ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2005. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-986 Proposed by Br. J. Mahon, Australia

Prove

$$\sum_{i=2}^{n} \frac{F_{2i-2}F_{2i}}{3\left(F_{2i}^2 - 1\right)\left(F_{2i+2}^2 - 1\right)} = \frac{-1}{8} + \frac{F_{2n}F_{2n+2}}{3\left(F_{2n+2}^2 - 1\right)}$$

B-987 Proposed by M.N. Deshpande and J.P. Shiwalkar, Nagpur, India

An unbiased coin is tossed n times. Let A be the event that no two successive heads occur. Show $Pr(A) = (F_{n+2})/2^n$.

B-988 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA

Show that for an odd integer n and any integer k, $5(F_k^2 - F_{k-n}^2) + 4(-1)^k$ is the product of two Lucas numbers, while if n is an even integer, then $F_k^2 - F_{k-n}^2$ is the product of two Fibonacci numbers.

B-989 Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Catalunya, Barcelona, Spain

Let n be a positive integer. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{F_k C(n,k)}\right)^n \geq e^{n-F_{2n}}.$$

B-990 Proposed by Mario Catalani, University of Torino, Torino, Italy

Let $F_n(x,y)$ and $L_n(x,y)$ be the bivariate Fibonacci and Lucas polynomials, defined, respectively, by $F_n(x,y) = xF_{n-1}(x,y)+yF_{n-2}(x,y), F_0(x,y) = 0, F_1(x,y) = 1$ and $L_n(x,y) = xL_{n-1}(x,y)+yL_{n-2}(x,y), L_0(x,y) = 2, L_1(x,y) = x$. Assume $x^2+4y \neq 0$. Prove the following identities

A) $$\sum_{k=0}^{n}\binom{n}{k}F_k(x,y)F_{n-k}(x,y) = \frac{1}{x^2+4y}(2^n L_n(x,y) - 2x^n),$$

B) $$\sum_{k=0}^{n}\binom{n}{k}L_k(x,y)L_{n-k}(x,y) = 2^n L_n(x,y) + 2x^n.$$

SOLUTIONS

A Golden Area

B-971 Proposed by Peter Jeuck, Hewitt, NJ (Vol. 42, no. 1, Feb. 2004)

If A_n is the area of the region under the curve given by

$$y = \sum_{i=0}^{p} a_i x^{p-i}$$

on the interval $[F_n, F_{n+1}]$, show that

$$\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right) = \alpha^{p+1}.$$

Solved by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

The end behaviour of the polynomial

$$y(x) = \sum_{i=0}^{P} a_i x^{p-i}$$

is determined by $f(x) = a_0 x^p$. Without loss of generality we may assume that $a_0 > 0$. Since $\lim_{x\to\infty} a_0 x^p = +\infty$ if $a_0 > 0$, we see that there is $n_0 \in \mathbb{N}$ such that for $n > n_0$, the graph of y is above the x axis on every interval of the form $[F_n, F_{n+1}]$. For $n > n_0$,

$$A_n = \int_{F_n}^{F_{n+1}} y(x)dx = \sum_{i=0}^{p} a_i \frac{F_{n+1}^{p-i+1} - F_n^{p-i+1}}{p-i+1} = F_{n+1}^{p+1}\cdot\sum_{i=0}^{P} a_i \cdot \frac{F_{n+1}^{-i} - \left(\frac{F_n}{F_{n+1}}\right)^{p+1}\cdot F_n^{-i}}{p-i+1}.$$

Thus

$$\frac{A_{n+1}}{A_n} = \left(\frac{F_{n+2}}{F_{n+1}}\right)^{p+1} \cdot \frac{\sum_{i=0}^{P} a_i \frac{F_{n+2}^{-i} - \left(\frac{F_{n+1}}{F_{n+2}}\right)^{p+1} \cdot F_{n+1}^{-i}}{p-i+1}}{\sum_{i=0}^{P} a_i \frac{F_{n+1}^{-i} - \left(\frac{F_n}{F_{n+1}}\right)^{p+1} \cdot F_n^{-i}}{p-i+1}}.$$

So when $n \to \infty$,

$$\frac{A_{n+1}}{A_n} \to \alpha^{p+1} \text{ since } \frac{F_{n+2}}{F_{n+1}} \to \alpha$$

and the other fraction converges to 1.

Also solved by Paul S. Bruckman, José Luis Diaz-Barrero, Steve Edwards, G.C. Greubel, Walther Janous, Harris Kwong, Angel Plaza, Jaroslave Seibert, H.-J. Seiffert, and the proposer.

Trace the "Trace"

B-972 Proposed by Mario Catalani, University of Torino, Torino, Italy (Vol. 42, no. 1, Feb. 2004)

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find an expression for $\mathrm{tr}(A^n)$, where $\mathrm{tr}(\cdot)$ means the trace operator. Let $t_n = \mathrm{tr}(A^n)$. Find a recurrence for t_n.

Solution by Steve Edwards, Southern Polytechnic State University, Marietta, GA

The characteristic polynomial of A is $x^3 - x^2 - x - 1$, so tr(A)= $x_1 + x_2 + x_3$, where x_1, x_2, x_3, are the roots of the characteristic polynomial (and the eigenvalues of A). Explicitly,

$$x_1 = \frac{1}{3}\left[1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\ \right]$$

$$x_2 = \frac{1}{3}\left[1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}\ \right]$$

$$x_3 = \frac{1}{3}\left[1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}\ \right], \text{ where } \omega = \frac{-1 + i\sqrt{3}}{2} \text{ (see [1]).}$$

The eigenvalues of A^n are the n^{th} powers of the eigenvalues of A, so $tr(A^n) = x_1^n + x_2^n + x_3^n$. Moreover, since A satisfies its characteristic equation, $A^3 = A^2 + A + I$, and more generally $A^n = A^{n-1} + A^{n-2} + A^{n-3}$. Since tr(A+B) = tr(A) + tr(B), $t_n = t_{n-1} + t_{n-2} + t_{n-3}$.

Reference:

1. M. Elia. "Derived Sequences, the Tribonacci Recurrence and Cubic Forms." *The Fibonacci Quarterly* **39.2** (2001): 107-115.

Also solved by Paul S. Bruckman, Charles K. Cook, Ovidiu Furdui, George Greubel, Pentti Haukkanen, Walther Janous, Harris Kwong, David Manes, Jaroslave Seibert, H.J. Seiffert, and the proposer.

A Series Estimate

B-973 Proposed by José Luis Díaz-Barrero & Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain (Vol. 42, no. 1, Feb. 2003)

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ 1 + \sqrt{\frac{F_{2k}}{F_{k+1}^2}} \right\}^{-1} > 2^{n-1}.$$

Solution by Paul S. Bruckman, Charles K. Cook and H.-J. Seiffert (independently)

¿From eqn. (I_{10}) of [1], we know that $F_{2k} = F_{k+1}^2 - F_{k-1}^2$, which implies $F_{2k} \leq F_{k+1}^2$ Φr+ $\sqrt{F_{2k}/F_{k+1}^2} \leq 2$, $k \in N_0$. There is equality only when $k = 1$. Hence, if S_n, $n \in N_0$, denotes the sum on the left hand side of the desired inequality, then

$$S_n > \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} = 2^{n-1},$$

where the Binomial theorem was used.

Reference:

1. V.E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Ovidiu Furdui, G.C. Greubel, Walther Janous, Harris Kwong, Angel Plaza, Jaroslave Seibert, and the proposer.

Minkowski Meets Fibonacci

B-974 Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya Barcelona, Spain (Vol. 42, no. 1, Feb. 2003)

Let n be a positive integer. Prove that

$$\sqrt{\sum_{k=1}^{n} F_{k+1}^2} \leq \frac{1}{2n} \left\{ \sqrt{\sum_{k=1}^{n} [F_k + (n^2-1)L_k]^2} + \sqrt{\sum_{k=1}^{n} [(n^2-1)F_k + L_k]^2} \right\}.$$

Solution by G.C. Greubel, Newport News, VA

Starting with Minkowski's inequality,

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p \right)^{1/p},$$

let

$$a_k = F_k + (n^m - 1)L_k$$

and

$$b_k = (n^m - 1)F_k + L_k.$$

By adding these two quantities together we have

$$a_k + b_k = 2n^m F_{k+1}.$$

Using these quantities in the given inequality we have the result

$$\left(\sum_{k=1}^{n}(F_{k+1})^p\right)^{1/p} \leq \frac{1}{2n^m}\left\{\left(\sum_{k=1}^{n}[F_k + (n^m - 1)L_k]^p\right)^{1/p} + \left(\sum_{k=1}^{n}[(n^m - 1)F_k + L_k]^p\right)^{1/p}\right\}.$$

If we let $m = 2$ and $p = 2$ then we have the required result.

Also solved by Paul S. Bruckman, Ovidiu Furdui, Walther Janous, Peter Jeuck, Harris Kwong, Jaroslave Seibert, H.-J. Seiffert, and the proposer.

Close Them Up!

B-975 Proposed by N. Gauthier, Royal Military College of Canada (Vol. 42, no. 1, Feb. 2004)

For l, m and n nonnegative integers, find closed-form expressions for the following sums:

$$A. \quad \sum_{k=0}^{n} k\binom{n}{k} L_{km+l}L_{mn-km};$$

$$B. \quad \sum_{k=0}^{n} k\binom{n}{k} L_{km+l}F_{mn-km};$$

$$C. \quad \sum_{k=0}^{n} k\binom{n}{k} F_{km+l}L_{mn-km};$$

$$D. \quad \sum_{k=0}^{n} k\binom{n}{k} F_{km+l}F_{mn-km}.$$

Solution by Jaroslav Seibert, University Hradec Kralove, The Czech Republic

We shall use some identities which are given, for example, in T. Koshy. "Fibonacci and Lucas Numbers with Applications":

$$L_{m+n} = \frac{1}{2}L_mL_n + \frac{5}{2}F_mF_n \text{ (page 91)} \tag{1}$$

$$F_{m+n} = \frac{1}{2}F_mL_n + \frac{1}{2}F_nL_m \text{ (page 91)} \tag{2}$$

$$\sum_{k=0}^{n}\binom{n}{k}L_{km}L_{mn-km} = 2^nL_{mn} + 2L_m^n \text{ (page 235)} \tag{3}$$

$$\sum_{k=0}^{n}\binom{n}{k}F_{km}L_{mn-km} = \sum_{k=0}^{n}\binom{n}{k}L_{km}F_{mn-km} = 2^nF_{mn} \text{ (page 235)} \tag{4}$$

$$\sum_{k=0}^{n}\binom{n}{k}F_{km}F_{mn-km} = \frac{1}{5}(2^nL_{mn} - 2L_m^n) \text{ (page 235)} \tag{5}$$

A. Applying (1) we can write

$$S_A = n\sum_{k=1}^{n}\binom{n-1}{k-1}L_{km+l}L_{mn-km} = n\sum_{k=0}^{n-1}\binom{n-1}{k}L_{(k+1)m+l}L_{mn-(k+1)m}$$

$$= n\sum_{k=0}^{n-1}\binom{n-1}{k}L_{km+m+l}L_{m(n-1)-km}$$

$$= n\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{1}{2}L_{km}L_{m+l} + \frac{5}{2}F_{km}F_{m+l}\right)L_{m(n-1)-km}$$

$$= \frac{1}{2}nL_{m+l}\sum_{k=0}^{n-1}\binom{n-1}{k}L_{km}L_{m(n-1)-km} + \frac{5}{2}mF_{m+l}\sum_{k=0}^{n-1}F_{km}L_{m(n-1)-km}.$$

It follows from (2) and (3) that

$$S_A = \frac{1}{2}nL_{m+l}(2^{n-1}L_{m(n-1)} + 2L_m^{n-1}) + \frac{5}{2}nF_{m+l}2^{n-1}F_{m(n-1)}$$

$$= \frac{1}{2}n2^{n-1}(L_{m+l}L_{m(n-1)} + 5F_{m+l}F_{m(n-1)}) + nL_{m+l}L_m^{n-1}$$

and finally using (1)

$$S_A = n2^{n-1}L_{mn+l} + nL_{m+l}L_m^{n-1}.$$

B. Analogously, using (1), (4), (5) we have after simplification

$$S_B = \sum_{k=0}^{n} k \binom{n}{k} L_{km+l} F_{mn-km} = n2^{n-1} F_{mn+l} - nF_{m+l} L_m^{n-1}.$$

C. Now, using (2), (3), (4) we get

$$S_C = \sum_{k=0}^{n} k \binom{n}{k} F_{km+l} L_{mn-km} = n2^{n-1} F_{mn+l} + nF_{m+l} L_m^{n-1}.$$

D. Using (2), (4), (5) we obtain

$$S_D = \sum_{k=0}^{n} k \binom{n}{k} F_{km+l} F_{mn-km} = \frac{1}{5} n2^{n-1} L_{mn+l} - \frac{1}{5} nL_{m+l} L_m^{n-1}.$$

Similar arguments were used by most solvers. Also solved by Paul S. Bruckman, Ovidiu Furdui, G.C. Greubal, Walther Janous, Harris Kwong, and the proposer.

We would like to belatedly acknowledge the solutions to problems B-966 through B-970 by Paul S. Bruckman, problems B-967 through B-969 by Kenneth Davenport.