ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-615 Proposed by Paul S. Bruckman, Sointula, Canada

Given $n \geq 1$ and complex numbers $x_0,\ x_1,\ \ldots,\ x_{n-1}$, define the "cyclical" matrix

$$\boldsymbol{A}_n = \begin{vmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \ldots & x_{n-4} & x_{n-3} \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_0 \end{vmatrix}.$$

Let D_n denote the determinant of $\boldsymbol{A}_n$, and $s_n = x_0 + x_1 + \cdots + x_{n-1}$. Prove that if the x_k's are integers such that $s_n \neq 0$, then $s_n | D_n$.

H-616 Proposed by Paul S. Bruckman, Sointula, Canada

Let $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, $n = 0,\ 1,\ \ldots$ be the nth Catalan number (known to be an integer).

Prove that C_n is odd if and only if $n = 2^u - 1$, where $u = 0,\ 1,\ \ldots$.

H-617 Proposed by H.-J. Seiffert, Berlin, Germany

The sequence of Fibonacci polynomials is defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Show that, for all real numbers x and all nonnegative integers n,

(a) $$\sum_{k=0}^{2n} (-1)^{\lfloor k/2 \rfloor} \binom{2n}{k} F_k(x) = \sqrt{2}(-1)^n (x^2+4)^{n/2} F_n(x) \cos\Big(ny + \frac{\pi}{4}\Big),$$

(b) $$\sum_{k=0}^{2n} (-1)^{\lceil k/2 \rceil} \binom{2n}{k} F_k(x) = \sqrt{2}(-1)^n (x^2+4)^{n/2} F_n(x) \sin\Big(ny + \frac{\pi}{4}\Big),$$

where $y = \arccos\dfrac{x}{\sqrt{x^2+4}}$. Here, $\lfloor . \rfloor$ and $\lceil . \rceil$ denote the floor and ceiling function, respectively.

SOLUTIONS

A ratio of Gammas

H-602 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 41, no. 4, August 2003)

Find the limit

$$\lim_{n\to\infty} \frac{\Gamma\left(-k\frac{F_{n+1}}{\alpha F_n}\right)}{\Gamma\left(-\ell\frac{L_{n+1}}{\alpha L_n}\right)},$$

where k and ℓ are fixed positive integers, Γ is the Euler function, and α is the golden section.

Solution by V. Mathe, Marseille, France

It is well known (see [1], for example) that the gamma function Γ has simple poles at $z = -n$ for $n = 0,\ 1,\ 2,\ \ldots$, with the respective residues $(-1)^n/n!$; that is:

$$\lim_{z\to -n}(z+n)\Gamma(z) = \frac{(-1)^n}{n!}.$$

We also know that $F_{n+1}/(\alpha F_n)$ and $L_{n+1}/(\alpha L_n)$ tend to 1 as $n \to \infty$.
We therefore have:

$$\lim_{n\to\infty}\left(k - k\frac{F_{n+1}}{\alpha F_n}\right)\Gamma\left(-k\frac{F_{n+1}}{\alpha F_n}\right) = \frac{(-1)^k}{k!},$$

and

$$\lim_{n\to\infty}\left(\ell - \ell\frac{L_{n+1}}{\alpha L_n}\right)\Gamma\left(-\ell\frac{L_{n+1}}{\alpha L_n}\right) = \frac{(-1)^\ell}{\ell!}.$$

Then

$$\frac{\left(k - k\frac{F_{n+1}}{\alpha F_n}\right)\Gamma\left(-k\frac{F_{n+1}}{\alpha F_n}\right)}{\left(\ell - \ell\frac{L_{n+1}}{\alpha L_n}\right)\Gamma\left(-\ell\frac{L_{n+1}}{\alpha L_n}\right)} \to (-1)^{k-\ell}\frac{\ell!}{k!}$$

as $n \to \infty$.
However,

$$\frac{\left(k - k\frac{F_{n+1}}{\alpha F_n}\right)}{\left(\ell - \ell\frac{L_{n+1}}{\alpha L_n}\right)} = \frac{k}{\ell}\cdot\frac{\alpha F_n - F_{n+1}}{\alpha L_n - L_{n+1}}\cdot\frac{L_n}{F_n} = -\frac{k}{\ell}\cdot\frac{L_n}{\sqrt{5}F_n} \to -\frac{k}{\ell}$$

as $n \to \infty$.

Therefore the desired limit is $(-1)^{k-\ell+1}\dfrac{\ell\cdot\ell!}{k\cdot k!}$.

[1] *CRC standard mathematical tables and formulae.* 30^{th} edition. Edited by Daniel Zwillinger, Steven G. Krantz and Kenneth H. Rosen. CRC Press, Boca Raton, FL, 1996: 494.

Also solved by Paul Bruckman, W. Janous, H.-J. Seiffert and the proposer.

Sums of reciprocals of Fibonacci numbers

H-603 Proposed by the E. Herrmann, Siegburg, Germany (Vol. 41, no. 5, November 2003)

Show that if $n \geq 3$ and $n \equiv 1 \pmod 2$, then

$$\frac{1}{F_n} < \sum_{k=0}^{\infty}\frac{1}{F_{n+2k}} < \frac{1}{F_{n-1}}.$$

However, if $n \geq 4$ and $n \equiv 0 \pmod 2$, then

$$\frac{1}{F_{n-1}} < \sum_{k=0}^{\infty}\frac{1}{F_{n+2k}} < \frac{1}{F_{n-2}}.$$

Solution by Harris Kwong, Fredonia, NY

It is obvious that $1/F_n < \sum_{k=0}^{\infty} 1/F_{n+2k}$. We will use Binet's formulas to establish the other three inequalities. Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Since $\beta < 0$, if n is odd, then

$$\sqrt{5}F_{n+2k} = \alpha^{n+2k} - \beta^{n+2k} > \alpha^{n+2k} \quad \text{and} \quad \sqrt{5}F_{n-1} = \alpha^{n-1} - \beta^{n-1} < \alpha^{n-1}.$$

Hence, for odd integers $n \geq 3$,

$$\sum_{k=0}^{\infty}\frac{1}{F_{n+2k}} < \sum_{k=0}^{\infty}\frac{\sqrt{5}}{\alpha^{n+2k}} = \frac{\sqrt{5}}{\alpha^{n-2}(\alpha^2-1)} = \frac{\sqrt{5}}{\alpha^{n-1}} < \frac{1}{F_{n-1}}.$$

In a similar manner, if n is even and $n \geq 4$, then

$$\sqrt{5}F_{n+2k} > \sqrt{5}F_{n+2k-1} > \alpha^{n+2k-1}, \qquad \sqrt{5}F_{n-2} < \alpha^{n-2},$$
$$\sqrt{5}F_{n+2k} < \alpha^{n+2k} \quad \text{and} \quad \sqrt{5}F_{n-1} > \alpha^{n-1}.$$

Therefore, for even integers $n \geq 4$,

$$\sum_{k=0}^{\infty} \frac{1}{F_{n+2k}} < \sum_{k=0}^{\infty} \frac{\sqrt{5}}{\alpha^{n+2k-1}} = \frac{\sqrt{5}}{\alpha^{n-2}} < \frac{1}{F_{n-2}},$$

and

$$\sum_{k=0}^{\infty} \frac{1}{F_{n+2k}} > \sum_{k=0}^{\infty} \frac{\sqrt{5}}{\alpha^{n+2k}} = \frac{\sqrt{5}}{\alpha^{n-1}} > \frac{1}{F_{n-1}}.$$

Also solved by Paul Bruckman, V. Mathe, H.-J. Seiffert and the proposer.

An Already Encountered Matrix

H-604 Proposed by Mario Catalani, Torino, Italy
(Vol. 41, no. 5, November 2003)

In **H-592**, the proposers introduced, for $n \geq 2$, a non diagonal $n \times n$ matrix A such that $A^2 = xA + yI$, where $x,\ y$ are indeterminates and I is the identity matrix.

a) State the conditions under which all the eigenvalues of A are equal.

b) Assume now that not all the eigenvalues of A are equal. Assume that A is a $2n \times 2n$ matrix, and that $\operatorname{tr}(A) = nx$. Consider the Hamilton-Cayley equation for A

$$\sum_{k=0}^{2n} (-1)^k \lambda_k A^{2n-k} = 0,$$

where $\lambda_0 = 1$. Find $\sum_{k=0}^{2n} \lambda_k$.

Solution by the proposer

If λ is an eigenvalue of A, it must satisfy

$$\lambda^2 = x\lambda + y;$$

hence, the eigenvalues are

$$\alpha \equiv \alpha(x, y) = \frac{x + \sqrt{x^2 + 4y}}{2} \quad \text{and} \quad \beta \equiv \beta(x, y) = \frac{x - \sqrt{x^2 + 4y}}{2}.$$

We are going to show that all the eigenvalues are equal *iff* $x^2 + 4y = 0$; that is, $\alpha = \beta$. If $\alpha = \beta$, then trivially all the eigenvalues are equal. Now assume that all the eigenvalues are equal and $\alpha \neq \beta$. Using the result from [1], we obtain

$$\sum_{k=0}^{n} (-1)^k \lambda_k F_{n-k} = 0, \tag{1}$$

where $F_n \equiv F_n(x, y)$ are the Fibonacci polynomials

$$F_n = xF_{n-1} + yF_{n-2}, \quad F_0 = 0, \quad F_1 = 1,$$

and λ_i are the elementary symmetric functions of the eigenvalues. If all the eigenvalues are equal (say, all equal to α), then

$$\lambda_i = \binom{n}{i} \alpha^i,$$

so that the identity (1) becomes

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^k F_{n-k} = 0.$$

If $\alpha \neq 0$, we can use Binet's form for F_n and the identity becomes (after simplifying the denominator)

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^k (\alpha^{n-k} - \beta^{n-k}) = 0.$$

Manipulation of the left hand side leads to

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^n - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha^k \beta^{n-k} = -\sum_{k=0}^{n} \binom{n}{k} (-\alpha)^k \beta^{n-k} = -(\beta - \alpha)^n,$$

where we used Identity 1.25 in [2] and the binomial theorem. It follows that $\alpha = \beta$, contradicting our hypothesis.

For part **b.**, imagine that we have k eigenvalues equal to α and $2n - k$ eigenvalues equal to β; then,

$$k\alpha + (2n - k)\beta = nx.$$

¿From this, we obtain $k = n$, using the fact that $\alpha + \beta = x$. Now note that $\lambda_0 = 1$ and if $\{\mu_1, \ldots, \mu_{2n}\}$ are the eigenvalues, then

$$\lambda_k = \sum_{\substack{i_1,\ldots,i_k \\ i_1<i_2<\cdots<i_k}} \mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}, \qquad 1 \leq k \leq 2n.$$

In this case, with n eigenvalues equal to α and n eigenvalues equal to β, this becomes

$$\lambda_k = \sum_{h=k-n}^{\min(k,n)} \binom{n}{h}\binom{n}{k-h}\alpha^h\beta^{k-h},$$

where the summands corresponding to negative h are taken equal to zero. Thus,

$$\sum_{k=1}^{2n}\lambda_k = \sum_{k=1}^{2n}\sum_{h=k-n}^{\min(k,n)} \binom{n}{h}\binom{n}{k-h}\alpha^h\beta^{k-h}.$$

When $k = 0$, we take the sum to be 1. We can then write

$$\sum_{k=0}^{2n}\lambda_k = \sum_{k=0}^{2n}\sum_{h=k-n}^{\min(k,n)} \binom{n}{h}\binom{n}{k-h}\alpha^h\beta^{k-h}.$$

Now consider

$$(1+\alpha)^n(1+\beta)^n = \sum_{h=0}^{n}\sum_{i=0}^{n}\binom{n}{h}\binom{n}{i}\alpha^h\beta^i.$$

Setting $h + i = k \leq 2n$, so that $k - n \leq h = k - i \leq \min(k, n)$, upon substitution we obtain

$$(1+\alpha)^n(1+\beta)^n = [(1+\alpha)(1+\beta)]^n = (1+x-y)^n = \sum_{k=0}^{2n}\sum_{h=k-n}^{\min(k,n)}\binom{n}{h}\binom{n}{k-h}\alpha^h\beta^{k-h} = \sum_{k=0}^{2n}\lambda_k,$$

where we used $\alpha + \beta = x,\ \alpha\beta = y$.

1. N. Gauthier and J.R. Gosselin, "Problem H-592", The Fibonacci Quarterly **40.5** (2002): 473.

2. H.W. Gould, "Combinatorial Identities", Morgantown, W. Va. 1972.

Also solved by Paul Bruckman and V. Mathe.

Please Send in Proposals!