FACTORIAL BINET FORMULA AND DISTRIBUTIONAL MOMENT FORMULATION OF GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

Let us consider two finite sequences $a_0, a_1, \ldots, a_{r-1}$ and $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ of real or complex numbers.[1] The sequence $\{V_n\}_{n\geq 0}$ defined by $V_n = \alpha_n$ for $0 \leq n \leq r-1$ and the linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + \cdots + a_{r-1} V_{n-r+1}, \quad n \geq r-1, \tag{1.1}$$

is called a (weighted) r*-generalized Fibonacci sequence*. Furthermore, the sequences

$$a_0, a_1, \ldots, a_{r-1} \quad \text{and} \quad \alpha_0, \alpha_1, \ldots, \alpha_{r-1}$$

are called the *coefficients* and the *initial conditions* of the sequence $\{V_n\}_{n\geq 0}$ respectively (see [7, 12, 13, 14] for example). In the sequel we shall refer to it as a *sequence of type* (1.1). For such a sequence, the polynomial $P(X) = X^r - a_0 X^{r-1} - \cdots - a_{r-2}X - a_{r-1}$ is called the *characteristic polynomial* and its roots are called the *characteristic roots*. It is known that the Binet formula (see (2.2) below) expresses V_n in terms of the characteristic roots and the initial conditions (see [7, 12, 13]).

Sequences of type (1.1) have interested many authors because of their applications in various aspects of mathematics, physics and engineering. In particular, they appeared in the papers of Curto and Fialkow [6, 9] on the K*-moment problem*, where K is a closed subset of $\mathbf{R}$. For a given sequence $\{V_n\}$ of real numbers, this problem consists of finding a positive Borel measure μ whose support is contained in K such that

$$V_n = \int_K t^n d\mu(t) \tag{1.2}$$

for all n, where the right hand side of (1.2) is called a *moment* of the measure μ. Recently such a close connection between sequences $\{V_n\}_{n\geq 0}$ of type (1.1) and the K-moment problem has also been studied in [2, 5, 8]: for example, if the characteristic roots are all real and simple, then the Binet formula shows that V_n can be written as in (1.2) for all $n \geq 0$, for some discrete

[1] It is often assumed that $a_{r-1} \neq 0$ in the literature; however, we allow a_{r-1} to be zero in this paper.

(but not necessarily positive) measure μ, where K is a compact subset of $\mathbf{R}$ which contains the support of μ (see [5]). On the other hand, if there exists a discrete measure μ whose support is contained in a compact set K such that (1.2) holds for all $n \geq 0$, then $\{V_n\}_{n\geq 0}$ is a sequence of type (1.1). When the characteristic roots are real but not necessarily simple, the representation of V_n as in (1.2) is not possible in general.

The aim of this paper is to introduce the notion of moments of distributions of compact support and study its close connection with sequences of type (1.1). For this purpose, we will first present a factorial Binet formula (see (2.4) below), which is a modification of the usual Binet formula. We then establish the correspondence between the moments of distributions of discrete support and sequences of type (1.1) whose characteristic roots are real but not necessarily simple. In other words, we present here a novel technique for obtaining a new expression of the Binet formula for sequences of type (1.1) with real characteristic roots.

The paper is organized as follows. In §2 we consider the Binet formula for sequences of type (1.1) and introduce the notion of their factorial Binet formula. In §3 we define the notion of a generating distribution and give its connection with sequences of type (1.1). Finally, §4 is devoted to the distributional moment formulation of sequences of type (1.1).

2. A FACTORIAL BINET FORMULA

Let $\mathcal{F}_r(a_0, a_1, \dots, a_{r-1})$ denote the vector space over $\mathbf{R}$ (or over $\mathbf{C}$) consisting of all real (resp. complex) r-generalized Fibonacci sequences with coefficients $a_0, a_1, \dots, a_{r-1}$. Note that we do not assume $a_{r-1} \neq 0$. It is well known that this vector space is of dimension r (see [7, 12, 13] for example). Let

$$P(X) = X^r - a_0X^{r-1} - \cdots - a_{r-2}X - a_{r-1} \tag{2.1}$$

be the characteristic polynomial associated with the sequence of type (1.1), and let $\lambda_1, \lambda_2, \dots, \lambda_s$ be its roots with multiplicities $m_1, m_2, \dots, m_s$ respectively. Note that λ_i can possibly be zero.

Let $\{V_n\}_{n\geq 0}$ be a sequence in $\mathcal{F}_r(a_0, a_1, \dots, a_{r-1})$. It is well known in the classical literature that its *Binet formula* is given by

$$V_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n \tag{2.2}$$

for $n \geq 0$, where $\beta_{i,j}$ are determined uniquely by the initial conditions $\{\alpha_j\}_{0\leq j\leq r-1}$ (see [7, 12, 13] for example).[2] More precisely, we can determine $\beta_{i,j}$ by solving the system of r linear equations

$$\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n = \alpha_n, \quad n = 0, 1, \dots, r-1.$$

For $n \geq r$, consider the polynomial $X^{n-r}P(X)$. Every characteristic root λ_i ($1 \leq i \leq s$) with multiplicity $m_i \geq 1$ is a root of $X^{n-r}P(X)$ with multiplicity $\geq m_i$. Hence, we see that

[2] In this paper, we adopt the convention that $0^0 = 1$.

for every j with $0 \leq j \leq m_i - 1$, λ_i is also a root of the j^{th} derivative $(X^{n-r}P(X))^{(j)}$ of $X^{n-r}P(X)$. Since $X^{n-r}P(X) = X^n - a_0X^{n-1} - \cdots - a_{r-2}X^{n-r+1} - a_{r-1}X^{n-r}$, the process of derivation until the j^{th} order implies that λ_i $(1 \leq i \leq s)$ satisfy the following relation: for $n \geq r + j$,

$$\frac{n!}{(n-j)!}\lambda_i^{n-j} - a_0\frac{(n-1)!}{(n-1-j)!}\lambda_i^{n-1-j} - \cdots - a_{r-1}\frac{(n-r)!}{(n-r-j)!}\lambda_i^{n-r-j} = 0,$$

and for $r \leq n < r + j$,

$$\frac{n!}{(n-j)!}\lambda_i^{n-j} - a_0\frac{(n-1)!}{(n-1-j)!}\lambda_i^{n-1-j} - \cdots - a_{n-j-1}\frac{j!}{0!}\lambda_i^0 = 0.$$

For every $n \geq 0$, we set

$$y_n^{(i,j)} = \begin{cases} \frac{n!}{(n-j)!}\lambda_i^{n-j}, & n \geq j, \\ 0, & 0 \leq n < j, \end{cases}$$

where $1 \leq i \leq s$ and $0 \leq j \leq m_i - 1$. The above observation suggests the following.

Proposition 2.1: *The set* $\{\{y_n^{(i,j)}\}_{n\geq 0} \ : \ 1 \leq i \leq s, 0 \leq j \leq m_i - 1\}$ *constitutes a basis of the vector space* $\mathcal{F}_r(a_0, a_1, \ldots, a_{r-1})$.

Proof: First, by the above observation, we see that each $\{y_n^{(i,j)}\}_{n\geq 0}$ is an element of $\mathcal{F}_r(a_0, a_1, \ldots, a_{r-1})$.

Let $\{V_n\}_{n\geq 0}$ be an arbitrary element of $\mathcal{F}_r(a_0, a_1, \ldots, a_{r-1})$. Then we have the Binet formula (2.2) for uniquely determined numbers $\beta_{i,j}$. On the other hand, for a real number x and an integer $n \geq 0$, set

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1),$$

where we define $[x]_0 = 1$ (see [3, p. 19]). Note that for an integer m with $0 \leq m < n$, we always have $[m]_n = 0$. Then for any real number x, we have the well known formula

$$x^n = \sum_{k=0}^{n} S_n^k [x]_k,$$

where S_n^k is the Stirling number of the second kind which is the number of partitions of a set of n objects into k classes (see [3, §1.10] or [10], for example). Note that for convenience, we put

$$S_n^0 = \begin{cases} 0, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Note also that we have $y_n^{(i,j)} = [n]_j \lambda_i^{n-j}$, where for $\lambda_i = 0$, the right hand side is understood to be zero for $n < j$, since $[n]_j = 0$. Then by (2.2), we have

$$V_n = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \beta_{i,k} n^k \right) \lambda_i^n = \sum_{i=1}^{s} \sum_{k=0}^{m_i-1} \beta_{i,k} \sum_{j=0}^{k} \lambda_i^j S_k^j [n]_j \lambda_i^{n-j}$$

$$= \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \left(\sum_{k=j}^{m_i-1} \beta_{i,k} \lambda_i^j S_k^j \right) y_n^{(i,j)} \right). \tag{2.3}$$

Therefore, the set $\{\{y_n^{(i,j)}\}_{n\geq 0} \ : \ 1 \leq i \leq s, 0 \leq j \leq m_i - 1\}$ generates the vector space $\mathcal{F}_r(a_0, a_1, \ldots, a_{r-1})$ (over $\mathbf{R}$ or $\mathbf{C}$). By counting the dimension, we see that the set must be a basis of the vector space. This completes the proof. $\square$

¿From Proposition 2.1 we derive the following factorial Binet formula for sequences of type (1.1), which is similar to the usual Binet formula (2.2).

Theorem 2.2 (Factorial Binet Formula): *Let $\{V_n\}_{n\geq 0}$ be a sequence of type* (1.1) *in $\mathcal{F}_r(a_0, a_1, \ldots, a_{r-1})$. Then there exist unique numbers $\alpha_{i,j}$ $(1 \leq i \leq s, 0 \leq j \leq m_i - 1)$ such that*

$$V_n = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \alpha_{i,j} \frac{n!}{(n-j)!} \lambda_i^{n-j} \tag{2.4}$$

for all $n \geq 0$, where for $n < j$, we treat $(n!/(n-j)!)\lambda_i^{n-j}$ as zero.

Remark 2.3: The connection between the two expressions (2.2) and (2.4) of V_n is given as follows. If we are given the expression (2.2), then $\alpha_{i,j}$ in (2.4) can be obtained by the equation (2.3) by using the Stirling numbers of the second kind. Conversely, if we are given the expression (2.4), then $\beta_{i,j}$ in (2.2) can be obtained by using the formula

$$[n]_j = \sum_{k=0}^{j} s_j^k n^k,$$

where s_j^k are the Stirling numbers of the first kind (see [3, §1.5] or [10], for example). Note that

$$s_j^0 = \begin{cases} 0, & j > 0, \\ 1, & j = 0. \end{cases}$$

More precisely, we have

$$V_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \left(\sum_{k=j}^{m_i-1} \alpha_{i,k} s_k^j \lambda_i^{-k} \right) n^j \right) \lambda_i^n. \tag{2.5}$$

Here, when $\lambda_i = 0$, the term $\alpha_{i,k} s_k^j \lambda_i^{-k} n^j \lambda_i^n$ is understood to be zero for $n > 0$ or for $n = 0$ and $j > 0$. It is understood to be zero also for $n = 0, j = 0$ and $k > 0$, since $s_k^j = 0$.

3. MOMENT OF DISTRIBUTIONS

Let $\Gamma = \{\gamma_n\}_{0 \le n < p}$ with $1 \le p \le +\infty$ be a sequence of real numbers. Let K be a closed subset of $\mathbf{R}$. The *classical* K-moment problem associated with Γ consists of finding a positive Borel measure μ such that

$$\gamma_n = \int_K t^n d\mu(t), \quad 0 \le \forall n < p, \quad \text{and} \quad \operatorname{Supp} \mu \subset K, \tag{3.1}$$

where $\operatorname{Supp} \mu$ denotes the support of μ. If this problem has a solution μ, then we say that μ is a *representing measure* of the sequence $\Gamma = \{\gamma_n\}_{0 \le n < p}$. For $p = +\infty$ the problem (3.1) is called the *full K-moment problem* (see [1, 4] for example). For $p < +\infty$ the problem (3.1) is called the *truncated K-moment problem*, which has been studied by Curto and Fialkow [6,9]. It is well known that if Γ has a representing measure of finite support, then it is a sequence of type (1.1). Furthermore, if Γ has a representing measure and is a sequence of type (1.1) with $p \ge 2r + 1$, then this measure is of finite support [6, 8, 9].

The *generalized K-moment problem* associated with Γ consists of finding a general measure μ which is not necessarily nonnegative and which satisfies (3.2) below. Such a measure μ is called a *generating measure* of the sequence Γ. Recall that this problem, issued from the many body problem in quantum physics, has been studied in [5] by using sequences of type (1.1). It has been shown that if Γ has a generating measure, then this measure is of finite support if and only if Γ is a sequence of type (1.1) [5].

The aim of this section is to introduce a concept of distributional moment problem in order to generalize these situations.

Let μ be a distribution. Recall that if μ is of compact support, then it is defined on the space of functions of class C^∞ (see [11] for example), and hence we can consider the value $\langle \mu | t^n \rangle$ of μ on $t \mapsto t^n$.

Definition 3.1: Let μ be a distribution of compact support K. The number $V_n = \langle \mu | t^n \rangle$ is called the *moment* (or *power moment*) of order n of the distribution μ.

The above definition is a natural extension of the moment of a measure. We get also the following notion of generating distributions.

Definition 3.2: Let $\Gamma = \{\gamma_n\}_{0 \le n < p}$ with $1 \le p \le +\infty$ be a sequence of real numbers and μ a distribution of compact support K. We say that μ is a *generating distribution of* Γ if we have

$$\gamma_n = \langle \mu | t^n \rangle \tag{3.2}$$

for every n with $0 \le n < p$.

Let $a_0, a_1, \ldots, a_{r-1}$ be a sequence of real numbers and let $\lambda_1, \ldots, \lambda_s$ be the roots of the characteristic polynomial (2.1) with multiplicities $m_1, m_2, \ldots, m_s$ respectively. Note that a_{r-1} or λ_i can possibly be zero. In the following, we assume that the roots are all real numbers. Let δ_i be the Dirac measure at the point λ_i and $\delta_i^{(j)}$ its j^{th} derivative. Then δ_i and $\delta_i^{(j)}$ define distributions of compact support. Let us consider the following classical properties on the differentiation of distributions:

$$\langle \phi\mu | f \rangle = \langle \mu | \phi f \rangle \quad \text{and} \quad \langle D^k \mu | f \rangle = (-1)^k \langle \mu | D^k f \rangle.$$

Here, ϕ and f are real functions of class C^∞ and $D = d/dt$, where $t \in \mathbf{R}$ denotes the usual coordinate. Consider the distribution of finite support

$$\mu = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \alpha_{i,j}(-1)^j \delta_i^{(j)}, \tag{3.3}$$

where $\alpha_{i,j}$ are real numbers. Then a straightforward computation allows us to obtain

$$\langle \mu | t^n \rangle = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \alpha_{i,j} \frac{n!}{(n-j)!} \lambda_i^{n-j} \tag{3.4}$$

for every $n \geq 0$. Hence, from Proposition 2.1 we derive the following.

Proposition 3.3: *Let $\lambda_1, \dots, \lambda_s$ and $m_1, \dots, m_s$ be as above. Consider the distribution*

$$\mu = \sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \alpha_{i,j}(-1)^j \delta_i^{(j)},$$

where $\alpha_{i,j}$ $(1 \leq i \leq s, 0 \leq j \leq m_i - 1)$ are real numbers. Then the sequence $\{V_n\}_{n\geq 0}$ of moments $V_n = \langle \mu | t^n \rangle$ of the distribution μ is a sequence of type (1.1) *of order $r = m_1 + \cdots + m_s$. More precisely, its initial conditions and characteristic polynomial are given by $V_n = \langle \mu | t^n \rangle$ for $n = 0, 1, \dots, r-1$ and $P(X) = \prod_{i=1}^{s}(X - \lambda_i)^{m_i}$ respectively.*

Example 3.4: For $\lambda \in \mathbf{R}$, $m \geq 1$, and $\beta_0, \dots, \beta_{m-1} \in \mathbf{R}$, set

$$T_{\lambda,m} = \sum_{j=0}^{m-1} \beta_j \delta_\lambda^{(j)},$$

where $\delta_\lambda^{(j)}$ is the j^{th} derivative of the Dirac measure δ_λ. For $n \geq 0$, set

$$V_n = \langle T_{\lambda,m} | t^n \rangle = \sum_{j=0}^{m-1} (-1)^j \beta_j \frac{n!}{(n-j)!} \lambda^{n-j} = \left(\sum_{j=0}^{m-1} \left(\sum_{k=j}^{m-1} s_k^j \frac{(-1)^k \beta_k}{\lambda^k} \right) n^j \right) \lambda^n,$$

where s_k^j are the Stirling numbers of the first kind (see (2.5) or [3]). Then the sequence of moments $\{V_n\}_{n\geq 0}$ of the distribution $T_{\lambda,m}$ is a sequence of type (1.1) of order $r = m$, whose initial conditions are given by $V_n = \langle T_{\lambda,m} | t^n \rangle$ for $n = 0, 1, \dots, m-1$ and whose characteristic polynomial is $P(X) = (X - \lambda)^m$.

The distributional K-moment problem can be stated as follows. Let $\Gamma = \{\gamma_n\}_{0 \leq n < p}$ $(1 \leq p \leq +\infty)$ be a sequence of real numbers. Let K be a compact subset of $\mathbf{R}$. The *distributional K-moment problem associated with* Γ consists of finding a distribution μ such that

$$\gamma_n = \langle \mu | t^n \rangle, \quad 0 \leq \forall n < p, \quad \text{and} \quad \operatorname{Supp} \mu \subset K, \tag{3.5}$$

where $\operatorname{Supp} \mu$ denotes the support of μ.

The next section is devoted to the study of the distributional K-moment problem associated with a sequence of type (1.1).

Remark 3.5: We see that the distribution μ given by (3.3) is a (general) measure if and only if $\alpha_{i,j} = 0$ for every $j \neq 0$. In this case, the discrete measure $\mu = \sum_{i=1}^{s} \alpha_{i,0}\delta_i$ can be written as $\mu = \mu^+ - \mu^-$ for some positive measures μ^+ and μ^-.

In fact, all the above arguments work even if we allow V_n to be complex numbers. In that case, $\alpha_{i,j}$ are complex numbers. If the distribution μ given by (3.3) is a measure, then we have $\mu = (\mu_1^+ - \mu_1^-) + (\mu_2^+ - \mu_2^-)\sqrt{-1}$ for some positive measures $\mu_1^\pm$ and $\mu_2^\pm$.

4. DISTRIBUTIONAL MOMENT FORMULATION OF SEQUENCES OF TYPE (1.1)

It is known that a sequence $\{V_n\}_{n\geq 0}$ of real or complex numbers admits a generating measure of finite support if and only if it is of type (1.1) and all the roots of the characteristic polynomial are real and simple (see [2, 5, 6]). The following theorem generalizes this result.

Theorem 4.1: *Let $\{V_n\}_{n\geq 0}$ be a sequence of real or complex numbers. Then the following two are equivalent.*

(i) *The sequence $\{V_n\}_{n\geq 0}$ is an element of $\mathcal{F}_r(a_0, a_1, \ldots, a_{r-1})$ for some real numbers $a_0, \ldots, a_{r-1}$ such that all the roots of the characteristic polynomial* (2.1) *are real numbers.*

(ii) *The sequence $\{V_n\}_{n\geq 0}$ has a generating distribution of finite support.*

For the proof of Theorem 4.1, we will use the following classical lemma (see [11] for example).

Lemma 4.2: *A distribution whose support consists of at most one point y is a finite linear combination of the Dirac measure at y and its derivatives.*

Proof of Theorem 4.1: It is easy to check that

$$\frac{n!}{(n-j)!}\lambda_i^{n-j} = \langle (-1)^j \delta_i^{(j)} | t^n \rangle$$

for every $n \geq 0$. Note that the above equation is valid also for $\lambda_i = 0$. Then the implication (i) $\Longrightarrow$ (ii) follows from Theorem 2.2 and (3.3).

For the converse, set $\operatorname{Supp}\mu = \{\lambda_1, \ldots, \lambda_s\}$. We have $\mu = \sum_{i=1}^{s} \mu_i$, where μ_i is a distribution whose support consists of the point λ_i. By Lemma 4.2 we have

$$\mu_i = \sum_{j=0}^{m_i - 1} \alpha_{i,j}(-1)^j \delta_i^{(j)}$$

and hence

$$\mu = \sum_{i=1}^{s} \sum_{j=0}^{m_i - 1} \alpha_{i,j}(-1)^j \delta_i^{(j)}$$

for some real or complex numbers $\alpha_{i,j}$. Therefore, the conclusion follows from Proposition 3.3. More precisely, we have

$$V_n = \langle \mu | t^n \rangle = \sum_{i=0}^{s} \sum_{j=0}^{m_i - 1} \alpha_{i,j} \frac{n!}{(n-j)!}\lambda_i^{n-j}.$$

Therefore, $\{V_n\}_{n\geq 0}$ is a sequence of type (1.1). $\square$

Example 4.3: Let $\{V_n\}_{n\geq 0}$ be the 2-generalized Fibonacci sequence defined by $V_0 = 1, V_1 = 1$ and $V_{n+1} = 4V_n - 4V_{n-1}$ for $n \geq 1$. The characteristic polynomial of $\{V_n\}_{n\geq 0}$ is $P(X) = (X-2)^2 = X^2 - 4X + 4$. From the factorial Binet formula (2.4) (or the usual Binet formula (2.2)), we derive that

$$V_n = 2^n - n2^{n-1} = \left(1 - \frac{n}{2}\right) 2^n$$

for $n \geq 0$. The sequence $\{V_n\}_{n\geq 0}$ has the generating distribution $\mu = \delta_2 + \delta_2^{(1)}$ defined by $\langle \mu | \varphi \rangle = \varphi(2) - \varphi'(2)$.

More generally, for complex numbers a, b and a real number λ, let $\{V_n\}_{n\geq 0}$ be the 2-generalized Fibonacci sequence defined by $V_0 = a, V_1 = (a+b)\lambda$, and $V_{n+1} = a_0 V_n + a_1 V_{n-1}$ for $n \geq 1$, where $a_0 = 2\lambda$ and $a_1 = -\lambda^2$. It is easy to verify that the Binet formula is given by $V_n = (a+bn)\lambda^n$ for $n \geq 0$. A straightforward verification shows that $V_n = \langle \mu | t^n \rangle$ for every $n \geq 0$, where $\mu = a\delta_\lambda - b\lambda\delta_\lambda^{(1)}$.

Example 4.4: Let $\{V_n\}_{n\geq 0}$ be the 5-generalized Fibonacci sequence defined by $V_0 = 1, V_1 = 3, V_2 = 9, V_3 = 29, V_4 = 102$, and $V_{n+1} = 13V_n - 67V_{n-1} + 171V_{n-2} - 216V_{n-3} + 108V_{n-4}$ for $n \geq 4$. Then we can verify that the factorial Binet formula is given by

$$V_n = 2^n + n2^{n-1} + \frac{1}{2}\frac{n!}{(n-2)!}3^{n-2}$$

for $n \geq 0$. We see easily that $V_n = \langle \mu | t^n \rangle$ for every $n \geq 0$, where

$$\mu = \delta_2 - \delta_2^{(1)} + \frac{1}{2}\delta_3^{(2)}.$$

Example 4.5: For $r \geq 1$, let $\{V_n\}_{n\geq 0}$ be the r-generalized Fibonacci sequence defined by $V_0 = \cdots = V_{r-2} = 0, V_{r-1} = 1$, and

$$V_{n+1} = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k+1} V_{n-k} \tag{4.1}$$

for $n \geq r-1$. Note that the characteristic polynomial is $P(X) = (X-1)^r$. Then we can verify that the factorial Binet formula is given by

$$V_n = \frac{1}{(r-1)!}\frac{n!}{(n-r+1)!} = \binom{n}{r-1}$$

for $n \geq 0$, where the last term is understood to be zero for $n < r-1$. A straightforward verification shows that $V_n = \langle \mu | t^n \rangle$ for every $n \geq 0$, where

$$\mu = \frac{1}{(r-1)!}\delta_1^{(r-1)}.$$

Note that the equation (4.1) gives the formula

$$\binom{n+1}{r-1} = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k+1} \binom{n-k}{r-1}$$

for $n \geq r-1$.

Example 4.6: Let $\{V_n\}_{n \geq 0}$ be the 3-generalized Fibonacci sequence defined by $V_0 = 2, V_1 = 1, V_2 = 0$, and $V_{n+1} = 4V_n - 4V_{n-1} + 0V_{n-2}$ for $n \geq 2$. Then we can verify that the factorial Binet formula is given by $V_n = 2^n - n2^{n-1} + 0^n$ for $n \geq 0$. We see easily that $V_n = \langle \mu | t^n \rangle$ for every $n \geq 0$, where $\mu = \delta_2 + \delta_2^{(1)} + \delta_0$. Compare this example with that of Example 4.3.

Remark 4.7: We have not assumed $a_{r-1} \neq 0$ from the beginning so as to allow a characteristic root λ_i to be zero. This is necessary in Theorem 4.1. If we assume $a_{r-1} \neq 0$, then the characteristic roots should be nonzero, and we would have to exclude $0 \in \mathbf{R}$ as a support for the distribution in Theorem 4.1. See Example 4.6 for the effect of a support containing zero.

Remark 4.8: As we have noted in §3, it has been known that if $\{V_n\}_{n \geq 0}$ has a generating measure, then this measure is of finite support if and only if $\{V_n\}_{n \geq 0}$ is a sequence of type (1.1) [5]. The corresponding statement for distributions is also correct: i.e., if $\{V_n\}_{n \geq 0}$ has a generating distribution, then this distribution is of discrete support if and only if $\{V_n\}_{n \geq 0}$ is a sequence of type (1.1). The proof will be given in a forthcoming paper.

So far, we have always assumed that all the roots of the characteristic polynomial (2.1) are real numbers. We need this assumption here, since we cannot define a distribution in the complex plane. We do not know if we can remove this assumption or not.

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