INTRINSIC PALINDROMES

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To our friend Cristian Jansenson

Natural numbers occur everywhere in our daily life: a bus ticket, a car plate, an id number, a timetable, etc. These numbers are mostly expressed in the decimal system. That is, for a natural numbers $n \in \mathbb{N}$ we write $n = (a_{k-1} \ldots a_0)_{10}$ for some $0 \leq a_i \leq 9$ and $a_{k-1} \neq 0$, which means that

$$n = a_{k-1}10^{k-1} + a_{k-2}10^{k-2} + \cdots + a_0.$$

Of particular interest are the so-called palindromic numbers. These are the numbers whose decimal expansion is the same when read from left to right, and from right to left, that is $(a_{k-1} \ldots a_0)_{10} = (a_0 \ldots a_{k-1})_{10}$.

This kind of number appears already in the Ganitasârasamgraha, a sanskrit manuscript dated around 850 AD. Therein the Indian mathematician Mahâvîrâchârya described 12345654321 as the quantity which "beginning with one until it reaches six, then decreases in reverse order" [1, p. 399]. This is a curious palindromic number, and in particular it is the square of another palindrome: $12345654321 = 111111^2$. There are many ways of generating palindromic numbers (see for instance [2]) including an interesting conjectural one [3].

Running across a palindrome by chance is a rare occurrence: for instance, the probability of picking at random a number $10^4 \leq n < 10^5$ and obtaining a palindrome is $1/10^2$, a fact well-known by the collectors of palindromic bus tickets. Hence we tend to feel pretty lucky when one of these rare numbers crosses our path. And the more digits it has, the luckier we feel, and the luckier the number itself seems to be.

But, is this feeling really justified? The truth has to be said: this property is *not* intrinsic to the number, but also depends on the base used to express it. So the number $n := (894111498)_{10}$ is lucky (or palindromic) in base 10, but is not in base 13 as $n = (113314377)_{13}$. Hence each time we encounter 894111498 we have to – at least in principle – thank heaven for this occurrence *together* with the fact that the human race has five fingers in each hand.

However, we can easily get independent of the base by defining a number to be intrinsically palindromic if it is palindromic in *some* base. A little reflection shows that this definition is meaningless as it stands: every number is palindromic in any base $m > n$, as $n = (n)_m$. Indeed it is much more natural to take into account the number of digits. So we define a number $n \in \mathbb{N}$ to be *k-palindromic* if there is a base b such that the b-expansion of n is palindromic of length k.

The previous observation shows that every number if 1-palindromic. Also note that $n = (11)_{n-1}$ provided that $n \geq 3$, that is every $n \geq 3$ is 2-palindromic.

What about k-palindromic numbers for $k \geq 3$? Our thesis is that very few numbers are k-palindromic, at least for $k \geq 4$. The probability of a number in an appropriate range being,

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say, intrinsically 9-palindromic is small, and indeed quite close to the probability of being 9-palindromic in base 10. This justifies our first impression that 894111498 is lucky, regardless of the base chosen to represent it.

To write down our results we first need to introduce the following counting functions. Take $k, N, b \in \mathbb{N}$ and set

$$\Phi_k(N, b) := \#\{n \leq N; n \text{ is } k\text{-palindromic in base } b \}.$$

Also set

$$\Phi_k(N) := \#\{n \leq N; n \text{ is } k\text{-palindromic}\}.$$

Then $\Phi_k(N, b)/N$ and $\Phi_k(N)/N$ represent the *density* (or probability) of numbers below N which are k-palindromic in base b and intrinsically k-palindromic, respectively.

Theorem 1: *Let $k \geq 4$, and write $k = 2i + r$ with $i \in \mathbb{N}$ and $r = 0, 1$. Then*

$$\Phi_k(N) \leq 4(N + 1)^{\frac{i+r+1}{k}}.$$

Proof: A base b contributes to $\Phi_k(N)$ if and only if $\Phi_k(N, b) > 0$, namely if and only if there exists a number

$$n = a_{k-1}b^{k-1} + \cdots + a_0 \leq N$$

palindromic in base b of length k, that is such that $0 \leq a_j \leq b - 1$ and $a_{k-j} = a_j$ for $j = 0, \ldots, i + r$, and $a_{k-1} \neq 0$. Then $b^{k-1} + 1 \leq n \leq N$, and hence b contributes to $\Phi_k(N)$ if and only if $b \leq (N - 1)^{\frac{1}{k-1}}$.

We consider separately the cases $b \leq (N + 1)^{\frac{1}{k}}$ and $(N + 1)^{\frac{1}{k}} < b \leq (N - 1)^{\frac{1}{k-1}}$. In the first case, the largest k-palindrome in base b is

$$(b - 1)(b^{k-1} + \cdots + 1) = b^k - 1 \leq N,$$

and so $\Phi_k(N, b) = \Phi_k(\infty, b) = (b - 1)b^{i+r-1}$.

Now for the second case we let $\theta(b) \in \mathbb{N}$ be the largest integer such that $\theta(b)(b^{k-1}+1) \leq N$, that is $\theta(b) = [N/(b^{k-1} + 1)] \leq N/b^{k-1}$. Then $(\theta(b) + 1)(b^{k-1} + 1) > N$ and so every k-palindromic number in base b begins with $a_{k-1} \leq \theta(b)$. Hence

$$\Phi_k(N, b) \leq \theta(b)b^{i+r-1} \leq N/b^i.$$

Set

$$L := [(N + 1)^{\frac{1}{k}}] \quad \text{and} \quad M := [(N - 1)^{\frac{1}{k-1}}].$$

We split the sum $\Phi_k(N) = \varphi + \chi + \psi$ with $\varphi := \sum_{b=2}^{L-1} \Phi_k(N, b)$, $\psi := \sum_{b=L+2}^{M} \Phi_k(N, b)$, and $\chi := \Phi_k(N, L) + \Phi_k(N, L + 1)$. From the previous considerations we conclude that

$$\varphi \leq \sum_{b=2}^{L-1} b^{i+r} \leq \int_2^L t^{i+r}dt \leq \frac{L^{i+r+1}}{i + r + 1} \leq \frac{(N + 1)^{\frac{i+r+1}{k}}}{i + r + 1}.$$

On the other hand

$$\psi \leq \sum_{L+2}^{M} \frac{N}{b^i} \leq N \int_{L+1}^{M} \frac{dt}{t^i} \leq \frac{N}{(i - 1)(L + 1)^{i-1}} \leq \frac{(N + 1)^{\frac{i+r+1}{k}}}{i - 1}.$$

Finally $\chi \leq L^{i+r} + N/(L + 1)^i \leq 2(N + 1)^{\frac{i+r}{k}}$ and thus

$$\Phi_k(N) \leq \left(\frac{2}{(N + 1)^{\frac{1}{k}}} + \frac{1}{i + r + 1} + \frac{1}{i - 1} \right)(N + 1)^{\frac{i+r+1}{k}} \leq 4(N + 1)^{\frac{i+r+1}{k}}. \quad \square$$

Let $k = 2i + r \geq 4$, and let $b \geq 2$ be a base. Set $N := b^k - 1$, so that N is larger than every number whose representation in base b has length k. Then

$$\Phi_k(N, b) = \Phi_k(\infty, b) = (b - 1)b^{i+r-1},$$

and so the density of numbers below N which are k-palindromic in base b is $(b-1)b^{i+r-1}/N \sim 1/b^i$. On the other hand, the previous result shows that the density of intrinsic k-palindromes below N is bounded by $4(b^k)^{\frac{i+r+1}{k}}/(b^k - 1) \leq 4/b^{i-1}$.

For instance, the probability of a number $n < 10^9$ being 9-palindromic in base 10 is 0.00009, while the probability of it being 9-palindromic in any base is below 0.004.

¿From the point of view of probability, the situation is then – in most cases – quite clear. For $k \leq 2$ every number is k-palindromic, while for $k \geq 4$ almost every number is not. The critical case is $k := 3$. Consider the following table:

$$
\begin{aligned}
\Phi_3(10^2 + 100) - \Phi_3(100) &= 61 \\
\Phi_3(10^3 + 100) - \Phi_3(10^3) &= 70 \\
\Phi_3(10^4 + 100) - \Phi_3(10^4) &= 83 \\
\Phi_3(10^5 + 100) - \Phi_3(10^5) &= 86 \\
\Phi_3(10^6 + 100) - \Phi_3(10^6) &= 89 \\
\Phi_3(10^7 + 100) - \Phi_3(10^7) &= 94
\end{aligned}
$$

This suggests that almost every number is 3-palindromic, but *not* every sufficiently large number. To tackle this problem we consider the following reformulation. We recall that $\{\xi\} := \xi - [\xi] \in [0, 1)$ denotes the fractional part of a real number $\xi \in \mathbb{R}$.

Lemma 2: *Let $n, b \in \mathbb{N}$ such that the b-expansion of n has length 3. Then n is 3-palindromic in base b if and only if*

$$\left\{ (n + 1)\frac{b}{b^2 + 1} \right\} < \frac{b}{b^2 + 1}.$$

Proof: First note that the hypothesis that the b-expansion of n has length 3 is equivalent to the fact that $b^2 + 1 \leq n \leq b^3 - 1$. Now n is 3-palindromic in base b if and only if there exists $0 < e < b$ and $0 \leq f < b$ such that

$$n = e(b^2 + 1) + fb.$$

Solving the associated Diophantine linear equation $n = x(b^2 + 1) + yb$ with respect to x, y we see that the above representation is equivalent to the existence of $\ell \in \mathbb{Z}$ satisfying

$$0 < n - \ell b < b \quad \text{and} \quad 0 \leq \ell(b^2 + 1) - nb < b.$$

The second pair of inequalities is equivalent to $nb/(b^2 + 1) \leq \ell < (n + 1)b/(b^2 + 1)$, and so this implies that $\{(n+1)b/(b^2 + 1)\} < b/(b^2 + 1)$. Then this condition is necessary for n to be 3-palindromic in base b.

Let's check that it is also sufficient. The integer $\ell := [(n+1)b/(b^2 + 1)]$ satisfies the second pair of inequalities. Then it only remains to prove that it also satisfies the first pair, which is equivalent to $\ell < n/b < \ell + 1$. This follows from the inequalities

$$(n + 1)\frac{b}{b^2 + 1} < \frac{n}{b} < n\frac{b}{b^2 + 1} + 1,$$

which are in turn a consequence of the hyposthesis $b^2 + 1 \leq n \leq b^3 - 1$. $\quad\square$

After the first version of this paper was written, N.J.A. Sloane included the sequences of k-palindromic numbers for $k = 3, \ldots, 9$ in his on-line Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences). We refer to it for this additional numerical data.

Now we begin to look at palindromicity as an intrinsic property – not attached to any particular base – nothing stops us from considering the fact that a given number can be palindromic in several different bases. For instance

$$3074 = (44244)_5 = (22122)_6.$$

Common sense dictates that multiple palindromicity should be a much more rare occurrence than a simple one, which is also rare as we have already shown. In fact it even seems unclear whether there are numbers which are k-palindromic in as many bases as desired. To formalize this, let

$$\mu_k(n) := \#\{b; n \text{ is } k\text{-palindromic in base } b\}.$$

We first propose the problem of determining whether μ_k is unbounded or not. Again the cases $k = 1, 2$ are easy. In the first case $n = (n)_m$ for any base $m > n$, and so $\mu_1(n) = \infty$ for every n. In the second case, set $n := 2^{2u+1}$ for some $u \in \mathbb{N}$. Then $n = 2^v(2^w - 1) + 2^v = (2^v, 2^v)_{2^w-1}$ for $v < w$ such that $v + w = 2u + 1$. Then $\mu_2(n) \geq u$. The following solves the case $k = 3$.

Theorem 3: *There exists an infinite sequence $n_1 < n_2 < n_3 < \ldots$ such that*

$$\mu_3(n_j) \geq \frac{1}{7} \log(n_j + 1).$$

Proof: Take $N \gg 0$ and assume that $\mu_3(n) < (1/7) \log(N + 1)$ for all $n \leq N$, so that

$$\sum_b \Phi_3(N, b) = \sum_{n=1}^{N} \mu_3(n) < \frac{1}{7} N \log(N + 1). \tag{1}$$

We will soon see that this is contradictory:

Set $L := [(N + 1)^{\frac{1}{3}}]$ and $M := [(N - 1)^{\frac{1}{2}}]$. For $L \leq b \leq M$ we let $\zeta(b) \in \mathbb{N}$ be the largest integer such that

$$\zeta(b)(b^2 + 1) + (b - 1)b \leq N.$$

Then $1 \leq \zeta(b) \leq b - 1$, and also every 3-palindromic number $n := eb^2 + fb + e$ with $e \leq \zeta(b)$ is less or equal that N. Hence $\Phi_3(N, b) \geq \zeta(b)b$ which implies that

$$\sum_b \Phi_3(N, b) \geq \sum_{b=L}^{M} \Phi_3(N, b) \geq \sum_{b=L}^{M} \zeta(b)b \geq \sum_{b=L}^{M} b\left(\frac{N}{b^2 + 1} - 2\right),$$

as $\zeta(b) + 2 \geq N/(b^2 + 1)$. We have that

$$\sum_{b=L}^{M} b\left(\frac{N}{b^2 + 1} - 2\right) \geq N \int_{L}^{M+1} \frac{t}{t^2 + 1} dt - 2M^2 \geq \frac{N}{2}(\log((M + 1)^2 + 1) - \log(L^2 + 1)) - 2N.$$

We have that $(M + 1)^2 + 1 \geq N$ and $L^2 + 1 \leq 2N^{2/3}$ and thus we conclude $\sum_p \Phi_3(N, p) > \frac{N}{6} \log N - 2N - \log 2$, which contradicts Inequality (1) for N large enough.

It follows that for each (sufficiently large) $N \in \mathbb{N}$ there exists an $n \leq N$ such that

$$\mu_3(n) \geq \frac{1}{7} \log(N + 1) \geq \frac{1}{7} \log(n + 1).$$

The fact that $\mu_3(n) < \infty$ implies that the set of such n's is infinite. □

Here is some sample data for the cases $k := 4, 5$.

$$\mu_4(624) = \mu_4(910) = 2, \quad \mu_4(19040) = 3, \quad \mu_5(2293) = 2.$$

For $k, \ell, N \in \mathbb{N}$, we let $\Phi_{k,\ell}(N)$ be the number of $n \leq N$ which are k-palindromes in ℓ different bases, that is

$$\Phi_{k,\ell}(N) := \#\{n \leq N; \mu_k(n) \geq \ell\}.$$

In particular $\Phi_{k,1} = \Phi_k$. The following table gives some more informative data.

k	ℓ	N	$\Phi_{k,\ell}(N)$
4	2	10^4	13
4	3	10^5	2
4	4	10^5	0
5	2	10^4	10
5	3	10^5	0
6	2	10^5	0

This suggests that for $k \geq 4, \ell \geq 2$ and $k + \ell \geq 8$, there are no k-palindromic numbers with multiplicity ℓ whatsoever.

Finally we can also consider

$$\mu_{\geq k}(n) := \#\{b; n \text{ is } j\text{-palindromic in base } b \text{ for some } j \geq k\} = \sum_{j \geq k} \mu_j(n),$$

that is the number of different bases in which n is a palindrome of length *at least* k. It is easy to see that this function is unbounded; we have that

$$n_L := 2^{2^L} - 1 = \overbrace{(2^{2^\ell} - 1, \ldots, 2^{2^\ell} - 1)}^{2^{L-\ell}}{}_{2^{2^\ell}}$$

and so n_L is $2^{L-\ell}$-palindromic in base 2^{2^ℓ} for $\ell = 0, \ldots, L$. Hence $\mu_{\geq k}(n_L) \geq L - \log_2 k$.

A further problem is to determine the *density* of k-palindromic numbers in ℓ different bases. From this point of view, Theorem 1 is an important advance towards the solution of the cases $k \geq 4$ and $\ell = 1$.

The cases when $\ell \geq 2$ seem to be much more elusive, but also interesting. A solution of them would allow us, for instance, to know how lucky we are when the number of the taxi-cab we are riding is the

$$19040 = (8888)_{13} = (5995)_{15} = (2, 14, 14, 2)_{19}.$$

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