ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-606 Proposed by Mario Catalani, University of Torino, Italy

Let us consider, for a nonnegative integer n, the following sum

$$S_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{2\lfloor \frac{k}{2} \rfloor} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-1-k}{2\lfloor \frac{k}{2} \rfloor + 1}.$$

A summation with a negative upper limit is taken to be equal to zero. Express S_n both in closed form and as a recurrence.

H-607 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let n be a positive integer greater than or equal to 3. Evaluate the sum

$$\sum_{i=1}^{n} \Big[\Big(\frac{F_{i+1} - F_{i-1}}{F_{i+2}^2 - F_{i-2}^2} \Big)^{n-2} \prod_{\substack{j=1 \\ j \neq i}} \Big(1 - \frac{F_{j+2} - F_{j-2}}{F_{i+2} - F_{i-2}} \Big)^{-1} \Big].$$

H-608 Proposed by Mario Catalani, University of Torino, Italy

Let P_n denote the Pell numbers

$$P_n = 2P_{n-1} + P_{n-2}, \;\; P_0 = 0, \;\; P_1 = 1.$$

Find

$$\lim_{n \to \infty} \prod_{k=1}^{n} \Big(1 + \frac{1}{\sqrt{2P_{2^k}^2 + 1}} \Big).$$

SOLUTIONS

Fibonacci Polynomials and Binomial Coefficients

H-594 Proposed by Mario Catalani, University of Torino, Italy (Vol. 41, no. 1, February 2003)

Consider the generalized Fibonacci and Lucas polynomials:

$$F_{n+1}(x,y) = xF_n(x,y) + yF_{n-1}(x,y),\ F_0(x,y) = 0,\ F_1(x,y) = 1;$$

$$L_{n+1}(x,y) = xL_n(x,y) + yL_{n-1}(x,y),\ L_0(x,y) = 2,\ L_1(x,y) = x.$$

Assume $y \neq 0$, $2x^2 - y \neq 0$. We will write F_n and L_n for $F_n(x,y)$ and $L_n(x,y)$, respectively. Show that:

1. $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k y^{-2k} F_{3k} = \frac{xF_{2n+1} - yF_{2n} + (-x)^{n+2}F_n + (-x)^{n+1}yF_{n-1}}{y^n(2x^2-y)};$$

2. $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k y^{-2k} L_{3k} = \frac{xL_{2n+1} - yL_{2n} + (-x)^{n+2}L_n + (-x)^{n+1}yL_{n-1}}{y^n(2x^2-y)}.$$

Solution by H.-J. Seiffert, Berlin, Germany

Define the Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ F_1(x) = 1, \quad \text{and} \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \quad \text{for } n \geq 1,$$

and

$$L_0(x) = 2,\ L_1(x) = x, \ \text{and}\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \quad \text{for } n \geq 1,$$

respectively. It is known (see [1]) that

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k F_{3k}(x) = \frac{xF_{2n+1}(x) - F_{2n}(x) + (-x)^{n+2}F_n(x) + (-x)^{n+1}F_{n-1}(x)}{(2x^2-1)} \tag{1}$$

and

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^k L_{3k}(x) = \frac{xL_{2n+1}(x) - L_{2n}(x) + (-x)^{n+2}L_n(x) + (-x)^{n+1}L_{n-1}(x)}{(2x^2-1)}. \tag{2}$$

Simple induction arguments, show that, for all integers n, $F_n = F_n(x,y) = (\sqrt{y})^{n-1}F_n(x/\sqrt{y})$ and $L_n = L_n(x,y) = (\sqrt{y})^n L_n(x/\sqrt{y})$, where $\sqrt{y}$ can be any of the two possible square roots of y. Now, it is easily verified that 1 follows from (1) when replacing x by $x/\sqrt{y}$ and dividing the resulting equation by $\sqrt{y}$, and that 2 follows from (2) with x replaced by $x/\sqrt{y}$.

1. H.-J. Seiffert. "Problem H-586." The Fibonacci Quarterly **40.4** (2002): 379.

Also solved by Paul Bruckman, Kenneth Davenport, Walther Janous, Vincent Mathe and the proposer.

Binomial Coefficients and Pell Numbers

H-595 Proposed by José Díaz-Barrero & Juan Egozcue, Barcelona, Spain (Vol. 41, no. 1, February 2003)

Let $\ell,\ n$ be positive integers. Prove that

$$\sum_{k=0}^{n}\binom{k+\ell+1}{k+1}\Big\{\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}P_n^{j-k-1}\Big\} \le P_n^{\ell+1}-1,$$

where P_n is the nth Pell number, i.e., $P_0=0$, $P_1=1$, and $P_{n+2}=2P_{n+1}+P_n$for $n\ge 0$.

Solution by Kenneth Davenport, Frackville, PA

The inner sum is, by the well-known binomial formula,

$$\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}P_n^{j-k-1} = \frac{(-1)^{k+1}}{P_n^{k+1}}\sum_{j=0}^{k+1}\binom{k+1}{j}(-P_n)^j$$

$$= \frac{(-1)^{k+1}}{P_n^{k+1}}\cdot(1-P_n)^{k+1} = \Big(1-\frac{1}{P_n}\Big)^{k+1}.$$

We are then led to consider the sum

$$\sum_{k=0}^{n}\binom{k+\ell+1}{k+1}\Big(1-\frac{1}{P_n}\Big)^{k+1}.$$

Substituting $m=k+1$, we simplify the above expression to get

$$\sum_{m=1}^{n}\binom{m+\ell}{m}\Big(1-\frac{1}{P_n}\Big)^{m}.$$

Next, we make use of the power series

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{n} x^n, \quad \text{for } -1 < x < 1.$$

We let $x = 1 - 1/P_n$ obtaining that

$$P_n^{\ell+1} - 1 = \sum_{m=1}^{\infty} \binom{m+\ell}{m} \Big(1 - \frac{1}{P_n}\Big)^m,$$

which implies the desired inequality.

Note that the only feature of P_n that the above proof used is the fact that $P_n > 1$. In particular, the above inequality holds with P_n replaced by any real number $x > 1$.

Also solved by Paul Bruckman, Mario Catalani, Walther Janous, Vincent Mathe, Angel Plaza and Sergio Fálcon, Ling-Ling Shi, and the proposers.

Prime Factors of Fibonacci Numbers

H-596 Proposed by the Editor
(Vol. 41, no. 2, May 2003)

A beautiful result of McDaniel (*The Fibonacci Quarterly* **40.1**, 2002) says that F_n has a prime divisor $p \equiv 1 \pmod 4$ for all but finitely many positive integers n. Show that the asymptotic density of the set of positive integers n for which F_n has a prime divisor $p \equiv 3 \pmod 4$ is $1/2$. Recall that a subset $\mathcal{N}$ of all the positive integers is said to have anasymptotic density λ if the limit

$$\lim_{x\to\infty} \frac{\#\{1 \le n < x \mid n \in \mathcal{N}\}}{x}$$

exists and equals λ.

Solution by the Editor

Suppose that $n > 3$ is odd. Then F_n is either congruent to 2 modulo 4 or it is odd according to whether n is a multiple of 3 or not. In particular, F_n has odd prime factors and if q denotes anyone of these then reducing the relation

$$L_n^2 - 5F_n^2 = (-1)^n \cdot 4 = -4$$

modulo q, we read that $(-4|q) = 1$, therefore $(-1|q) = 1$, and thus $q \equiv 1 \pmod 4$. Here, for any integer a we used $(a|q)$ to denote the Legendre symbol of a in respect to q. This argument shows that F_n is never a multiple of a prime $q \equiv 3 \pmod 4$ if n is odd, so the set of positive integers n for which F_n might have a prime divisor $q \equiv 3 \pmod 4$ is contained in the set of even numbers, and as such it can have asymptotic density at most 1/2. To prove the result, it suffices to show therefore that most even numbers n have the property that F_n is a multiple of some prime $q \equiv 3 \pmod 4$. Write $n = 2m$. Assume that there exists a prime factor p of m with $p \equiv 2 \pmod 3$. Then, $2p \equiv 4 \pmod 6$. The Fibonacci sequence is periodic modulo 4 with period 6, and if $k \equiv 4 \pmod 6$, then $F_k \equiv F_4 \pmod 4$. In particular, $F_{2p} \equiv 3 \pmod 4$, therefore there must exist a prime factor $q \equiv 3 \pmod 4$ of F_{2p}. Since $2p|n$, it follows that $F_{2p}|F_n$, therefore q divides F_n as well. Thus, if $n = 2m$, then F_n is always divisible by a prime $q \equiv 3 \pmod 4$, except, eventually, when m is not divisible by any prime number $p \equiv 2 \pmod 3$. But it is known that these last numbers form a set of asymptotic density zero. In fact, a result of Landau (see [1]) shows that if x is a large positive real number, then the set of positive integers $m \leq x$ such that m is not a multiple of any prime $p \equiv 2 \pmod 3$ has cardinality $O(x/\sqrt{\log x}) = o(x)$, which completes the proof.

1. E. Landau. "Handbuch der Lehre von der verteilung der Primzahlen." 3rd Edition. Chelsea Publ. Co. (1974): 668-669.

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