each p_i divides N; whence by Lemma 5, each F_{p_i} divides m. It follows that P contains all the prime factors of every F_{p_i}. This is the situation dealt with in Theorem 4, and it can only occur in the five cases listed in (50).

By (2), (50), Lemma 8, and Theorem 1, if F_N has only mpfs, we see that $F_N = 2^r \cdot 3^s \cdot 5^t$. Further, $r \leq 4$; $s \leq 2$; $t \leq 1$; rt = 0; st = 0; if

$r = 0$ then $s = 0$ and $t = 1$; if $s = t = 0$ then $r = 3$; if $rs > 0$ then $r = 4$ and $s = 2$. Thus $F_N = 5, 8,$ or 144; whence $N = 5, 6,$ or 12; and all these cases are valid and stated in the theorem.

4. CARMICHAEL'S THEOREM

By using the theory of cyclotomic polynomials, Carmichael proved, for the general sequence* D_n, a theorem which, in our terminology, reads as follows [Compare [1], Theorem XXIII, pages $61 - 62$.]

<u>Carmichael's Theorem.</u> If $N \neq 1, 2, 6,$ or 12, then there is a prime p, such that $N = \alpha(p)$.

We shall proceed to derive this theorem, for the Fibonacci sequence, by the elementary considerations we have used so far. Let

$$(52) \qquad\qquad N = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k} \quad,$$

where the q_i are distinct primes and the $n_i \geq 1$. We shall write $N_{(1)}$ for any of the k integers $N_i = N/q_i$, and more generally $N_{(h)}$ for any of the $\binom{k}{h}$ integers $N/q_{i_1} q_{i_2} \cdots q_{i_h}$, with $\{i_1, i_2, \cdots, i_h\}$ a subset (without repetition) of $\{1, 2, \cdots, k\}$. We shall also write R_h for the product of the $\binom{k}{h}$ integers $F_{N_{(h)}}$.

<u>Lemma 20.</u> If N satisfies (52), then

$$(53) \qquad \left[F_{N_1}, F_{N_2}, \cdots, F_{N_k} \right] = \frac{R_1 R_3 R_5 \cdots}{R_2 R_4 R_6 \cdots} = \prod_{h=1}^{h} R_h^{(-1)^{h-1}} \quad .$$

[By repeated application of Lemma 4, we see that

$$(54) \qquad (F_{N_{i_1}}, F_{N_{i_2}}, \cdots, F_{N_{i_h}}) = F_{(N/q_{i_1}, N/q_{i_2}, \cdots, N/q_{i_h})} = F_{N/q_{i_1} q_{i_2} \cdots q_{i_h}} = F_{N_{(h)}} \quad;$$

---

*See footnote on page 227 above.

so that R_h is the product of the greatest common factors of all sets of h numbers $F_{N_{(1)}}$. Let a prime factor p divide exactly s_1 of the $F_{N_{(1)}}$; and let $p^2, p^3, \cdots, p^m$ divide $s_2, s_3, \cdots, s_m$ of the $F_{N_{(1)}}$, respectively; but let no $F_{N_{(1)}}$ be divisible by p^{m+1}. Then $k \geq s_1 \geq s_2 \geq \cdots \geq s_m \geq 1$ and $\mathrm{pot}_p[F_{N_1}, F_{N_2}, \cdots, F_{N_k}] = m$. Of the $\binom{k}{h}$ factors in R_h, (54) shows that $\binom{s_1}{h}, \binom{s_2}{h}, \cdots, \binom{s_m}{h}$ are respectively divisible by $p, p^2, \cdots, p^m$. (Note that $\binom{s}{h} = 0$ if $s < h$, and that the set of factors divisible by p includes those divisible by p^2, which include those divisible by p^3, and so on). Thus

$$\mathrm{pot}_p R_h = \binom{s_1}{h} + \binom{s_2}{h} + \cdots + \binom{s_m}{h}, \text{ whence}$$

$$\mathrm{pot}_p\left(\frac{R_1 R_3 R_5 \cdots}{R_2 R_4 R_6 \cdots}\right) = \sum_{t=1}^{m}\sum_{h=1}^{k}(-1)^{h-1}\binom{s_t}{h} = \sum_{t=1}^{m}\{1 - (1-1)^{s_t}\} = m,$$

which implies (53).]

It follows from Lemmas 5 and 20 that

$$(55) \qquad Q_N = \frac{F_N R_2 R_4 \cdots}{R_1 R_3 R_5 \cdots} = \frac{F_N}{[F_{N_1}, F_{N_2}, \cdots, F_{N_k}]}$$

is a positive integer. [Carmichael [1] writes D_N for my F_N, and $F_N(\alpha, \beta) = \beta^{\phi(N)} Q_N(\alpha/\beta)$ for my Q_N, where

$$(56) \qquad \phi(N) = q_1^{n_1-1}(q_1 - 1) q_2^{n_2-1}(q_2 - 1) \cdots q_k^{n_k-1}(q_k - 1)$$

in the Euler ϕ-function.]

By (55) and Theorem 2, if a prime p divides Q_N, it is either a factor of F_N which is prime to every $F_{N_{(1)}}$, or it also divides some $F_{N_{(1)}}$. In the former case, by Lemma 8, since $\alpha(p)$ divides N, but no $N_{(1)}$, necessarily $N = \alpha(p)$, and if $N \ne 5$, $p \ne 5$ and p is a spf of F_N. In the latter case, by Theorem 2, p is a mpf of F_N, and $\text{pot}_p Q_N = 1$, except if $N = 6$ (when $Q_N = Q_6 = F_6 F_1 / F_2 F_3 = 4$.)

Lemma 21. If N satisfies (52) and

$$(57) \qquad\qquad Q_N > q_1 q_2 \cdots q_k \quad,$$

then there is a prime p such that $N = \alpha(p)$.

[As explained above, if $N = 6$, $Q_N = 4 < 2 \cdot 3$, so this case does not arise. Thus $\text{pot}_p Q_N = 0$ or 1 and $Q_N / q_1 q_2 \cdots q_k$ cannot be divisible by any q_i. Thus if this quotient exceeds one, Q_N must be divisible by some prime other than the q_i, and such a prime p has $N = \alpha(p)$.]

Lemma 22. If N satisfies (52) and $k \ge 4$, then (57) holds.

[Since R_h has $\binom{k}{h}$ Fibonacci-number factors, and since

$$\sum_{h=0}^{k} \binom{k}{h} = (1 + 1)^k = 2^k \quad\text{and}\quad \sum_{h=0}^{k} (-1)^h \binom{k}{h} = (1 - 1)^k = 0 \quad,$$

we see that the numerator and denominator of Q_N, by (55), each has 2^{k-1} Fibonacci-number factors. Also, by (4), if $a = (1/2)(\sqrt{5} + 1)$ and $b = (1/2)(\sqrt{5} - 1)$ so that $a > 1 > b = 1/a$ [Carmichael writes α and $-\beta$ for my a and b],

$$(58) \quad a^n(1 - b^2) \le a^n(1 - b^{2n}) \le \sqrt{5} F_n \le a^n(1 + b^{2n}) \le a^n(1 + b^2) \quad.$$

Therefore, since $(1 - b^2)/(1 + b^2) = 1/\sqrt{5}$ and by (55) and (58), $Q_N \ge a^f (1/\sqrt{5})^{2^{k-1}}$, where, by (56),

$$f = N - \Sigma N_{(1)} + \Sigma N_{(2)} - \cdots = N \left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right) \cdots \left(1 - \frac{1}{q_k}\right) = \phi(N);$$

so that

$$(59) \qquad Q_N \geq a^{\phi(N)}(1/\sqrt{5})^{2^{k-1}} \geq a^{(q_1-1)(q_2-1)\cdots(q_k-1)}(1/\sqrt{5})^{2^{k-1}} .$$

Clearly $(q_1 - 1)(q_2 - 1) \cdots (q_k - 1)$ exceeds the value when we put $q_1 = 2$ and $q_i = 2i - 1$ $(i \geq 2)$, namely $q^{k-1}(k - 1)!$. The function

$$2^k + \sum_{i=1}^{k}(q_i - 1)$$

increases more slowly with each q_i than does the product, and its value at the minimal point is $2^k + k^2 - k + 1$. If $k \geq 4$, this is seen to be less than $2^{k-1}(k - 1)!$. Thus, by (59),

$$(60) \qquad Q_N \geq \left\{ \prod_{i=1}^{k} a^{q_i-1} \right\} (a^2/\sqrt{5})^8$$

The function a^{n-1}/n has a minimum for integer values of n when $n = 2$, and it exceeds one when $n \geq 4$. Thus, by (60),

$$(61) \quad Q_N / q_1 q_2 \cdots q_k \geq (a/2)(a^2/3)(a^4/5)(a^6/7)(a^2/\sqrt{5})^8 = a^{29}/131250 > 8 ,$$

and the lemma follows.]

Lemma 23. If N satisfies (52) and $k = 3$, then (57) holds if at least one $q_i \geq 11$, or if no $q_i = 2$, or if any $n_i \geq 2$.

[As in the proof of Lemma 21, (59) still holds. Now, if we suppose that $q_1 < q_2 < q_3$, we see that $q_1 \geq 2$, $q_2 \geq 3$, and, by the first supposition of the lemma, $q_3 \geq 11$. Thus

(62)

$$(q_1 - 1)(q_2 - 1)(q_3 - 1) = (q_1 - 1)(q_2 - 1)(q_3 - 2) + (q_1 - 1)(q_2 - 2) + (q_1 - 1)$$

$$\geq 2(q_3 - 2) + (q_2 - 2) + (q_1 - 1) \geq (q_1 - 1) + (q_2 - 1) + (q_3 - 1) + 7 ;$$

and so, by (59) and (62), as before,

$$(63) \qquad Q_N / q_1 q_2 q_3 \geq (a/2)(a^2/3)(a^{10}/11)(a^7/5^2) = a^{20}/1650 > 9 \ ,$$

and (57) follows.

Adopting the second supposition, we have $q_1 \geq 3$, $q_2 \geq 5$, and $q_3 \geq 7$
Then (62) is replaced by

$$(64) \quad (q_1 - 1)(q_2 - 1)(q_3 - 1) \geq 8(q_3 - 2) + 2(q_2 - 2) + q_1 - 1 \geq (q_1 - 1) + (q_2 - 1) +$$
$$+ (q_3 - 1) + 36 \ ,$$

and (63) by

$$(65) \qquad Q_N / q_1 q_2 q_3 \geq (a^2/3)(a^4/5)(a^6/7)(a^{36}/5^2) = a^{48}/2625 > 10^6 \ ,$$

and (57) follows again.

Finally, if any $n_i \geq 2$, $\phi(n) \geq 2(q_1 - 1)(q_2 - 2)(q_3 - 1)$. Thus, as before,
$q_1 \geq 2$, $q_2 \geq 3$, $q_3 \geq 5$, and $2(q_1 - 1)(q_2 - 1)(q_3 - 1) \geq (q_1 - 1) + (q_2 - 1) +$
$(q_3 - 1) + 9$; whence

$$(66) \qquad Q_N / q_1 q_2 q_3 \geq (a/2)(a^2/3)(a^4/5)(a^9/5^2) = a^{16}/750 \approx 2.9$$

and we get (57).]

Lemma 24. If N satisfies (52) and $k = 2$, then (57) holds if $N/q_1 q_2 \geq 3$, or if at least one $q_i \geq 11$.
[Let $N/q_1 q_2 = r$. Then by (55), $Q_N = F_{q_1 q_2 r} F_r / F_{q_2 r} F_{q_1 r}$, and by (8),

$$(67) \qquad Q_N = \sum_{h=1}^{q_1} {}' \binom{q_1}{h} F_{q_2 r}^{h-1} F_{q_2 r - 1}^{q_1 - h} F_h \Bigg/ \sum_{h=1}^{q_1} \binom{q_1}{h} F_r^{h-1} F_{r-1}^{q_1 - h} F_h \ ,$$

whence, by Lemma 2, $Q_N \geq (F_{q_2 r - 1} / F_r)^{q_1 - 1}$. Thus, by (58)

$$(68) \qquad Q_N / q_1 q_2 \geq \left\{ a^{(q_2 - 1)r - 1} (1 - b^{\frac{2}{3}(q_2 r - 1)})/(1 + b^{2r}) \right\}^{q_1 - 1} \Big/ q_1 q_2 \qquad .$$

First we assume that $r \geq 3$. Then, by the kind of argument used above, if $q_1 \geq 3$ and $q_2 \geq 2$, and by (68),

$$(69) \quad Q_N /q_1 q_2 \geq (a^{q_1-3}/q_1)(a^{3q_2-4}/q_2)\left\{a(1 - b^{10})/(1 + b^6)\right\}^{q_1-1}$$
$$> a^4(0.94)^2/6 > 1 \ .$$

Next, we assume that $q_1 \geq 2$, $q_2 \geq 11$, $r \geq 1$. Then, by (68),

$$(70) \quad Q_N /q_1 q_2 \geq (a^{8q_1-17}/q_1)(a^{q_2-2}/q_2)\left\{a(1 - b^{20})/(1 + b^2)\right\}^{q_1-1} > a^9(0.72)/22 > 2.$$

The results (69) and (70) establish the lemma.]

   Lemma 25. If q is prime and $q \geq 3$, then there is a prime p such that $q = \alpha(p)$.
[If $q \geq 3$, $F_q \geq 2$, by Lemma 2, and so F_q has a prime factor p. By Lemma 8, $\alpha(p)$ divides q, whence, since q is prime, $\alpha(p) = q$.]

   Lemma 26. If q is prime and $\lambda \geq 2$, then there is a prime $p \neq 5$, such that $q^\lambda = \alpha(p)$.
[By Lemma 16 and Theorem 1, if $q^{\lambda-1} = m$, $(F_{qm}/F_m, F_m) = 1$ if $q \neq 5$; and if $5^{\lambda-1} = m$, $(F_{5m}/F_m, F_m) = 5$. If $q \neq 5$, by Lemma 17, $F_{qm}/F_m \geq F_4/F_2 = 3$; so that F_{qm} must have a prime factor $p \neq 5$, prime to F_m. If $q = 5$, since $F_{25}/5F_5 = 3001$, by (2), Lemma 17 shows that again F_{qm} has a prime factor $p \neq 5$, prime to F_m. Thus, by Lemma 8, for any $q, \alpha(p)$ divides $qm = q^\lambda$ but not $m = q^{\lambda-1}$. Therefore $q^\lambda = \alpha(p)$.]

   We now have sufficient information to prove Carmichael's theorem.

   Theorem 6. If $N \neq 1, 2, 6$, or 12, then there is a prime p such that $N = \alpha(p)$.

   Proof. Let the (unique) prime-power expansion of N be given by (52). By Lemma 21, Lemmas 22, 23, and 24 show that the theorem holds in the following cases: (i) if $k \geq 4$, all N; (ii) if $k = 3$ and either (a) one $q_i \geq 11$, (b) no $q_i = 2$, or (c) one $n_i \geq 2$; and (iii) if $k = 2$ and either (a) $N/q_1 q_2 \geq 3$ or (b) one $q_1 \geq 11$. In addition, Lemmas 25 and 26 show that the theorem holds (iv) if $k = 1$ and $N \neq 2$. We see from (2) that, indeed, when $N = 1, 2$,

6, or 12, there is no prime p such that $N = \alpha(p)$. It therefore remains to show that such a p exists, (v) when $k = 3$, no $q_i \geq 11$, one $q_i = 2$, and no $n_i \geq 2$, and (vi) when $k = 2$, $N \neq 6$ or 12, $N/q_1q_2 = 1$ or 2, and no $q_i \geq 11$. We look for primes p which divide F_N but no corresponding $F_{N_{(1)}}$, for then $N = \alpha(p)$, as explained earlier.

Case (v). We have $N = 2 \cdot 3 \cdot 5 = 30$, $2 \cdot 3 \cdot 7 = 42$, and $2 \cdot 5 \cdot 7 = 70$. We see from (2) that $30 = \alpha(31)$, $42 = \alpha(211)$, and $70 = \alpha(71) = \alpha(911)$; so that the theorem holds.

Case (vi). We have $N = 2 \cdot 5 = 10$, $2^2 \cdot 5 = 20$, $2 \cdot 7 = 14$, $2^2 \cdot 7 = 28$, $3 \cdot 5 = 15$, $3 \cdot 7 = 21$, and $5 \cdot 7 = 35$. We see from (2) that $10 = \alpha(11)$, $20 = \alpha(41)$, $14 = \alpha(29)$, $28 = \alpha(281)$, $15 = \alpha(61)$, $21 = \alpha(421)$, and $35 = \alpha(141961)$. This completes the theorem.

Lemma 27. If $N = \alpha(p)$ and $N \neq 5$ whence $p \neq 5$), p is a simple prime factor of F_N.
[By Lemma 18, if p is a mpf of F_N, p divides both N and F_N. Thus, since, by Theorem 1, if $p \neq 5$, $(p, \alpha(p)) = 1$; N must be divisible by $p\alpha(p)$, so that $N \neq \alpha(p)$. The lemma follows.]

By Lemma 27, Theorem 5 is seen to follow from Theorem 6. We also see that Theorem 3 and its corollaries follow from Theorems 1, 2, and 6 (with the exception of the fact that the $\ell_s(m,p)$ increase with s).

For completeness, we also state the following result.

Lemma 28. If $f_1 = 1$, $f_2, f_3, \cdots, f_m = N$ are all the divisors of N, then

(71)
$$F_N = \prod_{r=1}^{m} Q_{f_r} .$$

[If N satisfies (52), its divisors are the $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1) = m$ integers

$$f = q_1^{s_1} q_2^{s_2} \cdots q_k^{s_k} ,$$

where $0 \leq s_i \leq n_i$, $i = 1, 2, \cdots, k$. By (55), a particular factor F_g can appear only once in Q_f; and this, when

$$g = q_1^{t_1} q_2^{t_2} \cdots q_k^{t_k}$$

and $t_i = n_i$ except when $i = i_1, i_2, \cdots, i_h$ (when $t_i < n_i$), only if $f = g$ or gq_{i_1} or $gq_{i_1} q_{i_2}$ or $\cdots$ or $gq_{i_1} q_{i_2} \cdots q_{i_h}$. It follows by (55) that F_g appears in

$$\prod_{r=1}^{m} Q_{f_r}$$

to the total power

$$1 - \binom{h}{i} + \binom{h}{2} - \cdots + (-1)^h \binom{h}{h} = (1 - 1)^h = 0 \text{ if } h \geq 1 \text{ ,}$$

and 1 if $h = 0$. This proves that the product is simply F_N.]

5. PERIODICITY OF RESIDUES

We shall complete this discussion of divisibility properties with a survey of results pertaining to the characteristic number $\mu(m,n)$ defined in Section 1.

Lemmas 13 and 14 show that we may limit the study of the functions $\alpha(m,n)$ and $\mu(m,n)$ to that of $\alpha(p,n)$ and $\mu(p,n)$, where p is prime. We have established the essential properties of $\alpha(p,n)$ in Theorem 1. Thus, by (15), the corresponding behaviour of $\mu(p,n)$ is known if we know that of $\beta(p,n)$. So far, we have only stated, in Lemma 10, that $\beta(m)$ (and, in particular, $\beta(p,n)$) is always an integer. The papers of Robinson [5], Vinson [6], and Wall [7] have answered almost every question that may be asked about $\beta(p,n)$, and it is their work which will be outlined here. Proofs of all the results quoted below may be found in Vinson's paper [6], and so will be omitted here.

Theorem 7. If p is an odd prime and n a positive integer, then

$$(72) \qquad \beta(p,n) = \begin{cases} 4 & \text{if } \text{pot}_2 \alpha(p) = 0 \\ 1 & \text{if } \text{pot}_2 \alpha(p) = 1 \\ 2 & \text{if } \text{pot}_2 \alpha(p) \geq 2 \end{cases} \qquad ;$$

but

(73) $\beta(2,1) = \beta(2,2) = 1$,   and $\beta(2,n) = 2$ if $n \geq 3$.

We note that, with the two exceptions given in (73), $\beta(p,n)$ is independent of n. Also, $\beta(p,n)$ always takes one of the three values 1, 2, or 4 — a remarkably simple result.

Theorem 8. If m is a positive integer satisfying (27), then (i) $\beta(m) = 4$, if $m \geq 3$ and $\alpha(m)$ is odd; (ii) $\beta(m) = 1$, if $\text{pot}_2\alpha(p_i) = 1$ for every $p_i \neq 2$ ($i = 1, 2, \cdots, k$) and if $\text{pot}_2 m \leq 2$; and (iii) $\beta(m) = 2$ for all other m.

We note that Theorem 8 contains Theorem 7, as a special case, when $m = p^n$, where p is prime. (The connection is through Lemma 13.)

Theorem 9. If p is an odd prime, not equal to 5, and n a positive integer, then

(74) $$\beta(p,n) = \begin{cases} 1 \text{ if } p \equiv 11 \text{ or } 19 \pmod{20} \\ 2 \text{ if } p \equiv 3 \text{ or } 7 \pmod{20} \\ 4 \text{ if } p \equiv 13 \text{ or } 17 \pmod{20} \end{cases} ;$$

and (of the remaining values of $p \equiv 1$ or $9 \pmod{20}$)$\beta(p,n) \neq 2$ if $p \equiv 21$ or 29 (mod 40).

These results are connected with the foregoing by way of Lemma 12. Vinson points out that the theorem is "complete" in the sense that every remaining possibility occurs; he lists the examples:

(75)
$$\begin{cases} \beta(521) = 1, \ \beta(41) = 2, \ \beta(761) = 4, \ [p \equiv 1 \pmod{40}] ; \\ \beta(809) = 1, \ \beta(409) = 2, \ \beta(89) = 4, \ [p \equiv 9 \pmod{40}] ; \\ \beta(101) = 1, \ \beta(61) = 4, \ \ \ \ \ \ \ \ \ \ \ \ [p \equiv 21 \pmod{40}] ; \\ \beta(29) = 1, \ \beta(109) = 4, \ \ \ \ \ \ \ \ \ \ [p \equiv 29 \pmod{40}] . \end{cases}$$

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GENERALIZED FIBONACCI SEQUENCES
ASSOCIATED WITH A GENERALIZED PASCAL TRIANGLE

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1. INTRODUCTION

In this paper we introduce the numbers

$$(1.1) \quad \begin{cases} u_n = u_n(p,q,s) = \displaystyle\sum_{i=0}^{\left[\frac{n}{p+sq}\right]} \left(\begin{bmatrix} \left[\frac{n-ip}{s}\right] \\ i \quad q \end{bmatrix} \right) & n = 1, 2, \cdots \\ u_0 = u_0(p,q,s) = 1 \end{cases}$$

where n, p, q, s are positive integers and $[x]$ is the largest integer in x. The characteristic equation and a generating function are developed and the relation to a generalized Pascal's triangle is exhibited in Section 2. An interesting feature is the repetition of each term g times where g = (p,s). Certain sums and some properties relating to congruence are established in Sections 3 and 4.

The numbers corresponding to the case s = 1 are developed in our previous paper [2]. Thus the Fibonacci numbers are those for p = q = s = 1. The numbers in Dickinson [1] are the special case p = a, q = 1, s = c - a. By multiplying the binomial coefficients

$$\left(\begin{bmatrix} \left[\frac{n-ip}{s}\right] \\ i \quad q \end{bmatrix} \right)$$

by $a^{n-iq}b^{iq}$* before summing, the numbers could be generalized further.

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*A more appropriate choice of exponents, suggested by Dr. David Zeitlin, appears in a paper by him which will follow.

2. THE CHARACTERISTIC EQUATION AND GENERATING FUNCTION

We note that

$$\left(\begin{bmatrix} \dfrac{n+s-ip}{s} \\ i \quad q \end{bmatrix}\right) - \left(\begin{bmatrix} \dfrac{n-ip}{s} \\ i \quad q \end{bmatrix}\right) = \left(\begin{bmatrix} \dfrac{n-ip}{s} \\ iq-1 \end{bmatrix}\right)$$

or zero, from properties of binomial coefficients. Hence, if $Ef(x) = f(x+1)$ we have $(E^s - 1)u_n$ is a sum of binomial coefficients with first coefficient involving $iq - 1$. After repeating q times there results

$$(E^s - 1)^q u_n (p,q,s) = u_{n-p}(p,q,s), \quad n - p \geq 0$$

or

$$(2.1) \quad u_{n+p+qs} = \binom{q}{1} u_{n+p+(q-1)s} - \binom{q}{2} u_{n+p+(q-2)s} + \cdots + (-1)^{q+1} u_{n+p} + u_n$$

Hence the characteristic equation is

$$(2.2) \qquad\qquad x^p(x^s - 1)^q - 1 = 0$$

with initial conditions

$$(2.3) \qquad\qquad u_0 = u_1 = \cdots = u_{p+qs-1} = 1 \ .$$

It may be remarked that $u_{p+qs} = 2.$

Suppose the arithmetic triangle to be written but with each row repeated s times. Then one sees that $u_n(p,q,s)$ is the sum of the term in the first column and n^{th} row (counting the top row as the zeroth row) and the terms obtained by starting from this term and taking steps p, q — that is, p units up and q units to the right.

When $(p,s) = g > 1$ the sequences $\{u_{ng}\}$, $\{u_{ng+1}\}, \cdots, \{u_{ng+(g-1)}\}$ are the same since each sequence is determined by the same recursion formula and the same initial conditions.

Let $f(x) = x^p(x^s - 1)^q - 1$ so that $f'(x) = x^{p-1}(x^s - 1)^{q-1}[(p + qs)x^s - p]$. The roots of $f'(x) = 0$ are the roots of

$$x = 0, \quad x^s = 1 \quad \text{and} \quad x^s = \frac{p}{p + qs} \quad .$$

None of the roots of $f'(x) = 0$ is a root of $f(x) = 0$ and $f(x)$ has no multiple root. If the $p + sq$ roots of $f(x)$ are $x_1, x_2, x_3, \cdots, x_{p+sq}$ then the determinant of the coefficients $c_1, c_2, \cdots, c_{p+qs}$ in

$$\sum_{i=1}^{p+qs} c_i x_i^{n+1} = u_n \qquad n = 0, 1, \cdots, p + sq - 1$$

is different from zero. The system can be solved by Cramer's rule using Vandermondians. It results that $c_i = (x_i^s - 1)/[(x_i - 1)\{[p + sq]x_i^s - p\}]$ and hence

$$(2.4) \qquad u_n = \sum_{i=1}^{p+sq} \frac{(x_i^s - 1)x_i^{n+1}}{(x_i - 1)[(p + sq)x_i^s - p]} \quad , \quad n = 0, 1, 2, \cdots$$

To obtain a generating function, write

$$S = \sum_{i=0}^{\infty} u_i x^i \quad .$$

Then by multiplying S by each of

$$(-1)\binom{q}{1}x^s, \quad (-1)^2\binom{q}{2}x^{2s}, \cdots, (-1)^q\binom{q}{q}x^{qs} \quad \text{and} \quad -x^{p+qs}$$

and adding, one finds

$$[(1 - x^s)^q - x^{p+sq}] S = \sum_{k=0}^{q-1} \sum_{i=0}^{s-1} \sum_{j=0}^{k} (-1)^j \binom{q}{j}x^{ks+i} \quad .$$

Note we have used $u_0 = u_1 = \cdots = u_{p+sq-1} = 1$ and (2.1). Hence

$$\sum_{n=0}^{\infty} u_n x^n = \frac{\displaystyle\sum_{k=0}^{q-1}\sum_{i=0}^{s-1}\sum_{j=0}^{k} (-1)^j \binom{q}{j} x^{ks+i}}{(1 - x^s)^q - x^{p+sq}}$$

The numerator is equal to

$$\sum_{i=0}^{s-1} x^i \left\{ \sum_{k=0}^{q-1} (-1)^k \binom{q-1}{k} (x^s)^k \right\} = \sum_{i=0}^{s-1} x^i (1 - x^s)^{q-1}$$

$$= (1 - x^s)^q / (1 - x)$$

Hence

$$(2.5) \qquad \sum_{n=0}^{\infty} u_n x^n = \frac{(1 - x^s)^q / (1 - x)}{(1 - x^s)^q - x^{p+sq}}$$

As an example, for p = 2, q = 2, s = 3, this gives

$$\sum_{n=0}^{\infty} u_n(2,2,3) x^n = \frac{1 + x + x^2 - x^3 - x^4 - x^5}{1 - 2x^3 + x^6 - x^8}$$

This gives the sequence

$$\{u_n(2,2,3)\} = 1,1,1,1,1,1,1,1,2,2,2,4,4,4,7,7,8,12,12,16,21,21,31,37,\cdots$$

3. SUMS

$$(3.1) \qquad \sum_{i=0}^{n} u_i = \sum_{i=0}^{q-1} \left\{ (-1)^i \binom{q-1}{i} \left[u_{n+p+s(q-i)} + \right.\right.$$

$$u_{n+p+s(q-i)-1} + \cdots + u_{n+p+s(q-i)-s+1} \right] \right\} - s\delta_{1q}$$

where δ_{ij} is Kronecker's δ.

This is seen to be true for $q = 1$ and all n by summing $u_0 = u_{p+s} - u_p$; $u_1 = u_{p+s+1} - u_{p+1}$; $\cdots$; $u_n = u_{p+s+n} - u_{p+n}$. Since $u_0 = u_1 = \cdots = u_{p+sq-1} = 1$, this gives

$$\sum_{i=0}^{n} u_i = \sum_{i=0}^{n+p+s} u_i - (p + s) - \sum_{i=0}^{n+p} u_i + p$$

which is the result. Also this is true for $n = 0$ and all q. We have to show

$$u_0 = \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} [u_{p+s(q-i)} + \cdots + u_{p+s(q-i)-(s-1)}]$$

But $u_{p+sq} = 2$ so that by separating the term corresponding to $i = 0$ we get $1 + s(1 - 1)^{q-1} = 1 = u_0$.

It remains to show the result in general. Assume (3.1) to be true for $q \geq 2$ and $n = k$; then

$$\sum_{i=0}^{k+1} u_i = u_{k+1} + \sum_{i=0}^{k} u_i = \sum_{i=0}^{q} (-1)^i \binom{q}{i} u_{k+p+(q-i)s+1} +$$

$$\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} [u_{k+p+s(q-i)} + u_{k+p+s(q-i)-1} + \cdots$$

$$+ u_{k+p+s(q-i)-s+1}]$$

By combining terms

$$u_{k+1+p+s(q-i)} = u_{k+p+s(q-i)+1}$$

using

$$\binom{q}{j} - \binom{q-1}{j-1} = \binom{q-1}{j} ,$$

the result follows and the theorem is proved.

$$
\sum_{i=0}^{n} (-1)^{n-i} u_i = \begin{cases} \displaystyle\sum_{k=0}^{q} \sum_{i=n+1}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i \ , & s \text{ even} \\[3em] \dfrac{1}{1 - (-1)^{p+s} q_2^{\,q}} \left[\displaystyle\sum_{k=0}^{q} \sum_{i=n+1}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i \ + \right. \end{cases}
$$

$$
\left. + \begin{cases} 0 & p \equiv q \equiv 1 \pmod 2 \\ (-1)^{n+1}\, 2^{q-1} & p \equiv q \equiv 0 \pmod 2 \\ (-1)^{n}\, 2^{q-1} & p \not\equiv q \pmod 2 \end{cases} \right] \quad s \text{ odd}
$$

(3.2)

Proof: Solve (2.1) for u_n and write $(-1)^{n-i} u_i$ for $i = n, n-1, \cdots, 0$. The sum of the $(k+1)$st column formed by the expansions is

$$
\sum_{i=0}^{n} (-1)^{k+i} \binom{q}{k} u_{n+p+s(q-k)-i} = \sum_{i=0}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)-i+k} \binom{q}{k} u_i
$$

$$
- (-1)^{k+n+1} \binom{q}{k} \left[1 - 1 + \cdots + (-1)^{p+s(q-k)-1} \right]
$$

since the terms added to obtain the sum on the right have $u_i \equiv 1$ in each case. Hence this gives

$$
\sum_{i=0}^{n} (-1)^{k+i} \binom{q}{k} u_{n+p+s(q-k)-i} = \sum_{i=0}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i
$$

$$
+ (-1)^{n+k} \binom{q}{k} \cdot \epsilon
$$

where $\epsilon = 0$, $p + s(q-k) \equiv 0 \pmod 2$ and $= 1$ otherwise.

Summing for $k = 0, 1, \cdots, q$ gives

$$\sum_{i=0}^{n} (-1)^{n-i} u_i = \sum_{k=0}^{q} \sum_{i=n+1}^{n+p+s(q-k)} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i$$

$$+ \sum_{k=0}^{q} \sum_{i=0}^{n} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i +$$

$$\begin{cases} \sum_{k=0}^{q} (-1)^{n+k} \binom{q}{k} \epsilon(k) , & p + s(q - k) \not\equiv 0 \pmod 2 \\ \\ 0 & , \quad p + s(q - k) \equiv 0 \pmod 2 \end{cases}$$

But

$$\sum_{k=0}^{q} \sum_{i=0}^{n} (-1)^{n+p+s(q-k)+k-i} \binom{q}{k} u_i =$$

$$\begin{cases} 0 , & s \equiv 0 \pmod 2 \\ \\ (-1)^{p+sq} 2^q \sum_{i=0}^{n} (-1)^{n-i} u_i , & s \equiv 1 \pmod 2 \end{cases}$$

and

$$\sum_{n=0}^{q} (-1)^{n+k} \binom{q}{k} \epsilon(p,q,s,k) =$$

$$\begin{cases} (-1)^{n+1} 2^{q-1} , & s \text{ odd } p \equiv q \equiv 0 \pmod 2 \\ (-1)^{n} 2^{q-1} , & s \text{ odd } \quad p \not\equiv q \pmod 2 \\ 0 , & \text{otherwise} \end{cases}$$

Combining these results gives the theorem.

It may be remarked that

$$\sum_{i=0}^{n} u_{2i} \quad \text{and} \quad \sum_{i=0}^{n} u_{2i+1}$$

can be obtained from (3.1) and (3.2).

4. DIVISIBILITY PROPERTIES

Using methods similar to those of our previous paper, one can show the following: Any $p + sq$ consecutive terms are relatively prime. The least nonnegative residues modulo any positive integer m of $u_n(p, q, s)$ are periodic with a period P not exceeding m^{p+sq}. There is no preperiod and each period begins with $p + sq$ terms all unity. Any prime divides infinitely many $u_n(p, q, s)$ since

$$u_{P-p} \equiv \sum_{i=0}^{q} (-1)^i \binom{q}{i} u_{P+s(q-i)} \equiv 0 \pmod{m}.$$

REFERENCES

1. David Dickinson, "On Sums Involving Binomial Coefficients," *American Mathematical Monthly*, Vol. 57, 1950, pp 82 − 86.

2. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 2, 1964, pp 277 − 289.

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CORRIGENDUM FOR "SOME CONVERGENT RECURSIVE SEQUENCES, HOMEOMORPHIC IDENTITIES, AND INDUCTIVELY DEFINED COMPLEMENTARY SEQUENCES"

John C. Holladay, Institute for Defense Analysis, Washington, D.C.

On the above-entitled paper, appearing in the February 1966 volume of the Fibonacci Quarterly, please note the following changes:

Page 13. The last two lines of the Corollary should read:

... and only one homeomorphism g such that $g \geq 1$ and

$$(2.9) \qquad h + h^{-1} = g + g^{-1} \ .$$

Page 14. Equation (2.27) should read:

$$(2.27) \qquad (h \cup h^{-1})(t) = t \text{ for all } (h \cup h^{-1})^{-1}(x) \leq t \leq x.$$

Equation (2.30) should read:

$$(2.30) \qquad h_{n+1} = P - h_n^{-1} \qquad n > 1.$$

Equation (2.31) should read:

$$(2.31) \qquad h \left\uparrow \left(\frac{\alpha + \sqrt{\alpha^2 - 4}}{2} \right) I \ .$$

Page 15. Equation (2.36) should read

$$(2.36) \qquad h \left\downarrow \left(\frac{\beta + \sqrt{\beta^2 - 4}}{2} \right) I \ .$$

Page 16. Equations (2.39) and (2.40) should read as follows:

$$(2.39) \qquad v = \lim_{n \to \infty} v_n = \left(\frac{\beta + \sqrt{\beta^2 - 4}}{2} \right)$$

$$(2.40) \qquad h = \left(\frac{\alpha + \sqrt{\alpha^2 - 4}}{2} \right) I$$

Page 21. After proof for the Corollary, add a Reference [5].

Page 23. The first line of the Corollary should read:

Corollary: Let $P(n) \neq 2n$ for some integer $n > 0$. Then

Page 24. Change the last line of Theorem 16 to read:

$\{x_n\}$ be inductively defined by

Equation (3.26) should read:

$$(3.26) \qquad x_0 \leq a(x_{-1}) - x_{-1}$$

Page 26. Equation (4.13) should read:

$$(4.13) \qquad F(x) = (\alpha - \sqrt{\alpha^2 - 1})x + \beta - \beta\sqrt{\alpha^2 - 1}/(\alpha - 1)$$

Page 27. Equation (4.19) should read:

$$(4.19) \qquad (x - \beta)\{(\alpha - 1)x + \beta\alpha + \beta\} > \alpha^2 \epsilon/(\alpha + 1)$$

Equation (4.24) should begin with the line

$$(4.24) \qquad h^{-1}(x) = xF(1) \qquad 0 \leq x \leq 1$$

Page 28. The first line on the page should read:

If $\beta \geq 1\frac{1}{2}$, then $F(1) > 0$ implies that $-\epsilon < (\beta - 1)^2$. It may be

Page 29. Equation (5.3) should read:

$$(5.3) \qquad hh(t) < gh(t)$$

Page 33. Change the first line of Theorem 22 to read:

Let $\mu \leq 1$ and $P + \mu I > I$. Let g_1 be any

Page 34. The last three lines before the Corollary should read:

for h_1 has been proven. To prove convergence for g_1, insert μ into the proper positions of (1.39) and (1.40), and continue the argument of the paragraph containing (1.39) and (1.40). Uniqueness of h is obtained from Theorem 21.

Page 36. Add References below.

11. J. Lambek and L. Moser, Inverse and complementary sequences of natural numbers, <u>Amer. Math. Monthly</u>, 61 (1954), 454–458.

12. H. W. Gould, "Non-Fibonacci Numbers," <u>Fibonacci Quarterly</u>, Vol. 3, No. 3, October 1965, pp. 177–183.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

H-89 *Proposed by Maxey Brooke, Sweeny, Texas.*

Fibonacci started out with a pair of rabbits, a male and a female. A female will begin bearing after two months and will bear monthly thereafter. The first litter a female bears is twin males, thereafter she alternately bears female and male.

Find a recurrence relation for the number of males and females born at the end of the nth month and the total rabbit population at that time.

H-90 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Let the total population after n time periods be the sequence $\left\{F_n^3\right\}_{n=2}^{\infty}$ determine the common birth sequence for every female rabbit and tie it in with the value of the Fibonacci polynomials at $x = 2$. ($f_0(x) = 0$, $f_1(x) = 1$ and $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$; $n \geq 0$).

H-91 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Let $m = \left[\dfrac{k}{2}\right]$, then show

$$F_{kn}/F_n = \sum_{j=0}^{m=1} (-1)^{jn} L_{k-1-2j} + e_n \quad ,$$

where

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$$e_n = \begin{cases} (-1)^{mn} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

and $[x]$ is the greatest integer not exceeding x.

H-92 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Prove or Disprove: A part from F_1, F_2, F_3, F_4, no Fibonacci number, F_i $(i > 0)$ is a divisor of a Lucas Number.

H-93 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that

$$F_n = \sum_{k=1}^{\overline{n-1}} \left(3 + 2 \cos \frac{2k\pi}{n} \right)$$

$$L_n = \sum_{k=1}^{\overline{n-2}} \left(3 + 2 \cos \frac{(2k + 1)\pi}{n} \right) \qquad ,$$

where $\overline{n}$ is the greatest integer contained in $n/2$.

SOLUTIONS

H-50 *Proposed by Ralph Greenberg, Philadelphia, Pa. and H. Winthrop, University of So. Florida, Tampa, Fla.*

Show

$$\sum_{n_1+n_2+n_3+\cdots+n_i=n} \Pi n_i = F_{2n}$$

where the sum is taken over all partitions of n into positive integers and the order of distinct summands is considered.

A paper by D. A. Lind and V. E. Hoggatt, Jr. , "Composition Formulas Derived from Birth Sequences," will appear soon in the Fibonacci Quarterly, and will discuss this among many other examples.

H-22 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

If

$$P(x) = \prod_{i=1}^{\infty} (1 + x^{F_i}) = \sum_{n=0}^{\infty} R(n) x^n \quad ,$$

then show

(i) $\qquad R(F_{2n} - 1) = n$

(ii) $\quad R(N) > n$ if $N > F_{2n} - 1$.

H-53 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*
  and S. L. Basin, Sylvania Electronics Systems, Mt. View, Calif.

The Lucas sequence $L_1 = 1$, $L_2 = 3$; $L_{n+2} = L_{n+1} + L_n$ for n 1 is incomplete (see V. E. Hoggatt, Jr. and C. King, Problem E-1424 American Monthly, Vol. 67, No. 6, June–July 1960, p. 593, since every integer n is not the sum of distinct Lucas numbers. OBSERVE THAT 2, 6, 9, 13, 17, $\cdots$ cannot be so represented. Let M(n) be the number of positive integers less than n which cannot be so represented. Show

$$M(L_n) = F_{n-1} \qquad \circ$$

Find, if possible, a closed form solution for M(n).

A paper by David Klarner to appear soon in the Fibonacci Quarterly completely answers these questions.

H-55 *Proposed by Raymond Whitney, Lock Haven State College, Lock Haven, Pa.*

Let $F(n)$ and $L(n)$ denote the nth Fibonacci and n^{th} Lucas numbers, respectively.

Given $U(n) = F(F(n))$, $V(n) = F(L(n))$, $W(n) = L(L(n))$ and $X(n) = L(F_n)$, find recurrence relations for the sequences $U(n)$, $V(n)$, $W(n)$, and $X(n)$.

A paper by student Gary Ford to appear soon in the Fibonacci Quarterly offers several answers to this problem. Also a paper by R. Whitney deals with this and will appear shortly.

H-52   *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Prove that the value of the determinant

$$\begin{vmatrix} u_n^2 & u_{n+2}^2 & u_{n+4}^2 \\ u_{n+2}^2 & u_{n+4}^2 & u_{n+6}^2 \\ u_{n+4}^2 & u_{n+6}^2 & u_{n+8}^2 \end{vmatrix}$$

is $18(-1)^{n+1}$.

Solution by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Since $F_{2n}^2 = (L_{4n} - 2)/5$, the auxiliary polynomial satisfied by F_{2n}^2 is the product of the auxiliary polynomials for L_{4n}: $(x^2 - 7x + 1)$ and for $C_n = -2$: $(x - 1)$ or

$$(x^2 - 7x + 1)(x - 1) = x^3 - 8x^2 + 8x - 1 \ .$$

Therefore the recurrence relation for F_{2n}^2 is

$$u_{n+6}^2 = 8u_{n+4}^2 - 8u_{n+2}^2 + u_n^2 \ .$$

Thus $D_{n+1} = (-1)D_n$ by using the above recurrence relation after multiplying the first column of D_n by -1. The value of D_0 is -18, therefore $D_n = 18(-1)^{n+1}$.

TWO BEAUTIES

H-47   *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{n=0}^{\infty} \binom{n + k - 1}{n} L_n x^n = \frac{\psi_k(x)}{(1 - x - x^2)k} \quad ,$$

where

$$\psi_k(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} L_r x^r \quad .$$

H-51 *Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*
and L. Carlitz, Duke University, Durham, N.C.

Show that if

(i) $$\frac{xt}{1 - (2 - x)t + (1 - x - x^2)t^2} = \sum_{k=1}^{\infty} Q_k(x) t^k$$

and

(ii) $$\sum_{n=0}^{\infty} \binom{n + k - 1}{n} F_n x^n = \frac{\phi_k(x)}{(1 - x - x^2)^k}$$

that

$$\phi_k(x) = \sum_{r=0}^{k} (-1)^{r+1} \binom{k}{r} F_r x^r = Q_k(x)$$

Solutions by Kathleen Weland, Gary Ford, and Douglas Lind, Undergraduate Research Program, University of Santa Clara, Santa Clara, Calif.

It is familiar that

$$(1 - x)^{-k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{n} x^n .$$

Let W_n obey $W_{n+2} = pW_{n+1} - qW_n$, $p^2 - 4q \neq 0$, and let $a \neq b$ both satisfy $x^2 - px + q = 0$, so that $a + b = p$, $ab = q$. Then $W_n = Aa^n + Bb^n$ for some constants A and B and all n. It follows that

$$\sum_{n=0}^{\infty} \binom{n + k - 1}{n} W_n x^n = A(1 - ax)^{-k} + B(1 - bx)^{-k}$$

$$= \frac{A(1 - bx)^k + B(1 - ax)^k}{(1 - px + qx^2)^k}$$

$$= \frac{\left[A\sum_{j=0}^{k}(-1)^j \binom{k}{j} b^j x^j + B\sum_{j=0}^{k}(-1)^j \binom{k}{j} a^j \right]}{(1 - px + qx^2)^k}$$

$$= \frac{\left[\sum_{j=0}^{k}(-1)^j \binom{k}{j} (Aa^{-j} + Bb^{-j})(abx)^j \right]}{(1 - px + qx^2)^k}$$

$$= \frac{\left[\sum_{j=0}^{k}(-q)^j \binom{k}{j} W_{-j} x^j \right]}{(1 - px + qx^2)^k} = \frac{R_k(x)}{(1 - px + qx^2)^k} .$$

To get (ii) of H–51 we put $p = 1, q = -1$, $W_n = F_n$, and recalling $F_{-n} = (-1)^{n+1} F_n$ we have

$$\sum_{n=0}^{\infty} \binom{n + k - 1}{n} F_n x^n = \left[\sum_{j=0}^{k}(-1)^{j+1} \binom{k}{j} F_j x^j \right] (1 - x - x^2)^k .$$

To obtain H-47, we set $p = 1$, $q = -1$, $W_n = L_n$, and remembering that $L_{-n} = (-1)^n L_n$ we find

$$\sum_{n=0}^{\infty} \binom{n + k - 1}{n} L_n x^n = \left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} L_j x^j \right] \quad (1 - x - x^2)^k$$

We now generalize (i) of H-51. Since

$$R_k(x) = \sum_{j=0}^{k} (-q)^j \binom{k}{j} W_{-j} x^j$$

$$= A(1 - bx)^k + B(1 - ax)^k \quad ,$$

we have

$$\sum_{k=0}^{\infty} R_k(x) t^k = A \sum_{k=0}^{\infty} \left[(1 - bx)t \right]^k + B \sum_{k=0}^{\infty} \left[(1 - ax)t \right]^k$$

$$= \frac{A}{1 - (1 - bx)t} + \frac{B}{1 - (1 - ax)t}$$

$$= \frac{A + B - (A + B)t + (Aa + Bb)x^t}{1 - (2 - px)t + (1 - px + qx^2)t^2}$$
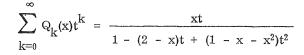
Now $W_0 = A + B$, $W_1 = Aa + Bb$, so we may write

$$(*) \quad \sum_{k=0}^{\infty} R_k(x) t^k = \frac{W_0 + (xW_1 - W_0)t}{1 - (2 - px) t + (1 - px + qx^2)t^2} \quad ,$$

Putting $p = 1$, $q = -1$, $W_n = F_n$ makes $R_k(x) = Q_k(x)$ of Problem H-51, and (*) becomes

$$\sum_{k=0}^{\infty} Q_k(x)t^k = \frac{xt}{1 - (2 - x)t + (1 - x - x^2)t^2}$$

the required result.

Also solved by the proposers.

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LATE PROBLEM ADDITIONS

A SIMPLE PROOF, PLEASE!

H-94 Submitted by Robert W. Floyd, Carnegie Institute of Technology, and Donald E. Knuth, California Institute of Technology.

Let α be any irrational number, and let the notation $\{x\}$ stand for the fractional part of x. Suppose a man has accurately marked off the points 1, 0, $\{\alpha\}$, $\{2\alpha\}$, $\cdots$, $\{(n - 1)\alpha\}$ on a line, $n \geq 1$. These $n + 1$ points divide the line segment between 0 and 1 into n disjoint intervals. Show that when the man adds the next point $\{n\alpha\}$, it falls in the largest of these n intervals; if there are several intervals which have the maximum length, the point $\{n\alpha\}$ falls in one of these maximal intervals. Furthermore, if α is the "golden ratio" $\phi^{-1} = \frac{1}{2}(\sqrt{5} - 1) = 0.618 \cdots$, then the point $\{n\alpha\}$ always divides the corresponding interval into two intervals whose lengths are in the golden ratio. A number α has the property that $\{n\alpha\}$ always divides its interval into two parts, such that the ratio of longer to shorter is less than 2, if and only if $\{\alpha\}$ = ϕ^{-1} or ϕ^{-2}. (Note: The fact that the fractional parts $\{n\alpha\}$ are asymptotically equidistributed in (0, 1) is well known; this problem shows the mechanism behind that theorem, since $\{n\alpha\}$ always chooses the largest remaining open place. Furthermore, the sequence $\{n\phi\}$ is the "most equidistributed" of all these sequences.)

H-95 Proposed by J. A. H. Hunter, Toronto, Canada.

Show

$$F_{n+k}^3 + (-1)^k F_{n-k}^3 = L_k[F_k^2 F_{3n} + (-1)^k F_n^3].$$

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A SINGULAR FIBONACCI MATRIX AND ITS RELATED LAMBDA FUNCTION

Curtis McKnight (student) & Dean Priest, Harding College, Searcy, Arkansas

After a very brief introduction to some of the extremely basic properties of Fibonacci numbers, a student of mine inductively produced the following identities concerning determinants of Fibonacci matrices:

(1)
$$\begin{vmatrix} F_n & F_{n+1} \\ F_{n+s+2} & F_{n+s+3} \end{vmatrix} = (-1)^{n+1} F_{s+2}$$

(2)
$$\begin{vmatrix} F_n & F_{n+m+1} \\ F_{n+m+2} & F_{n+2m+3} \end{vmatrix} = (-1)^{n+1} F_{m+1} F_{m+2}$$

(3)
$$\begin{vmatrix} F_n & F_{n+m+1} \\ F_{n+m+s+2} & F_{n+2m+s+3} \end{vmatrix} = (-1)^{n+1} F_{m+1} F_{m+s+2} \qquad .$$

Each row of the determinant is regarded as a pair of numbers, the subscript s refers to the number of terms in the Fibonacci sequence skipped between successive pairs, and the subscript m refers to the number of terms skipped between the two numbers of a pair.

It is simple exercise to establish the validity of (1), (2), and (3) using $F_{m+n} = F_{n-1} F_m + F_n F_{m+1}$. However, close inspection will show that (1), (2), and (3) are only special cases and/or variations of

(4)
$$F_p F_q - F_{p-k} F_{q+k} = (-1)^{p-k} F_k F_{q+k-p} \; ,$$

where $k = m + 1$ and $q - p = s + 1$.

This comparison is made easier when (4) is written as

(4')
$$\begin{vmatrix} F_{p-k} & F_p \\ F_q & F_{q+k} \end{vmatrix} = (-1)^{p-k+1} F_k F_{q+k-p}$$

thus suggesting a form for a related 3×3 matrix

$$P = \begin{bmatrix} F_{j-k} & F_j & F_{j+k} \\ F_{m-k} & F_m & F_{m+k} \\ F_{n-k} & F_n & F_{n+k} \end{bmatrix}$$

A singular property of the Det(P) presents itself.

Theorem: Det(P) = 0, k, j, m, n are integers.

Proof: There is no loss in generality to assume $j > m > n$ and it is simply convenient to assume $k \geq 0$. By applying (4) it is apparent that the columns of P are linearly dependent. We note by inspection that F_k (column 3) $- F_{2k}$ (column 2) $= (-1)^{k+1} F_k$ (column 1). Thus, the determinant is clearly zero (0).

Q. E. D.

Since the Det(P) = 0, a previous article of this Quarterly [3] suggests it would be interesting to consider the Det(P + a) where P + a means a matrix P with a added to each element of P. The generality of j, m, n, and k would almost prohibit the techniques used by Whitney [3]. Hence procedures discussed by Bicknell in [1] and by Bicknell and Hoggatt in a previous article of this Quarterly [2] are employed. Using the formula [2]

$$Det(P + a) = Det(P) + a \lambda(P) ,$$

where $\lambda(P)$ is the change in the value of the determinant of P, when the number 1 is added to each element of P, we have

$$Det(P + a) = a \lambda(P) ,$$

since Det(P) = 0. Now $\lambda(P)$ and the corresponding Det(P + a) are interesting in any one of the following forms. They are also derived with the aid of (4').

(a) $$\lambda(P) = \begin{vmatrix} 1 & 1 & 1 \\ F_{m-k} - F_{j-k} & F_m - F_j & F_{m+k} - F_{j+k} \\ F_{n-k} - F_{j-k} & F_n - F_j & F_{n+k} - F_{j+k} \end{vmatrix}$$

or

(b) $\lambda(P) = \left[F_{2k} - F_k - (-1)kF_k \right]\left[(-1)^{m-k}F_{n-m} + (-1)^{j-k}F_{n-j} - (-1)^{j-k}F_{m-j} \right]$

Therefore,

(c) $\text{Det}(P + a) = \begin{vmatrix} a & a & a \\ F_{m-k} - F_{j-k} & F_m - F_j & F_{m+k} - F_{j+k} \\ F_{n-k} - F_{j-k} & F_n - F_j & F_{n+k} - F_{j+k} \end{vmatrix}$

or

(d) $\text{Det}(P + a) = [F_{2k} - F_k - (-1)^k F_k][(-1)^{m-k}F_{n-m} + (-1)^{j-k}F_{n-j} - (-1)^{j-k}F_{m-j}] \, a.$

The first factors of (b) and (d) have a straightforward simplification if it is known in advance whether or not k is even or odd. The various forms of $\lambda(P)$ and Det(P + a) become much more intriguing once the interesting patterns in the subscripts and exponents and their relationship to P are observed. These patterns could easily serve as mnemonic devices.

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1. Marjorie Bicknell, "The Lambda Number of a Matrix: The Sum of Its n^2 Cofactors," Amer. Math. Monthly, 72 (1965), pp 260—264.

2. Marjorie Bicknell and V. E. Hoggatt, Jr. , "Fibonacci Matrices and Lambda Functions," Fibonacci Quarterly, Vol. 1, No. 2, April 1964, pp 47 — 50.

3. Problem B-24, Fibonacci Quarterly, Vol. 2, No. 2, April, 1964.

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EXPLORING SPECIAL FIBONACCI RELATIONS

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In a previous exploration section, readers were introduced to the problem of generalized Fibonacci-Lucas relations. We denoted the terms of the generalized Fibonacci sequence as f_n and those of the associated generalized Lucas sequence as g_n where

$$g_n = f_{n-1} + f_{n+1}$$

Recall also that the sequence $(2,9)$ means the Fibonacci sequence with $f_1 = 2$ and $f_2 = 9$. The following represent some curious results obtained in trying to express $f_n g_n$ as a linear combination of f's and g's.

Sequence	Formula for $f_n g_n$
$(2,9)$	$f_n g_n = f_{2n+5} - g_{2n-2}$
$(3,7)$	$f_n g_n = g_{2n+3} - f_{2n-1}$
$(3,10)$	$f_n g_n = g_{2n+3} + f_{2n} + g_{2n-1}$
$(3,11)$	$f_n g_n = g_{2n+4} - g_{2n} - f_{2n-3}$
$(4,13)$	$f_n g_n = g_{2n+4} + f_{2n-4}$
$(5,11)$	$f_n g_n = g_{2n+3} + f_{2n+3} + f_{2n-2}$
$(5,13)$	$f_n g_n = g_{2n+4} + f_{2n-1} + f_{2n-4}$
$(6,13)$	$f_n g_n = g_{2n+4} + g_{2n-1}$
$(6,17)$	$f_n g_n = g_{2n+5} - f_{2n+3} + f_{2n-4}$
$(7,17)$	$f_n g_n = g_{2n+4} + g_{2n+2} + f_{2n-4}$

Now, of course, it must be recognized that these linear expressions could be represented in an infinity of different ways. However, it does not seem that they are all one and the same relation. If not, then we have specific relations that characterize the individual sequences. So the following questions are raised:

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(1) Can the above formulas for $f_n g_n$ be unified into one formula?

(2) If not, can other instances be found of this type of phenomenon?

(3) When is it that we have particular formulas for each Fibonacci sequence rather than a general formula for all sequences?

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ALGEBRA THROUGH PROBLEM SOLVING
BOOK REVIEW

Brother U. Alfred, St. Mary's College, Calif.

A paperback entitled Algebra Through Problem Solving, written by Abraham P. Hillman, University of New Mexico, and Gerald L. Alexanderson, University of Santa Clara (1966, Allyn and Bacon) has appeared as one of a series entitled Topics in Contemporary Mathematics. The following sequence of topics is covered.

1. The Pascal Triangle
2. The Fibonacci and Lucas Numbers
3. Factorials
4. Arithmetic and Geometric Progressions
5. Mathematical Induction
6. The Binomial Theorem
7. Combinations and Permutations
8. Polynomial Equations
9. Determinants
10. Inequalities.

The manner of treatment is to give a presentation of the basic ideas in a succinct and cogent fashion and then allow the student to become familiar with these ideas by solving numerous and varied problems. This approach, it seems to the present reviewer, is highly commendable.

Of particular interest is the role given the Fibonacci and Lucas numbers. Note that these sequences are introduced early in the book. In Chapter 2, they are used in the main to provide the student with an opportunity to make mathematical conjectures on new and unfamiliar material. But the authors do not stop there. As might be expected, they play a key role in the chapter on mathematical induction and are brought in occasionally in other portions of the book as well.

An unofficial and rapid count of these loci apart from Chapter 2 is given below.

In the chapter on mathematical induction, p. 35, No. 4; p. 37, No. 17, p. 38, Nos. 19-25 inclusive; in the chapter on the binomial theorem, Nos. 33-36, pp. 49-50; in the chapter on determinants, No. 17, pp. 94-95; in the chapter on inequalities, No. 5, p. 107, No. 28, p. 117.

OPTIMALITY PROOF FOR THE SYMMETRIC FIBONACCI SEARCH TECHNIQUE

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An important problem in engineering, economics, and statistics is to find the maximum of a function. When the function has only one stationary point, the maximum, and when it depends on a single variable in a finite interval, the most efficient way to find the maximum is based on the Fibonacci numbers. The procedure, now known widely as "Fibonacci search," was discovered and shown optimal in a minimax sense by Kiefer [1]. S. Johnson [2] gave a different demonstration. Oliver and Wilde [3,4] extended the procedure to the case where, in order to distinguish between adjacent measurements, a non-negligible distance must be preserved between them. Although this modification, called the symmetric Fibonacci technique is described informally in [4], where numerous extensions are also discussed the present paper gives the first complete, precise description, and formal optimality proof.

Let $y(x)$ be a unimodal function, i.e., one with a unique relative maximum value obtained at $x = x^*$ on the closed interval $[a,b]$; thus

(1)
$$y(x^*) = \max_{a \leq x \leq b} y(x)$$

and $a \leq x_1 < x_2 \leq x^*$ implies $y(x_1) < y(x_2)$; $x^* \leq x_1 < x_2 \leq b$ implies $y(x_1) > y(x_2)$. As a practical consideration, assume that $y(x)$ can be measured only with finite accuracy, but there is a $\delta > 0$ such that if $|x_1 - x_2| \geq \delta$ the measurements of $y(x_1)$ and $y(x_2)$ will be equal only if x_1 and x_2 lie on opposite sides of x^*.

A search strategy $S(n, \delta)$ on $[a,b]$ is a plan for evaluating the function at n distinct points $x_1, x_2, \cdots, x_n$, where the location of x_{k+1} depends on $y(x_j)$ for $j \leq k$ and where

(2)
$$|x_j - x_k| \geq \delta \qquad j \neq k, \qquad 1 \leq j, \quad k \leq n$$

The plan terminates after successive reduction of a starting interval on which the function is defined to a final interval of a required length, containing x^*.

Suppose k function evaluations have been performed, and let m_k be such that

(3) $$y(m_k) = \max\{y(x_1), \cdots, y(x_k)\}$$

Let

(4) $$L_k = \{x_j : x_j < m_k, \ j \le k\} \cup \{a\}$$

(5) $$R_k = \{x_j : x_j > m_k, \ j \le k\} \cup \{b\}$$

and

(6) $$\ell_k = \operatorname{Sup} L_k$$

(7) $$r_k = \operatorname{Inf} R_k$$

Since y is unimodal, it is clear that x^* must be in the interval $[\ell_k, r_k]$.

The purpose of this paper is to derive a search strategy $S_{op}(n, \delta)$ that is optimal in the sense that it maximizes the length of the starting interval for whatever final interval is given. From (2) – (7) it follows that the final interval for any $n > 1$ is

(8) $$r_n - \ell_n \ge 2\delta$$

By choosing the required length ($\ge 2\delta$) of the final interval we rescale the system as follows:

(9) $$\bar{r}_n - \bar{\ell}_n = \lambda(r_n - \ell_n) = 1, \qquad \lambda > 0$$

(10) $$\bar{\delta} = \lambda\delta$$

(11) $$\bar{x} = \lambda(x - a)$$

Thus the new strategy $\bar{S}(n, \bar{\delta})$ is defined on $[0, \bar{D}_n]$, where $\bar{D}_n = \lambda(b - a)$, and it yields a final interval of unit length.

The <u>symmetric</u> <u>Fibonacci</u> <u>search</u> is a search strategy for a starting interval $[0, \overline{D}_n^F]$ where the right end point $\overline{D}_n^F$ is given by

(12) $$\overline{D}_n^F = F_{n+1} - F_{n-1}\delta, \quad n = 1, 2, \cdots$$

where

(13) $$F_{n+2} = F_{n+1} + F_n \quad (F_0 = 0, \quad F_1 = 1)$$

are the Fibonacci numbers. The strategy is defined by the rules

(14) $$\overline{x}_1^F = F_n - F_{n-2}\overline{\delta}$$

and

(15) $$\overline{x}_{k+1}^F = \overline{\ell}_k + \overline{r}_k - \overline{m}_k$$

<u>Lemma</u>: The symmetric Fibonacci search defined above is an $\overline{S}(n, \overline{\delta})$ search strategy for $[0, \overline{D}_n^F]$ with $\overline{r}_n - \overline{\ell}_n = 1$.

<u>Proof</u>: By induction on n. The lemma is trivially true for $n = 1$, since by (3) – (7) $\overline{\ell}_1 = 0$ and $\overline{r}_1 = 1$. For $n = 2$, $\overline{D}_2^F = 2 - \overline{\delta}$ and $\overline{x}_1^F = 1$, implying that $\overline{\ell}_1 = 0$, $\overline{r}_1 = 2 - \overline{\delta}$ and consequently $\overline{x}_2^F = 1 - \overline{\delta}$ by (15). Thus $|\overline{x}_1^F - \overline{x}_2^F| = \overline{\delta}$ and for any unimodal function either $\overline{m}_2 = \overline{x}_1^F$ or $\overline{m}_2 = \overline{x}_2^F$, and in both cases $\overline{r}_2 - \overline{\ell}_2 = 1$.

Now assume the lemma true for $n = N \ (\geq 2)$. Then $\overline{D}_{N+1}^F = F_{N+2} - F_N\overline{\delta}$ and $\overline{x}_1^F = F_{N+1} - F_{N-1}\overline{\delta} = \overline{D}_N^F$. Consequently, $\overline{\ell}_1 = 0$, $\overline{r}_1 = F_{N+2} - F_N\overline{\delta}$ and $\overline{x}_2^F = F_N - F_{N-2}\overline{\delta} = \overline{D}_{N-1}^F$. Also $|\overline{x}_1^F - \overline{x}_2^F| = \overline{D}_N^F - \overline{D}_{N-1}^F \geq 1 - \overline{\delta} \geq \overline{\delta}$. If $\overline{m}_2 = \overline{x}_2^F$ then $\overline{\ell}_2 = 0$ and $\overline{r}_2 = \overline{x}_1^F = \overline{D}_N^F$. Thus $0 \leq \overline{x}^* \leq \overline{D}_N^F$ with one evaluation of the function at $\overline{D}_{N-1}^F$ and G_y the induction hypothesis the lemma is true for this case. If $\overline{m}_2 = \overline{x}_1^F$, then $\overline{\ell}_2 = \overline{x}_2^F = \overline{D}_{N-1}^F$ and $\overline{r}_2 = \overline{D}_{N+1}^F$. Define the new variable $\hat{x}^F = \overline{D}_{N+1}^F - \overline{x}^F$ so that $0 \leq \hat{x}^F \leq \overline{D}_N^F$ with one function evaluation at $\overline{D}_{N-1}^F$. Thus by induction the lemma is true for this case, too.

Note that if $y(\overline{x}_{k+1}^F) = y(\overline{m}_k)$ and say $\overline{m}_k > \overline{x}_{k+1}^F$, $\overline{x}^*$ is known to lie in the interval $[\overline{x}_{k+1}^F, \overline{m}_k]$. As $\overline{m}_{k+1} = \overline{m}_k$ or $\overline{m}_{k+1} = \overline{x}_{k+1}^F$, the Fibonacci

search plan suggests reducing the interval $[\bar{\ell}_k, \bar{r}_k]$ to either $[\bar{\ell}_k, \bar{m}_k]$ or $[\bar{x}_{k+1}^F, \bar{r}_k]$. An improved choice could be made in this case by taking the next interval as $[\bar{\ell}_{k+1}, \bar{r}_{k+1}] = [\bar{x}_{k+1}^F, \bar{m}_k]$ which can be the starting interval for a new $\bar{S}(n - k - 2, \bar{\delta})$ Fibonacci plan. This fortuitous situation is deliberately excluded from the theorem following in order to simplify the proof.

Theorem: The symmetric Fibonacci search plan is the unique $\bar{S}_{op}(n, \bar{\delta})$ strategy among all $\bar{S}(n, \bar{\delta})$ strategies, provided $y(\bar{x}_{k+1}) \ne y(m_k)$ for every $k = 1, 2, \cdots, n - 1$.

Proof: by induction on n. For n = 1 the proof is obvious because all $\bar{D}_1 = \bar{D}_1^F = 1$. For n = 2 we can assume without loss of generality that $\bar{x}_1 > \bar{x}_2$. Then any $\bar{S}(n, \bar{\delta})$ must satisfy $\bar{x}_1 = 1$, $\bar{D}_2 - \bar{x}_1 \le 1 - \bar{\delta}$, yielding $\bar{D}_2 \le 2 - \bar{\delta} = \bar{D}_2^F$. Assume now that the theorem is true for n = N $\ge$ 2, then for an $\bar{S}(N + 1, \bar{\delta})$ strategy let $\bar{x}_1 > \bar{x}_2$ and $\bar{m}_2 = \bar{x}_2$. Hence $\bar{\ell}_2 = 0$, $\bar{r}_2 = \bar{x}_1$, and we have an $\bar{S}(N, \bar{\delta})$ strategy on $[0, \bar{x}_1]$. By induction

$$(16) \qquad\qquad \bar{x}_1 \le \bar{D}_N^F$$

If $\bar{m}_2 = \bar{x}_1$, then $\bar{\ell}_2 = \bar{x}_2$, $\bar{r}_2 = \bar{D}_{N+1}$, and by induction therefore

$$(17) \qquad\qquad \bar{D}_{N+1} - \bar{x}_2 \le \bar{D}_N^F$$

However,

$$(18) \qquad\qquad \bar{x}_2 \le \bar{D}_{N-1}^F$$

also by induction, since if $\bar{x}^* < \bar{x}_2$ then $\bar{x}^*$ has to be located by an $\bar{S}(N - 1, \bar{\delta})$ strategy on $[0, \bar{x}_2]$. Thus addition of (17) and (18) yields

$$(19) \qquad\qquad \bar{D}_{N+1} \le \bar{D}_N^F + \bar{D}_{N-1}^F$$

But $\bar{D}_N^F + \bar{D}_{N-1}^F = \bar{D}_{N+1}^F$ by (12), and this shows that the symmetric Fibonacci search is indeed optimal.

Moreover, we observe that

$$(20) \qquad\qquad \bar{D}_{N+1} - \bar{x}_1 \le \bar{D}_{N-1}^F$$

by an argument similar to that used in (18). Relations (16) - (20) show that the symmetric Fibonacci search is the only way to achieve the maximum value $\overline{D}_N = \overline{D}_N^F$.

ACKNOWLEDGEMENT

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ALGEBRA THROUGH PROBLEM SOLVING, Book Review, Cont'd from p. 264.

From the standpoint of the Fibonacci association, this text is a landmark in its recognition of the pedagogical value of the Fibonacci series. Let us hope that other authors will see the wisdom of incorporating such interesting and pregnant material into their textbooks.

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AN EXPRESSION FOR GENERALIZED FIBONACCI NUMBERS

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An interesting expression for the Fibonacci numbers is presented here which relies on the modulo three value of the subscript.

$$(1a) \qquad F_{3a} = 1 - \sum_{i=0}^{a-1} \binom{2i + a - 1}{3i} (-1)^{a-i} 8^i \qquad (a > 0)$$

$$(1b) \qquad F_{3a+1} = 1 - 2 \sum_{i=0}^{a-1} \binom{2i + a}{3i + 1} (-1)^{a-i} 8^i \qquad (a \geq 0)$$

$$(1c) \qquad F_{3a+2} = 1 - 4 \sum_{i=0}^{a-1} \binom{2i + a + 1}{3i + 2} (-1)^{a-i} 8^i \qquad (a \geq 0)$$

This is a special case of a more general expression for the generalized Fibonacci numbers [1].

$$(2a) \qquad V_{n,m} = 1 \qquad (m = -n + 1, \cdots, 0)$$

$$(2b) \qquad V_{n,m} = \sum_{k=1}^{n} V_{n,m-k}$$

It is seen that $F_m = V_{2,m-2}$.

It is interesting to note that these numbers arise in the analysis of polyphase merge-sorting with $n + 1$ tapes. The $V_{n,m}$ represent the total number of strings on all the tapes and also the length of strings (assuming initial length of 1) at each step of the polyphase merge process. A description of the polyphase merge-sort can be found in [2].

The general expression can be written as:

$$(3) \qquad V_{n,a(n+1)+b} = 1 + 2^{b-1}(n - 1) \sum_{i=0}^{a} \binom{in + a + b - 1}{a - i} (-1)^{a-i} (2^{n+1})^i$$

$$(b = 1, \cdots, n + 1), \quad (a \geq 0).$$

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Let

$$f_n(x) = \sum_{m=0}^{\infty} V_{n,m} x^m$$

It follows immediately that

$$(1 - x - x^2 - \cdots - x^n)f_n(x) = 1 + (n - 1)x + (n - 2)x^2 + \cdots + x^{n-1} .$$

Therefore

$$f_n(x) = \frac{1 + (n - 1)x + (n - 2)x^2 + \cdots + x^{n-1}}{1 - x - x^2 - \cdots - x^n}$$

(4)
$$= \frac{1 + (n - 2)x - x^2 - \cdots - x^n}{1 - 2x + x^{n+1}}$$

$$= \frac{1}{1 - x} + \frac{(n - 1)x}{1 - 2x + x^{n+1}}$$

If

$$\frac{1}{1 - 2x + x^{n+1}} = \sum_{m=0}^{\infty} W_{n,m} x^m$$

the sequence $W_{n,m}$ is defined by:

(5a) $\qquad\qquad W_{n,m} = 0 \qquad\qquad\qquad (m < 0)$

(5b) $\qquad\qquad W_{n,m} = 1 \qquad\qquad\qquad (m = 0)$

(5c) $\qquad W_{n,m} = 2W_{n,m-1} - W_{n,m-n-1} \qquad (m > 0)$

From Eq. (4) $\qquad V_{n,m} = 1 + (n - 1)W_{n,m-1} \qquad (m > 0)$

Theorem:

(6) $\qquad W_{n,a(n+1)+b} = \sum_{j=0}^{\infty} \binom{a + b + nj}{(n + 1)j + b} 2^{b+(n+1)j} (-1)^{a-j}$

(This formula immediately yields the identity (3).)

Proof:
$$\frac{1}{1 - 2x + x^{n+1}} = \sum_{m=0}^{\infty} (2x - x^{n+1})^m$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \binom{m}{k} 2^k (-1)^{m-k} x^{(n+1)m-nk}$$

Rearranging this sum in terms of powers of x, let $(n + 1)a + b = (n + 1)m - nk$. It follows that $k = b \pmod{(n + 1)}$, so $k = (n + 1)j + b$ for some $j \geq 0$. Changing the sum on m and k into a sum on a, b and j, and noting that $m = a + b + nj$, results in:

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} x^{(n+1)a+b} \sum_{j=0}^{\infty} \binom{a + b + nj}{(n + 1)j + b} 2^{(n+1)j+b} (-1)^{a-j}$$

This completes the proof. A similar method was used by Polya [3] to solve another recurrence relation.

Another interesting expression which arises from this analysis is the general expression for the numbers defined by:

(7a) $U_{n,m} = 0$ $(m = -n + 1, \cdots, -1)$

(7b) $U_{n,m} = 1$ $(m = 0)$

(7c) $U_{n,m} = \sum_{i=1}^{m} U_{n,m-i}$ $(m > 0)$

It is seen that $F_m = U_{2,m-1}$.

These numbers also arise in the analysis of polyphase merge-sorting; they represent the number of strings produced at each step of the process.

The general expression is:

$$U_{n,a(n+1)+b} = 2^{b-1} \sum_{i=0}^{a} \left\{ \binom{in + a + b}{a - i} + \binom{in + a + b - 1}{a - i - 1} \right\} (-1)^{a-m} (2^{n+1})^n$$

$$(b = 0, \cdots, n) , \quad (a + b > 0)$$

The proof is similar to the above and is omitted.

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FIBONACCI YET AGAIN

J. A. H. Hunter

Consider a triangle such that the square of one side equals the product of the other two sides.

Then we have sides: X, $\sqrt{XY}$, and Y; say $X > Y$.

Eliminating an common factor we may set $X = a^2$, $Y = b^2$, so that the "reduced" sides become a^2, ab, b^2.

Then, for a triangle, we must have $ab + b^2 > a^2$ which requires $(\sqrt{5} - 1)/2 < b/a < (\sqrt{5} + 1)/2$.

Hence a sufficient condition for a triangle that meets the requirements is

$$F_{2n-1}/F_{2n} < b/a < F_{2n}/F_{2n-1} \quad \text{with} \quad X = ka^2, \quad Y = kb^2 \quad .$$

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A POWER IDENTITY FOR SECOND-ORDER RECURRENT SEQUENCES

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1. INTRODUCTION

The following hold for all integers n and k:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n ,$$

$$F_{n+k}^2 = (F_k F_{k-1}) F_{n+2}^2 + (F_k F_{k-2}) F_{n+1}^2 - (F_{k-1} F_{k-2}) F_n^2 ,$$

$$F_{n+k}^3 = (F_k F_{k-1} F_{k-2} / 2) F_{n+3}^3 + (F_k F_{k-1} F_{k-3}) F_{n+2}^3 - (F_k F_{k-2} F_{k-3}) F_{n+1}^3$$

$$- (F_{k-1} F_{k-2} F_{k-3} / 2) F_n^3 .$$

These identities suggest that there is a general **expansion** of the form

$$(1.1) \qquad F_{n+k}^p = \sum_{j=0}^{p} a_j (k,p) F_{n+j}^p .$$

Here we show such an expansion does indeed exist, find an expression for the coefficients $a_j(k,p)$, and generalize (1.1) to second order recurrent sequences.

2. A FIBONACCI POWER IDENTITY

Define the Fibonomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$ by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1$$

Jarden [4] proved that the term-by-term product z_n of p sequences each obeying the Fibonacci recurrence satisfies

$$(2.1) \qquad \sum_{j=0}^{p+1} (-1)^{j(j+1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix} z_{n-j}$$

for integral n. In particular, $z_n = F_n^p$ obeys (2.1). Carlitz, [1, Section 1] has shown that the determinant

$$D_p = \left| F_{n+r+s}^p \right| \quad (r, s = 0, 1, \cdots, p)$$

has the value

$$D_p = (-1)^{p(p+1)(n+1)/2} \prod_{j=0}^{p} \binom{p}{j} \cdot (F_1^p F_2^{p-1} \cdots F_p)^2 \ne 0$$

implying that the $p+1$ sequences $\{F_n^p\}, \{F_{n+1}^p\}, \cdots, \{F_{n+p}^p\}$ are linearly independent over the reals. Since each of these sequences obeys the $(p+1)^{\text{th}}$ order recurrence relation (2.1), they must span the space of solutions of (2.1). Therefore an expansion of the form (1.1) exists.

To evaluate the coefficients $a_j(k,p)$ in (1.1) we first put $k = 0, 1, \cdots, p$, giving $a_j(k,p) = \delta_{jk}$ for $0 \le j, k \le p$, where δ_{jk} is the Kronecker delta defined by $\delta_{jk} = 0$ if $j \ne k$, $\delta_{kk} = 1$. Next we show that the sequence

$$\{a_j(k,p)\}_{k=0}^{\infty}$$

obeys (2.1) for $j = 0, 1, \cdots, p$. Indeed from (1.1) we find

$$0 = \sum_{r=0}^{p+1} (-1)^{r(r+1)/2} \begin{bmatrix} p+1 \\ r \end{bmatrix} F_{n+k-r}^p$$

$$= \sum_{j=0}^{p} \left\{ \sum_{r=0}^{p+1} (-1)^{r(r+1)/2} \begin{bmatrix} p+1 \\ r \end{bmatrix} a_j(k-r, p) \right\} F_{n+j}^p \quad .$$

But the F_{n+j}^p $(j = 0, 1, \cdots, p)$ are linearly independent, so that

$$\sum_{r=0}^{p+1} (-1)^{r(r+1)/2} \begin{bmatrix} p+1 \\ r \end{bmatrix} a_j(k-r, p) = 0 \quad (j = 0, 1, \cdots, p) \quad .$$

Now consider $b_j(k,p) = (F_k F_{k-1} \cdots F_{k-p})/F_{k-j}(F_j F_{j-1} \cdots F_1 F_{-1} \cdots F_{j-p})$ for $j = 0,1,\cdots,p-1$, $b_p(k,p) = \begin{bmatrix} k \\ p \end{bmatrix}$, together with the convention that $F_0/F_0 = 1$. Clearly $b_j(k,p) = \delta_{jk}$ for $0 \le j,k \le p$. Since $\{b_j(k,p)\}_{k=0}^{\infty}$ is the term-by-term product of p Fibonacci sequences, it must obey (2.1). Thus $\{a_j(k,p)\}_{k=0}^{\infty}$ and $\{b_j(k,p)\}_{k=0}^{\infty}$ obey the same $(p+1)^{\text{th}}$ order recurrence relation and have their first $p+1$ values equal $(j = 0,1,\cdots,p)$, so that $a_j(k,p) = b_j(k,p)$. Since $F_{-n} = (-1)^{n+1} F_n$, it follows that

$$F_{-1} \cdots F_{j-p} = F_{p-j} \cdots F_1 (-1)^{(p-j)(p-j+3)/2} \quad ,$$

so that for $j = 0,1,\cdots,p-1$, we have

$$a_j(k,p) = (-1)^{(p-j)(p-j+3)/2} \left(\frac{F_k F_{k-1} \cdots F_{k-p+1}}{F_p F_{p-1} \cdots F_1} \right) \left(\frac{F_p F_{p-1} \cdots F_1}{(F_j \cdots F_1)(F_{p-j} \cdots F_1)} \right) \left(\frac{F_{k-p}}{F_{k-j}} \right)$$

$$= (-1)^{(p-j)(p-j+3)/2} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} p \\ j \end{bmatrix} (F_{k-p}/F_{k-j}) \quad ,$$

which is also valid for $j = p$ using the convention $F_0/F_0 = 1$. Then (1.1) becomes

$$(2.2) \qquad F_{n+k}^p = \sum_{j=0}^{p} (-1)^{(p-j)(p-j+3)/2} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} p \\ j \end{bmatrix} (F_{k-p}/F_{k-j}) F_{n+j}^p$$

for all k. We remark that since consecutive p^{th} powers of the natural numbers obey

$$\sum_{j=0}^{p+1} (-1)^{p-j} \binom{p+1}{j} (n+j)^p = 0 \quad ,$$

a development similar to the above leads to

$$(2.3) \qquad (n+k)^p = \sum_{j=0}^{p} (-1)^{p-j} \binom{k}{p} \binom{p}{j} \left(\frac{k-p}{k-j} \right) (n+j)^p \quad ,$$

a result parallel to (2.2)

3. EXTENSION TO SECOND-ORDER RECURRENT SEQUENCES

We now generalize the result of Section 2. Consider the second-order linear recurrence relation

(3.1) $y_{n+2} = py_{n+1} - qy_n$ $(q \neq 0)$.

Let a and b be the roots of the auxiliary polynomial $x^2 - px + q$ of (3.1).
Let w_n be any sequence satisfying (3.1), and define u_n by $u_n = (a^n - b^n)/$
$(a - b)$ if $a \neq b$, and $u_n = na^{n-1}$ if a = b, so that u_n also satisfies (3.1).
Following [4], we define the u-generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}_u$ by

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{u_m u_{m-1} \cdots u_{m-r+1}}{u_1 u_2 \cdots u_r}   (r > 0) ;   \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1   .$$

Jarden [4] has shown that the product x_n of p sequences each obeying (3.1)
satisfies the $(p + 1)^{th}$ order recurrence relation

(3.2) $\displaystyle\sum_{j=0}^{p+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} p + 1 \\ j \end{bmatrix}_u x_{n-j} = 0$.

If all of these sequences are w_n, then it follows that $x_n = w_n^p$ obeys (3.2).

It is our aim to give the corresponding generalization of (1.1) for the
sequence w_n; that is, to show there exists coefficients $a_j(k,p,u) = a_j(k)$ such
that

(3.3) $\displaystyle w_{n+k}^p = \sum_{j=0}^{p} a_j(k) w_{n+j}^p$

and to give an explicit form for the $a_j(k)$. Carlitz [1, Section 3] proved that

$$D_p(w) = \left| w_{n+r+s}^p \right| (r, s = 0, 1, \cdots, p)$$

is nonzero, showing that the $p + 1$ sequences

$$\{w_n^p\} \ , \ \{w_{n+1}^p\} \ , \cdots , \{w_{n+p}^p\}$$

are linearly independent. Reasoning as before, we see these sequences span the space of solutions of (3.2), so that the expansion (3.3) indeed exists. Putting $k = 0, 1, \cdots , p$ in (3.3) gives $a_j(k) = \delta_{jk}$ for $0 \le j, k \le p$. It also follows as before that the sequence

$$\{a_j(k)\}_{k=0}^{\infty}$$

satisfies (3.2). Now consider

$$b_j(k,p,u) = b_j(k) = u_k u_{k-1} \cdots u_{k-p} / u_{k-j}(u_j u_{j-1} \cdots u_1 u_{-1} \cdots u_{j-p})$$

for $j = 0, 1, \cdots , p - 1$, $b_p(k) = \begin{bmatrix} k \\ p \end{bmatrix}_u$, along with the convention $u_0 / u_0 = 1$.
Then $b_j(k) = \delta_{jk}$ for $0 \le j, k \le p$. Also $\{b_j(k)\}_{k=0}^{\infty}$ obeys (3.2) because it is the product of p sequences each of which obeys (3.1). Since $\{a_j(k)\}_{k=0}^{\infty}$ and $\{b_j(k)\}_{k=0}^{\infty}$ $(j = 0, 1, \cdots , p)$ obey the same $(p + 1)^{th}$ order recurrence relation and agree in the first $p + 1$ values, we have $a_j(k) = b_j(k)$. Now $ab = a$. so that $u_{-n} = (a^{-n} - b^{-n}) / (a - b) = -q^n u_n$. Then

$$u_{-1} \cdots u_{j-p} = u_{p-j} \cdots u_1 (-1)^{p-j} q^{(p-j)(p-j+1)/2}$$

and thus for $j = 0, 1, \cdots , p - 1$ we see

(3.4)
$$a_j(k) = (-1)^{p-j} q^{(p-j)(p-j+1)/2} \left(\frac{u_k u_{k-1} \cdots u_{k-p+1}}{u_p u_{p-1} \cdots u_1} \right) \left(\frac{u_p u_{p-1} \cdots u_1}{(u_j \cdots u_1)(u_{p-j} \cdots u_1)} \right) \left(\frac{u_{k-p}}{u_{k-j}} \right)$$

$$= (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} k \\ p \end{bmatrix}_u \begin{bmatrix} p \\ j \end{bmatrix}_u (u_{k-p} / u_{k-j}) \quad ,$$

which is also valid for $j = p$ using the convention $u_0 / u_0 = 1$. Therefore (3.3) becomes

(3.5) $\quad w_{n+k}^p = \displaystyle\sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} k \\ p \end{bmatrix}_u \begin{bmatrix} p \\ j \end{bmatrix}_u (u_{k-p} / u_{k-j}) w_{n+j}^p \quad ,$

Carlitz has communicated and proved a further extension of this result. Let

$$x_n^{(p)} = w_{n+a_1} w_{n+a_2} \cdots w_{n+a_p} \quad,$$

where the a_j are arbitrary but fixed nonnegative integers. Then we have

$$(3.6) \qquad x_{n+k}^{(p)} = \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} k \\ p \end{bmatrix}_u \begin{bmatrix} p \\ j \end{bmatrix}_u (u_{k-p}/u_{k-j}) x_{n+j}^{(p)} \quad,$$

where u_0/u_0 still applies. We note that putting $a_1 = a_2 = \cdots = a_p = 0$ reduces (3.6) to (3.5).

To prove (3.6) using previous techniques requires us to show that the sequences

$$\left\{ x_n^{(p)} \right\}, \left\{ x_{n+1}^{(p)} \right\}, \cdots, \left\{ x_{n+p}^{(p)} \right\}$$

are linearly independent. To avoid this, we establish (3.6) by induction on k. Now (3.6) is true for $k = 0$ and all n. Assume it is true for some $k \geq 0$ and all n, and replace n by $n + 1$, giving

$$x_{n+k+1}^{(p)} = \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j \end{bmatrix}_u \frac{u_{k-p}}{u_{k-j}} x_{n+j+1}^{(p)}$$

$$= \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=1}^{p} (-1)^{p-j+1} q^{(p-j+1)(p-j+2)/2} \begin{bmatrix} p \\ j-1 \end{bmatrix}_u \frac{u_{k-p}}{u_{k-j+1}} x_{n+j}^{(p)} + \begin{bmatrix} k \\ p \end{bmatrix}_u x_{n+p+1}^{(p)} \quad.$$

It follows from (3.2) that

$$x_{n+p+1}^{(p)} = -\sum_{j=1}^{p+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix}_u x_{n+p+1-j}^{(p)}$$

$$= \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p+1 \\ j \end{bmatrix}_u x_{n+j}^{(p)} \quad.$$

Thus

$$
x_{n+k+1}^{(p)} = \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j-1 \end{bmatrix}_u \frac{x_{n+j}^{(p)}}{u_j u_{k-j+1}}
$$

$$
\cdot \, (u_{p+1} u_{k-j+1} - q^{p-j+1} u_{k-p} u_j).
$$

Since

$$
u_{p+1} u_{k-j+1} - q^{p-j+1} u_{k-p} u_j = u_{k+1} u_{p-j+1} \, ,
$$

we have

$$
x_{n+k+1}^{(p)} = u_{k+1} \begin{bmatrix} k \\ p \end{bmatrix}_u \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j-1 \end{bmatrix}_u \frac{u_{p-j+1}}{u_j u_{k-j+1}} x_{n+j}^{(p)}
$$

$$
= \begin{bmatrix} k+1 \\ p \end{bmatrix} \sum_{j=0}^{p} (-1)^{p-j} q^{(p-j)(p-j+1)/2} \begin{bmatrix} p \\ j \end{bmatrix}_u \frac{u_{k-p+1}}{u_{k-j+1}} x_{n+j}^{(p)} \, ,
$$

completing the induction step and the proof.

4. SPECIAL CASES

In this section we reduce (3.5) to a general Fibonacci power identity and to an identity involving powers of terms of an arithmetic progression. First if we let $w_n = F_{ns+r}$, $u_n = F_{ns}$, where r and s are fixed integers with $s \neq 0$, then both w_n and u_n satisfy

(4.1) $y_{n+2} - L_s y_{n+1} + (-1)^s y_n = 0$.

The roots of the auxiliary polynomial of (4.1) are distinct for $s \neq 0$, so that w_n and u_n satisfy the conditions of the previous section. In this case the u-generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}_u$ become the s-generalized Fibonacci coefficients $\begin{bmatrix} m \\ t \end{bmatrix}_s$ defined by

$$\begin{bmatrix} m \\ t \end{bmatrix}_s = \frac{F_{ms} F_{(m-1)s} \cdots F_{(m-t+1)s}}{F_{ts} F_{ts-s} \cdots F_s} \ (t > 0) \ ; \begin{bmatrix} m \\ 0 \end{bmatrix}_s = 1 \ .$$

A recurrence relation for these coefficients is given in [3]. Now here

$$q = (-1)^s, \ \text{so} \ (-1)^{p-j} q^{(p-j)(p-j+1)/2} = (-1)^{(p-j)[s(p-j+1)+2]/2} \ .$$

Then (3.5) yields

$$(4.3) \quad F^p_{(n+k)s+r} = \sum_{j=0}^{p} (-1)^{(p-j)[s(p-j+1)+2]/2} \begin{bmatrix} k \\ p \end{bmatrix}_s \begin{bmatrix} p \\ j \end{bmatrix}_s \frac{F_{(k-p)s}}{F_{(k-j)s}} F^p_{(n+j)s+r} \ .$$

Putting $s = 1$ and $r = 0$ gives equation (2.2).

On the other hand, if we let $w_n = ns + r$ and $u_n = n$, where r and s are fixed integers, then w_n and u_n obey

$$(4.4) \qquad\qquad y_{n+2} - 2y_{n+1} + y_n = 0 \ .$$

Since the characteristic polynomial of (4.4) has the double root $x = 1$, both w_n and u_n satisfy the conditions for the validity of (3.5). In this case we have $q = 1$ and $\begin{bmatrix} m \\ t \end{bmatrix}_u = \binom{m}{t}$, the usual binomial coefficient. Then (3.5) becomes

$$(4.5) \quad ([n+k]s+r)^p = \sum_{j=0}^{p} (-1)^{p-j} \binom{k}{p}\binom{p}{j}\left(\frac{k-p}{k-j}\right) ([n+j]s+r)^p \ .$$

This reduces to (2.3) by setting $s = 1$ and $r = 0$.

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. Hillman, University of New Mexico, Albuquerque, New Mex.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-94 **Proposed by Clyde A. Bridger, Springfield Jr. College, Springfield, III**

Show that the number N_n of non-zero terms in the expansion of

$$K_n = \begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots \\ \cdots \\ \cdots \\ 0 & 0 & 0 & \cdots & & 0 & -1 & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & & 0 & 0 & -1 & a_n \end{vmatrix}$$

is obtained by replacing each a_i and each b_i by 1 and evaluating K_n. Show further that $N_n = F_{n+1}$, the $(n+1)$st Fibonacci number.

B-95 **Proposed by Brother U. Alfred, St. Mary's College, Calif.**

What is the highest power of 2 that exactly divides

$$F_1 F_2 F_3 \cdots F_{100} \qquad ?$$

B-96 **Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mex.**

Let G_n be the number of ways of expressing the positive integer n as an ordered sum $a_1 + a_2 + \cdots + a_s$ with each a_i in the set $\{1,2,3\}$. (For

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example, $G_3 = 4$ since 3 has just the expressions $3, 2 + 1, 1 + 2, 1 + 1 + 1.$)
Find and prove the lowest order linear homogeneous recursion relation satisfied by the G_n .

B-97 **Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.**

Let $A = \{a_k\}_{k=1}^{\infty}$ be an increasing sequence of numbers and let $A(n)$ denote the number of terms of A not greater than n. The Schnirelmann density of A is defined as the greatest lower bound of the ratios $A(n)/n$ for $n = 1$, $2, \cdots$. Show that the Fibonacci sequence has density zero.

B-98 **Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.**

Let F_n be the nth Fibonacci number and find a compact expression for the sum $S_n(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots + F_n x^n$.

B-99 **Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.**

Find a compact expression for the infinite sum

$$T(x) = S_1(x) + \frac{S_2(x)}{2!} + \frac{S_3(x)}{3!} + \cdots ,$$

where $S_n(x)$ is as defined in B-98.

SOLUTIONS
DIFFERENCE AND DIFFERENTIAL EQUATIONS

B-76 **Proposed by James A. Jeske, San Jose State College, San Jose, Calif.**

The recurrence relation for the sequence of Lucas numbers is $L_{n+2} - L_{n+1} - L_n = 0$ with $L_1 = 1$, $L_2 = 3$.
Find the transformed equation, the exponential generating function, and the general solution.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

By (4.2) of Jeske's article, the transformed equation is

$$L_2(D)Y = Y'' - Y' - Y = 0, \quad Y(0) = 2, \quad Y'(0) = 1 \quad .$$

Now $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$ are the roots of $L_2(r) = 0$, hence with the given initial values we may determine the solution for the Lucas sequence to be

$$L_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n \quad .$$

The exponential generating function for the Lucas sequence is thus

$$Y(t) = e^{r_1 t} + e^{r_2 t} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!} \quad .$$

Also solved by Clyde A. Bridger, Howard L. Walton, and the Proposer.

IT PAYS TO CHECK

B-77   *Proposed by James A. Jeske, San Jose State College, San Jose, Calif.*

Find the general solution and the exponential generating function for the recurrence relation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0 \quad ,$$

with $y_0 = 0$, $y_1 = 0$, and $y_2 = -1$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Since $L_3(x) = x^3 - 5x^2 + 8x - 4 = (x - 1)(x - 2)(x - 2) = 0$, we have, using the notation of section 4 of Jeske's paper, $r_1 = 1$, $m_1 = 1$, $r_2 = 2$,

$m_2 = 2$, and $m = 2$. With the given initial values we find the exponential generating function to be

$$Y(t) = -e^t + e^{2t}(1 - t)$$

which, by applying the inverse transform (3.3), yields the general solution as

$$y_n = 2^n \left(1 - \frac{n}{2} \right) - 1 \ .$$

Also solved by Clyde A. Bridger and the Proposer. Douglas Lind also noted that formula (4.8) of Jeske's paper is not correct.

A LUCAS SUM

B-78 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show that

$$F_n = L_{n-2} + L_{n-6} + \cdots + L_{n-2-4m} + e_n, \quad n > 2 \ ,$$

where m is the greatest integer in $(n - 3)/4$, and $e_n = 0$ if $n \equiv 0 \pmod 4$, $e_n = 1$ if $n \not\equiv 0 \pmod 4$.

Solution by the Proposer.

Proof by induction: The proposition is easily shown true for $n = 3, 4, 5, 6$. Now assume the theorem true for $3 \le n \le k + 3$. Then one finds that

$$F_{k+4} = L_{k+2} + F_k = L_{k+2} + (L_{k-2} + L_{k-6} + \cdots + L_{k-2-4m} + e_k)$$

so that the theorem is true for $k + 4$, completing the induction step and the proof.

Also solved by David Zeitlin.

AN ALMOST LUCAS SUM

B-79 *Proposed by Brother U. Alfred, St. Mary's College, Calif.*

Let $a = (1 + \sqrt{5})/2$. Determine a closed expression for

$$X_n = [a] + [a^2] + \cdots + [a^n],$$

where the square brackets mean "greatest integer in."

Solution by the Proposer.

If $b = (1 - \sqrt{5})/2$, $a^k = L_k - b^k$ with b negative and $|b| < 1$. Hence $[a^k] = L_k$ if k is odd and $[a^k] = L_k - 1$ if k is even. It follows that

$$X_n = \sum_{k=1}^{n} L_k - \left[\frac{n}{2}\right] = L_{n+3} - 3 - \left[\frac{n}{2}\right]$$

Also solved by J.L. Brown, Jr. and Jeremy C. Pond.

OUR MAN OF PISA

B-80 *Proposed by Maxey Brooke, Sweeny, Texas.*

Solve the division alphametic

$$\text{FIB} \overline{)\text{XONACCI}}^{\text{PISA}}$$

where each letter represents one of the nine digits $1, 2, \cdots, 9$ and two letters may represent the same digit.

Solution by the Proposer.

$$382 \overline{)3764228}^{9854}$$

This solution may not be unique.

GAUSSIAN PRIMES

B-81   *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Prove that only one of the Fibonacci numbers $1, 2, 3, 5, \cdots$ is a prime in the ring of Gaussian integers.

Solution by L. Carlitz, Duke University, Durham, N.C.

Since

$$F_{2n+1} = F_{n+1}^2 + F_n^2 = (F_{n+1} + F_n i)(F_{n+1} - F_n i) \quad ,$$

it follows that F_{2n+1} is composite in the Gaussian ring for all $n > 0$. Since $F_{2n} = F_n L_n$ it follows that F_{2n} is composite in the ring of integers (and therefore in the Gaussian ring) for $n > 2$. For $n = 2$, $F_4 = 3$; for $n = 1$, $F_2 = 1$.

Also solved by Sidney Kravitz and the proposer.

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