COMPOSITIONS OF UNIONS OF GRAPHS

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1. INTRODUCTION

The concept of a composition of a graph was introduced by Arnold Knopfmacher and M.E. Mays in [2] and denotes a partition of the vertices such that the induced subgraph on each part is connected. Alternatively, a partition can be viewed as the set of connected components of a subgraph including all the vertices of the graph but only a subset of the edges. Most attention hitherto has been devoted to counting the numbers of compositions of various families of graphs, usually by finding some *ad hoc* recurrence relation based on their structure. We shall develop a more systematic method of analysis of how the composition of the union of two graphs can be obtained from the compositions of the two subgraphs. This provides new proofs giving greater insight into some known results (ladder graphs and wheel graphs), a new result on $3 \times n$ grid graphs, and a general result for the cartesian product of an arbitrary graph and a path with n vertices.

All graphs considered will be finite and undirected, with no loops or multiple edges. If G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set, where each edge can be thought of as an unordered pair of vertices. We shall, however, also use the more symbolic notation x–y to denote the edge between vertices x and y. The cartesian product $A \times B$ has vertex set $V(A) \times V(B)$, and $\{(a_1, b_1), (a_2, b_2)\}$ is an edge if and only if $a_1 = a_2$ and $\{b_1, b_2\} \in V(B)$, or $\{a_1, a_2\} \in V(A)$ and $b_1 = b_2$.

Since any composition of the graph is a partition of the vertex set, it defines an equivalence relation. Vertices x and y that are in the same part in a given composition will therefore simply be said to be related, written $x \sim y$. It is convenient to define a distance function associated with any composition of a connected graph G. If x and y are vertices of G, then $d(x, y)$ denotes the minimum number of additional edges of G that must be incorporated into the composition in order for x and y to become related (that is, $d(x, y)$ is the length of the shortest path in G joining their equivalence classes). For example, if $x \sim y$, then $d(x, y) = 0$, and if x and y are adjacent in G but $x \not\sim y$, then $d(x, y) = 1$.

In general, more than one subset of $E(G)$ may define the same composition of G. (For example, with the cycle C_n having n vertices and n edges, $E(G)$ obviously defines the composition with only one part, as does any subset of $E(G)$ containing $n - 1$ edges.) However, we shall say that an edge x–y belongs to a particular composition if its two endpoints are in the same part, that is, if $x \sim y$. Thus the edges belonging to a given composition of G form the largest subset of $E(G)$ defining that composition.

The number of compositions of a graph G will be denoted by $C(G)$. Suppose graph $G = A \cup B$, where $E(A) \cap E(B) = \emptyset$, so $A \cap B$ is a null graph. The problem is to determine $C(G)$ from $C(A)$ and $C(B)$. If P_n denotes the path with n vertices, then $C(A \times P_n)$ can be found by iteration, since $A \times P_1 \simeq A$, and $A \times P_n = (A \times P_{n-1}) \cup B$ for $n > 1$, where B is obtained from A by inserting an extra vertex of degree one adjacent to each vertex of A.

In this section, we discuss some easy cases, most of which are familiar, but derive new proofs of two known formulae. The first result appears in [2, Theorem 3].

Proposition 1.1: *Suppose $G = A \cup B$, as above. If $A \cap B = \emptyset$, or if $A \cap B$ is a singleton, then $C(G) = C(A)C(B)$.*

It follows immediately that attention may be restricted to connected graphs, and that inserting an extra vertex of degree one adjacent to any vertex doubles the number of compositions. Thus every tree with n vertices, including the path P_n, has 2^{n-1} compositions [2, Theorems 1 and 5].

The result for P_n can also be seen immediately by noting that every subset of the edge set defines a different partition of the vertices. This naive approach is also effective when counting compositions of the cycle C_n: every subset of the edge set defines a distinct composition, except, as remarked above, for the $(n-1)$-element subsets, which define the same composition as the whole edge set. Thus the total number of compositions of C_n is $2^n - n$, as stated in [2, Theorem 7]. It follows that the characteristic polynomial of the recurrence relation for $C(C_n)$ is $(\lambda - 2)(\lambda - 1)^2$, though a direct proof is probably difficult.

Proposition 1.2: *If W_n^* denotes the broken wheel (or fan) graph with n vertices, where $n \geq 2$, then $C(W_n^*) = F_{2n-1}$, the $(2n-1)^{th}$ Fibonacci number.*

Proof: The broken wheel W_n^* consists of a path P_{n-1} (the incomplete periphery of the wheel) together with one other vertex (the hub) adjacent to every vertex on the periphery. Thus $W_2^* = P_2$, and $W_n^* = W_{n-1}^* \cup B$, where $B = P_3 = b_1 - b_2 - b_3$, since the extra vertex (b_2) is adjacent to the hub b_1 and to the last vertex b_3 on the periphery.

Broken wheel, developing clockwise. Primed labels
refer to the previous iteration.

We classify the compositions of W_n^* according as the last vertex on the periphery is or is not related to the hub, and denote the corresponding numbers of compositions by $C_0(W_n^*)$ and $C_1(W_n^*)$ respectively. At each iteration the previous b_2 becomes the new b_3 (the last vertex on the periphery). Each composition of W_{n-1}^* in which b_1 and b_3 were related leads to one composition of W_n^* in which b_1 and b_2 are related and one in which they are not, whereas each composition in which b_1 and b_3 were unrelated leads to one of the former and two of the latter. The resulting partitions of B in W_n may be tabulated thus:

	$b_1 \sim b_3$	$b_1 \not\sim b_3$
$b_1 \sim b_2$	$\{\{b_1, b_2, b_3\}\}$	$\{\{b_1, b_2\}, \{b_3\}\}$
$b_1 \not\sim b_2$	$\{\{b_1, b_3\}, \{b_2\}\}$	$\{\{b_1\}, \{b_2, b_3\}\}$ and $\{\{b_1\}, \{b_2\}, \{b_3\}\}$

This leads to the vector recurrence relation

$$\begin{pmatrix} C_0(W_n^*) \\ C_1(W_n^*) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C_0(W_{n-1}^*) \\ C_1(W_{n-1}^*) \end{pmatrix}$$

for $n > 2$, whence

$$\begin{pmatrix} C_0(W_n^*) \\ C_1(W_n^*) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{n-2} \begin{pmatrix} C_0(W_2^*) \\ C_1(W_2^*) \end{pmatrix}.$$

Since clearly $C_0(W_2^*) = C_1(W_2^*) = 1$, and since $C(W_n^*) = C_0(W_n^*) + C_1(W_n^*)$, we have

$$C(W_n^*) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} C_0(W_n^*) \\ C_1(W_n^*) \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The result now follows by standard diagonalization techniques [1, p. 347] and Fibonacci identities. (The characteristic polynomial of the transition matrix M is $\lambda^2 - 3\lambda + 1$ and its eigenvalues are $\frac{1}{2}(3 + \sqrt{5})$, the square of the golden ratio, and its conjugate surd.) $\square$

The transition matrix M is familiar from Arnold's cat map [1, p. 678] and elsewhere. An alternative proof is based on the recurrence relation $C(W_n^*) = 3C(W_{n-1}^*) - C(W_{n-2}^*)$, which can be obtained directly. In fact, it can be shown that the same recurrence applies if, at each stage, the copy of P_3 is adjoined to an arbitrary edge on the boundary of the previous graph, instead of always being adjoined to the "last spoke" in the broken wheel.

Proposition 1.3: *If L_{2n} denotes the ladder graph with $2n$ vertices, which is isomorphic to $P_2 \times P_n$, then*

$$C(L_{2n}) = \frac{(3 + \sqrt{10})^n - (3 - \sqrt{10})^n}{\sqrt{10}}.$$

Ladder graph.

Proof: The proof is similar to that of Proposition 1.2, and is sketched only briefly. An alternative proof using recurrence relations is given in [2, Theorem 9]. For $n > 1$, the graph L_{2n} is obtained from L_{2n-2} by adjoining $b_1 - b_2 - b_3 - b_4$, with b_1 and b_4 replacing the endpoints of the last rung of L_{2n-2} and $b_2 - b_3$ becoming the last rung of L_{2n}. Compositions are classified according as the endpoints of the last rung are or are not related. The transition matrix is therefore obtained from the following table giving the numbers of compositions in each case:

	$b_1 \sim b_4$	$b_1 \not\sim b_4$
$b_2 \sim b_3$	2	3
$b_2 \not\sim b_3$	3	4.

It follows as in Proposition 1.2 that

$$C(L_{2n}) = (\,1 \quad 1\,) \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which leads to the final result. (The characteristic polynomial of the transition matrix is $\lambda^2 - 6\lambda - 1$ and its eigenvalues are $3 \pm \sqrt{10}$.) $\square$

The results of this section can be generalized to the situation where $B = P_m$ for any integer $m \geq 3$, and its endpoints are identified with two adjacent points in A. The arguments need not be repeated, but the transition matrix for iterating the procedure is

$$M = \begin{pmatrix} 2^{m-2} - m + 2 & 2^{m-2} - 1 \\ 2^{m-2} - 1 & 2^{m-2} \end{pmatrix}$$

and the coefficients in the expression for $C(A \cup B)$ are the column sums $2^{m-1} - m + 1$ and $2^{m-1} - 1$. We now pass to the situation where $A \cap B$ can be more than just a pair of points.

2. THE GENERAL RESULT

Suppose a general graph G is expressed as a union $G = A \cup B$, with $E(A \cap B) = \emptyset$. If $\sim_G$ is the relation associated with some composition of G, then the restriction of $\sim_G$ to the vertex set of A does not necessarily define a composition of A, since one or more of the equivalence classes of the restriction may not be connected if edges from A only are used. However, there is a unique composition of A defined by all edges a_i–a_j belonging to the composition of G (that is, such that $a_i \sim_G a_j$), and similarly for B. We shall say that the given composition of G is valid for this unique pair of compositions (the largest from which it can be obtained).

The procedure in counting compositions of G without duplication will therefore be to consider all pairs of compositions of A and B, and count the resulting composition of G only if it is valid. The examples in the previous section illustrate the method (considering whether pairs of vertices in $A \cap B$ are or are not related in A), but in general unrelated pairs need to be further subdivided, depending on the distance between their equivalence classes.

We therefore classify each pair of vertices $\{c_i, c_j\}$ in $A \cap B$ into one of these three types:

$$(0)\ d_A(c_i, c_j) = 0 (\text{i.e., } c_i \sim_A c_j); \quad (1)\ d_A(c_i, c_j) = 1; \quad (2)\ d_A(c_i, c_j) \geq 2.$$

Note that $\{c_i, c_j\}$ is of type (1) if and only if $c_i \not\sim_A c_j$ and there exist adjacent vertices a_i and a_j in A such that $c_i \sim_A a_i$ and $c_j \sim_A a_j$. (Naturally, the edge a_i–a_j cannot belong to the composition of A in question.) In all the examples above, the only pairs of vertices that arose were themselves adjacent in A, so had to be of type (0) or (1).

Proposition 2.1: *Suppose $G = A \cup B$ and we have a valid composition of G arising from compositions of A and B. If a pair (c_i, c_j) of vertices in $A \cap B$ is of type (1), then c_i and c_j remain unrelated in G. Furthermore, if every pair of vertices in $A \cap B$ is of type (0) or (1) and if the composition of G is valid, then the restriction of $\sim_G$ to $V(A)$ is $\sim_A$.*

Proof: First, if (c_i, c_j) is of type (1), then from the definition there exist adjacent vertices a_i and a_j in A such that $c_i \sim_A a_i$ and $c_j \sim_A a_j$, so a *fortiori* $c_i \sim_G a_i$ and $c_j \sim_G a_j$. If $c_i \sim_G c_j$, then by transitivity $a_i \sim_G a_j$. Since the composition of G is valid, it follows that the edge a_i–a_j belongs to the composition of A, which contradicts the definition of type (1).

Next, if the restriction of $\sim_G$ to $V(A)$ is not equal to $\sim_A$, then there must be a pair of vertices in A, say x and y, that are related in G but not in A. Thus there must be a path from x to y made up of edges belonging to the compositions of A or B (not both, since $E(A \cap B) = \emptyset$). At least one edge must belong to the composition of B, since $x \not\sim_A y$. In this path, starting from x, let c_i be the first vertex such that all edges (if any) from x to c_i belong to the composition of A, but the edge from c_i to the next vertex belongs to the composition of B. Define c_j similarly, starting from y. Then c_i and c_j must be in $A \cap B$, and clearly $x \sim_A c_i$ and $y \sim_A c_j$, whence $c_i \not\sim_A c_j$. From the hypothesis it follows that the pair (c_i, c_j) is of type (1). However, since $x \sim_G y$, we have by transitivity that $c_i \sim_G c_j$, which contradicts the first part of the proposition. $\square$

In contrast to Proposition 2.1, if a pair $\{c_i, c_j\}$ is of type (2) in A and $c_i \sim_B c_j$, then the resulting composition of G may still be valid, in spite of the fact that c_i and c_j do become related under G, and the restriction of $\sim_G$ to $V(A)$ is different from $\sim_A$. Since this change will in general affect other pairs in $A \cap B$, the validity of the resulting composition of G cannot be determined without investigating all pairs originally of type (2) in A.

An estimate of the number of cases needing to be considered (taking into account all compositions of A) may be obtained as follows. Suppose $A \cap B$ has p vertices and has q edges coming from $E(A)$, and let $q' = \frac{1}{2}p(p-1) - q$. For any given composition of A, each of the q pairs of adjacent vertices in $A \cap B$ must be of type (0) or type (1), whereas the q' pairs of non-adjacent vertices can be of any one of the three types. Thus the total number of cases to be considered is at most $2^q 3^{q'}$. This is not a good bound, since the transitivity of the relation $\sim_A$ ensures that the types for different pairs are not all independent.

The general result may be summarized as follows, using the notation above.

Theorem 2.2: *Let $C_0(A), \ldots, C_{k-1}(A)$ denote the numbers of compositions of A in the k cases determined by the possible types of pairs of vertices in $A \cap B$, and for each case $r = 0, 1, \ldots, k-1$ let M_r denote the number of valid compositions of G obtained by considering all compositions of B. Then $C(G) = M_0 C_0(A) + \cdots + M_{k-1} C_{k-1}(A)$.*

Corollary 2.3: *If $G_n = A \times P_n$ for $n = 1, 2, 3, \ldots,$ then $C(G) = \boldsymbol{z} M^{n-1} \boldsymbol{w}_1$, where $\boldsymbol{z}$ is the $1 \times k$ row vector $(1 \ldots 1)$, $\boldsymbol{w}_1$ is a $k \times 1$ column vector, and M is a $k \times k$ matrix. Hence $C(G_n)$ satisfies a k^{th} order recurrence relation, and if M is a diagonalizable matrix, then $C(G_n)$ is a linear combination of the n^{th} powers of the eigenvalues of M.*

Proof: Let B denote the graph obtained from A by inserting a new vertex of degree one adjacent to each vertex in A. Then $G_1 = A \times P_1 \simeq A$, and $G_n \simeq G_{n-1} \cup B$ for $n > 1$, where the intersection consists of the new vertices in B being identified with an isomorphic copy of A in G_{n-1}. In order for the iteration to proceed, the compositions of G_n must be classified into cases as well. Let m_{rs} denote the number of case r compositions of G_n that arise from case s compositions of G_{n-1}, and let $M = (m_{rs})$. If $\boldsymbol{w}_n$ denotes the column vector $(C_1(G_n) \cdots C_k(G_n))'$, then $\boldsymbol{w}_n = M\boldsymbol{w}_{n-1}$ for all n, by Theorem 2.2, so $\boldsymbol{w}_n = M^{n-1}\boldsymbol{w}_1$.

Clearly $C(G_n) = \sum_{r=1}^{k} C_r(G_n) = \boldsymbol{z}\boldsymbol{w}_n$, and the result follows. The consequences are standard results from linear algebra [1, p. 347]. □

3. TWO APPLICATIONS

We illustrate the general result with two applications of Theorem 2.2 and Corollary 2.3 to illustrate the distinction between types (1) and (2). The first is a straightforward example on wheel graphs, and the second is a more intricate application to $3 \times n$ grid graphs.

Proposition 3.1: *If W_n denotes the wheel graph with n vertices, then $C(W_n) = L_{2n-2} - n + 1$ for $n \geq 4$, where L_k denotes the k^{th} Lucas number.*

Proof: The wheel graph W_n is obtained from the broken wheel W_n^* by adjoining the missing edge on the periphery. To avoid having multiple edges, we consider it only for $n \geq 4$. We apply Theorem 2.1 with $A = W_n^*$ and $B = b_2$–b_n, say. For every composition of A in which $\{b_2, b_n\}$ is of type (0) or (1), the relationship between b_2 and b_n is unchanged in W_n (related or unrelated, respectively), and the final composition of W_n is the same as the original composition of W_n^*. On the other hand, every composition of A in which $\{b_2, b_n\}$ is of type (2) gives rise to two compositions of W_n, one in which b_2 and b_n are related in b, and one in which they are not. Thus, in the obvious notation,

$$C(W_n) = C_0(W_n^*) + C_1(W_n^*) + 2C_2(W_n^*) = C(W_n^*) + C_2(W_n^*).$$

Since $C(W_n^*) = F_{2n-1}$ by Proposition 1.2, it is sufficient to find $C_2(W_n^*)$, so suppose $d_A(b_2, b_n) = 2$, that is, the distance between their equivalence classes is 2 or more. Then neither b_2 nor b_n can be related to the hub b_1, so their equivalence classes must be of the form $\{b_2, \ldots, b_i\}$ and $\{b_j, \ldots, b_n\}$, where $2 \leq i < i+1 < j \leq n$. The remaining equivalence classes form a composition of the broken wheel graph W_{j-i}^* with b_1 at the hub, and $b_{i+1}, \ldots, b_{j-1}$ on the periphery. Thus

$$\begin{aligned} C_2(W_n^*) &= \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} C(W_{j-i}^*) = \sum_{i=2}^{n-2} \sum_{k=2}^{n-i} C(W_k^*) \\ &= \sum_{k=2}^{n-2} \sum_{i=2}^{n-k} C(W_k^*) = \sum_{k=2}^{n-2} (n-k-1) F_{2k-1}. \end{aligned}$$

A straightforward induction (or a more elegant analysis using generating functions) shows that this last sum is $F_{2n-3} - n + 1$, and leads to the final result

$$C(W_n) = C(W_n^*) + C_2(W_n^*) = F_{2n-1} + F_{2n-3} - n + 1 = L_{2n-2} - n + 1. \quad \square$$

The characteristic polynomial for the recurrence relation for $C(W_n)$ is therefore

$$(\lambda^2 - 3\lambda + 1)(\lambda - 1)^2 = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 5\lambda + 1,$$

though a direct proof is unlikely to be illuminating, if one can be found. An implicit formula for $C(W_n)$ appears in [2, Theorem 8].

Theorem 3.2: *Let $G_{3,n}$ denote the $3 \times n$ grid graph $P_3 \times P_n$. Then $C(G_{3,n}) = c_1\alpha_1^{n-1} + \cdots + c_5\alpha_5^{n-1}$, where the approximate numerical values of the constants are as follows*:

r	c_r	α_r
1	0.157907	-0.221357
2	0.000396095	0.191796
3	3.82152	19.3717
4	$0.0100861 - 0.0126952i$	$1.82895 - 1.23229i$
5	$0.0100861 + 0.0126952i$	$1.82895 + 1.23229i$

$3 \times n$ Grid Graph.

Proof: Since $G_{3,n} = P_3 \times P_n$, we have $A = P_3 = a_1 - a_2 - a_3$, using the notation of Corollary 2.3. The graph B that is adjoined at the n^{th} stage is a tree with six vertices (in the shape of a letter E), whose end vertices b_1, b_2, b_3 replace those in the last row of $G(3, n-1)$, and whose other vertices b_4, b_5, b_6 form the last row of $G(3, n)$, as shown. The possible partitions of the intersection $\{b_1, b_2, b_3\}$ fall into six cases, which are listed below, including the type for each pair of vertices, and the resulting equivalence classes:

Case	$\{b_1, b_2\}$ type	$\{b_1, b_3\}$ type	$\{b_2, b_3\}$ type	Parts
I	0	0	0	$\{b_1, b_2, b_3\}$
II	0	1	1	$\{b_1, b_2\}, \{b_3\}$
III	1	1	0	$\{b_1\}, \{b_2, b_3\}$
IV	1	1	1	$\{b_1\}, \{b_2\}, \{b_3\}$
V	1	2	1	$\{b_1\}, \{b_2\}, \{b_3\}$
VI	1	0	1	$\{b_1, b_3\}, \{b_2\}$

Note that the partitions in cases IV and V have the same parts, and can only be distinguished by the type of the pair $\{b_1, b_3\}$ of end vertices. Also note that cases IV and VI can arise in $G_{3,n}$ only for $n > 1$, as they are not compositions of P_3 itself, so the initial vector $\boldsymbol{w}_1 = (1, 1, 1, 0, 1, 0)'$, with zeros in positions 4 and 6.

The transition matrix is

$$M = \begin{pmatrix} 2 & 3 & 3 & 4 & 5 & 3 \\ 3 & 4 & 5 & 6 & 6 & 5 \\ 3 & 5 & 4 & 6 & 6 & 5 \\ 2 & 1 & 1 & 2 & 0 & 4 \\ 2 & 5 & 5 & 6 & 8 & 2 \\ 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

whose entries were verified using Mathematica. In general, a column of M can be obtained by taking the partition of $\{b_1, b_2, b_3\}$ for a particular case, combining it with each of the 16 compositions of B, then listing the distinct final compositions that result, as well as their restrictions to $\{b_1, b_2, b_3\}$ and $\{b_4, b_5, b_6\}$. Those whose restrictions to $\{b_1, b_2, b_3\}$ differ from the initial partition of $\{b_1, b_2, b_3\}$ are duplicates and must not be counted (except for one in case V, where there is a type 2 pair of vertices). The remaining compositions can be grouped into cases according to their restrictions to $\{b_4, b_5, b_6\}$ (which replaces $\{b_1, b_2, b_3\}$ for the next iteration), and the number in each case gives the required entry in the matrix.

Unfortunately, since in both case IV and case V the restrictions of the compositions to $\{b_1, b_2, b_3\}$ or $\{b_4, b_5, b_6\}$ are trivial, inspection of the unrestricted final composition is required to distinguish them. First consider columns 4 and 5, where the initial partition of $\{b_1, b_2, b_3\}$ is trivial. The entries in row 1 count final compositions in which b_4, b_5, b_6 are all related. There are four of these whose restriction to $\{b_1, b_2, b_3\}$ is trivial, giving $m_{14} = m_{15} = 4$ so far. There is one final composition whose restriction is $\{\{b_1, b_3\}, \{b_2\}\}$: this is valid in case V when $\{b_1, b_3\}$ is type 2 but not in case IV, so m_{15} increases to 5 but m_{14} stays at 4. In rows 2, 3 and 6 the difference of type between cases IV and V is irrelevant, so $m_{r4} = m_{r5}$ for $r = 2, 3, 6$. In rows 4 and 5, when b_4, b_5, b_6 are unrelated in the final composition, there are six for which $\{b_4, b_6\}$ is of type 2 (i.e., case V), and two for which $\{b_4, b_6\}$ is of the same type as $\{b_1, b_3\}$ (i.e., the case remains the same). Thus $m_{44} = 0 + 2, m_{45} = 0 + 0, m_{54} = 6 + 0$, and $m_{55} = 6 + 2$. Now consider any other entry in rows 4 and 5, say in column s. By inspection of the unrestricted composition, it is easy to see whether the equivalence classes of b_4 and b_6 can be linked with one extra edge or require more than one, i.e., whether the pair $\{b_4, b_6\}$ is of type 1 or 2, i.e., whether the final composition falls into case IV or case V, i.e., whether it should be counted towards m_{4s} or m_{5s}.

Once M has been established, the remaining calculations are straightforward linear algebra, carried out using Mathematica. The characteristic polynomial is

$$(\lambda + 1)(\lambda^5 - 23\lambda^4 + 75\lambda^3 - 91\lambda^2 - 6\lambda + 4) = (\lambda + 1)p(\lambda), \quad \text{say.}$$

The eigenvalues of M are -1 together with the zeros $\alpha_1, \ldots, \alpha_5$ of $p(\lambda)$, which is irreducible over the rationals. The constants $c_1, \ldots, c_5$ come from the diagonalization process. □

The powers of -1 do not appear in the formula for $C(G_{3,n})$, since their coefficient is zero; presumably the eigenvalue -1 simply reflects the symmetry between cases II and III. It is also noteworthy that $p(\lambda)$ is not only irreducible over $\mathbb{Q}$, it is not even solvable by radicals, being of prime degree and having exactly two non-real zeros [3, Lemma 14.7] so there is no expression for $C(G_{3,n})$ in terms of surds. This is markedly dissimilar to the examples in the previous section. However, there is one similarity, in that the formula is again dominated by one of the eigenvalues, in this case α_3, as is easily seen by inspection of the magnitudes of the eigenvalues and their coefficients. From an inspection of the coefficients of $p(\lambda)$ it is apparent that direct determination of the fifth order recurrence relation satisfied by the sequence $\{C(G_{3,n})\}$ would be a daunting task.

The accuracy of the formula was checked by comparing its values for $n = 1, 2, 3, 4$ with the actual number of compositions of $G_{3,n}$, obtained by listing and counting them individually (the computation for $n = 4$ taking over four hours). The largest difference is less than 10^{-10}, and is obviously due solely to round-off errors. The values are as follows:

n	1	2	3	4
$C(G_{3,n})$	4	74	1434	27080.

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