CONVOLVING THE m-TH POWERS OF THE CONSECUTIVE INTEGERS WITH THE GENERAL FIBONACCI SEQUENCE USING CARLITZ'S WEIGHTED STIRLING POLYNOMIALS OF THE SECOND KIND

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1. INTRODUCTION

Summation rules for various types of convolutions have been the subject of much interest in The Fibonacci Quarterly. The following references are of particular relevance to this topic: Cohen and Hudson [8], Corley [9], Filipponi and Freitag [11], Gould [14], Haukkanen [15], Hsu [19], and finally Philippou and Georghiou [23]. Also consult Neuman and Schonbach's SIAM Review article [22], where sums of convolved powers of the integers are determined with the help of Bernoulli numbers.

In a separate area of research, the Stirling numbers and their various generalizations have also been the subject of sustained attention in The Fibonacci Quarterly. The reader is referred to Branson [2], Cacoullos and Papageorghiou [3], Cakić [4], Carlitz [5] and [6], Charalambides [7], El-Desouky [10], Fray [12], Hillman, Mana and McAbee [16], Howard [17] and [18], Khan and Kwong [20], Sitgreaves [24], Toscano [25], and finally Yu [26].

In [19], Hsu relates Stirling numbers of the second kind to a summation formula. In [14], Gould makes use of Stirling numbers of the second kind to reconsider the sums of convolved powers of the integers of Neuman and Schonbach [22]. In [7], Charalambides discusses some combinatorial applications of the weighted Stirling numbers introduced by Carlitz in [5] and [6].

In the present Note, the weighted Stirling numbers of the second kind introduced by Carlitz in [5] and [6] are used to formulate a convolution of the general Fibonacci sequence $\{G_n \equiv A\alpha^n + B\beta^n\}_{n=-\infty}^{+\infty}$ with the sequence of the integral powers of the consecutive integers, $\{(a+n)^m\}_{n=-\infty}^{+\infty}$. A few applications are also presented at the end of the Note.

The following **Theorem** is established in the present Note:

Theorem: "*For $m \geq 0$, a, b integers and for A, B, α, β real numbers, with $\alpha + \beta = 1, \alpha\beta = -1$, the generalized convolution of the sequence of powers of the consecutive integers, $\{(a+n)^m\}_{n=-\infty}^{+\infty}$, with the general Fibonacci sequence,*

$$\{G_n \equiv A\alpha^n + B\beta^n\}_{n=-\infty}^{+\infty}$$

is

$$\sum_{k=0}^{n}(a+k)^m G_{b-a-k} = \sum_{l=0}^{m} l![c_m^{(l)}(a)G_{b-a+2+l} - c_m^{(l)}(a+n+1)G_{b-a-n+1+l}], \tag{1}$$

where, for v an arbitrary variable, the set of coefficients $\{c_m^{(l)}(v); 0 \leq m; 0 \leq l \leq m\}$ is the set of Carlitzs weighted Stirling polynomials of the second kind."

The general Fibonacci sequence $\{G_n\}_{n=-\infty}^{+\infty}$ obeys the usual second order recurrence relation

$$G_{n+2} = G_{n+1} + G_n; \quad G_n = A\alpha^n + B\beta^n; \tag{2}$$

the real numbers A and B are assumed known. Also note that one recovers the set of Fibonacci numbers, $\{G_n = F_n\}_{n=-\infty}^{+\infty}$, by choosing $A = -B = (\alpha - \beta)^{-1}$ in the above Binet form for the general Fibonacci numbers, whereas one gets the set of Lucas numbers, $\{G_n = L_n\}_{n=-\infty}^{+\infty}$, by choosing A = B = 1.

Theorem (1) refers to a generalized convolution because one recovers the standard form for the convolution of the two sequences of interest here, i. e. $\sum_{k=0}^{n} k^m G_{n-k}$, by putting $a = 0$ and $b = n$; see [15] for the definition of the standard form of the convolution of two sequences.

The polynomials introduced by Carlitz in [5] and [6] will be shown to appear in Theorem (1) in Section II below; Section III will present some applications to generalized and to standard convolutions.

2. PROOF OF THE THEOREM

The definitions introduced earlier will be used without further reference in what follows. Let $D \equiv x\frac{d}{dx}$ be a differential operator and, for $x \neq 1$, consider

$$\sum_{k=0}^{n} x^{k+a} = \frac{x^a}{1-x} - \frac{x^{a+n+1}}{1-x}. \tag{3}$$

Acting m times on this equation with D, for m a nonnegative integer, gives

$$\sum_{k=0}^{n} (k+a)^m x^{k+a} = D^m \frac{x^a}{1-x} - D^m \frac{x^{a+n+1}}{1-x}. \tag{4}$$

To determine $D^m \frac{x^v}{1-x}$, for v arbitrary, note that

$$\begin{aligned} &D^0 \frac{x^v}{1-x} = \frac{x^v}{1-x}; \qquad D^1 \frac{x^v}{1-x} = v\frac{x^v}{1-x} + \frac{x^{v+1}}{(1-x)^2}; \\ &D^2 \frac{x^v}{1-x} = v^2 \frac{x^v}{1-x} + (2v+1)\frac{x^{v+1}}{(1-x)^2} + 2\frac{x^{v+2}}{(1-x)^3} \ldots; \end{aligned} \tag{5}$$

the general term being

$$D^m \frac{x^v}{1-x} = \sum_{l=0}^{m} d_m^{(l)}(v) \frac{x^{v+l}}{(1-x)^{l+1}}, \tag{6}$$

for m a nonnegative integer.

The coefficients $\{d_m^{(l)}(v);\ 0 \leq m;\ 0 \leq l \leq m\}$ may be found as follows:
First put $m+1$ in place of m in (6) and get

$$D^{m+1} \frac{x^v}{1-x} = \sum_{l=0}^{m+1} d_{m+1}^{(l)}(v) \frac{x^{v+l}}{(1-x)^{l+1}}. \tag{7}$$

Then also consider the direct action of D on (6)

$$
\begin{aligned}
D\left[D^m \frac{x^v}{1-x}\right] &= D\left[\sum_{l=0}^{m} d_m^{(l)}(v)\frac{x^{v+l}}{(1-x)^{l+1}}\right] \\
&= \sum_{l=0}^{m} d_m^{(l)}(v)\left[(v+l)\frac{x^{v+l}}{(1-x)^{l+1}} + (l+1)\frac{x^{v+l+1}}{(1-x)^{l+2}}\right] \qquad (8) \\
&= \sum_{l=0}^{m+1}\left[(v+l)d_m^{(l)}(v) + l d_m^{(l-1)}(v)\right]\frac{x^{v+l}}{(1-x)^{l+1}},
\end{aligned}
$$

where the last line was arrived at by shifting the dummy index in the second sum of the second line of (8), $l+1 \to l$, and by defining $d_m^{(-1)}(v) \equiv 0$ and $d_m^{(m+1)}(v) \equiv 0$. One can now equate the right-hand member of (7) to that of the last line of (8), since $D[D^m \ldots] = D^{m+1}[\ldots]$. Furthermore, because $x(\neq 1)$ is an arbitrary variable, one can then equate the coefficients of $x^{v+l}/(1-x)^{l+1}$ on both sides of the resulting equation and this gives the following recurrence for the $\{d_m^{(l)}\}$ coefficients [13]:

$$
\begin{aligned}
&d_{m+1}^{(l)}(v) = (v+l)d_m^{(l)}(v) + l d_m^{(l-1)}(v); \quad 0 \le m; 0 \le l \le m; \\
&d_{m=0}^{(l=0)}(v) = 1; \quad d_m^{(-1)}(v) \equiv 0, \; d_m^{(m+1)}(v) \equiv 0.
\end{aligned} \qquad (9)
$$

Next, introduce a set of new coefficients, as follows,

$$\{d_m^{(l)}(v) \equiv l! c_m^{(l)}(v)\},$$

and get

$$
\begin{aligned}
&c_{m+1}^{(l)}(v) = (v+l)c_m^{(l)}(v) + c_m^{(l-1)}(v); \quad 0 \le m; \; 0 \le l \le m; \\
&c_{m=0}^{(l=0)}(v) = 1, \quad c_m^{(-1)}(v) \equiv 0, \; c_m^{(m+1)}(v) \equiv 0.
\end{aligned} \qquad (10)
$$

This recurrence is now examined.

First note that, for $v = 0$, (10) is the same as the recurrence for the Stirling numbers of the second kind, $\{S_m^{(l)}; 0 \le m, 0 \le l \le m\}$; see Gould [14]. In fact, Gould's convention [14] for the Stirling numbers of the second kind is used, whereby the usual set of Stirling numbers of the second kind, $\{S_m^{(l)}; 1 \le m, 1 \le l \le m\}$, as defined in Abramowitz and Stegun [1] for example, is augmented by including $S_0^{(0)} \equiv 1$. This is the set $\{S_m^{(l)}; 0 \le m, 0 \le l \le m\}$ which is referred-to here.

It can thus be seen that (10) generalizes the Stirling numbers of the second kind to a set of polynomials in the variable v. The corresponding sets of numbers seem to have originally appeared in the work of Leonard Carlitz, who referred to them as "*weighted Stirling numbers* *of the second kind*". In the notation of Carlitz, the coefficient $c_m^{(l)}(v)$ would be written as $R(m,l,v)$: see [5], *Section* 3, and [6] in its entirety. The notation used in the present Note is adopted for two main reasons:

a. To maintain the indices in $c_m^{(l)}$ as close as possible to the now-standardized nomenclature for the Stirling numbers of the second kind, $S_m^{(l)}$, as given in Abramowitz and Stegun [1].

b. To ascribe the letter c to these polynomials, in honor of Carlitz.

A closed-form for Carlitz's weighted Stirling polynomials of the second kind as a function of the arbitrary variable v is

$$c_m^{(l)}(v) = \sum_{k=0}^{m-l} \binom{m}{k} S_{m-k}^{(l)} v^k; \tag{11}$$

a detailed proof was given by Carlitz in [5], *Section* 3. These polynomials should probably be called the "Carlitz-Stirling polynomials of the second kind" if it can be ascertained that Carlitz was indeed the first to study them in [5] and [6]. The first few polynomials are, for $0 \le m \le 3$:

$$\begin{aligned}
&m = 0: \quad c_0^{(0)}(v) = 1;\\
&m = 1: \quad c_1^{(0)}(v) = v; \quad c_1^{(1)}(v) = 1;\\
&m = 2: \quad c_2^{(0)}(v) = v^2; \quad c_2^{(1)}(v) = 2v + 1; \quad c_2^{(2)}(v) = 1;\\
&m = 3: \quad c_3^{(0)}(v) = v^3; \quad c_3^{(1)}(v) = 3v^2 + 3v + 1; \quad c_3^{(2)}(v) = 3v + 3; \quad c_3^{(3)}(v) = 1.
\end{aligned} \tag{12}$$

The general term is given in terms of the augmented Stirling numbers by the closed-form expression (11) but for $l = 0$ and for $l = m$, one gets the following simple expressions, $c_m^{(0)}(v) = v^m$ and $c_m^{(m)}(v) = 1$, respectively. In general one can use (11) to evaluate these polynomials for any value of the variable v. In particular, the Carlitz-Stirling polynomials of the second kind can be determined for the values of v which are of interest in (3) and (4): $v = a$ and $v = a + n + 1$.

Now return to (4) and make the following substitution: $x \to x^{-1}$. With the help of (6), for $v = a$ and for $v = a + n + 1$, (4) then becomes

$$\sum_{k=0}^{n} (k + a)^m x^{-k-a} = \sum_{l=0}^{m} l! \left[c_m^{(l)}(a) \frac{x^{-a+1}}{(x-1)^{l+1}} - c_m^{(l)}(a + n + 1) \frac{x^{-a-n}}{(x-1)^{l+1}} \right]. \tag{13}$$

Next multiply both sides of this expression by Ax^b, set $x = \alpha$ and use $x - 1 = -\beta = +\alpha^{-1}$ to get

$$\sum_{k=0}^{n} (k + a)^m A\alpha^{b-k-a} = \sum_{l=0}^{m} l! \left[c_m^{(l)}(a) A\alpha^{b-a+2+l} - c_m^{(l)}(a + n + 1) A\alpha^{b-a-n+1+l} \right]. \tag{14}$$

Finally return to (13) and multiply both sides by Bx^b but this time set $x = \beta$, $x - 1 = +\beta^{-1}$. This procedure then gives the β-complement of (14), whereby α and A are simply replaced everywhere in (14) by β and B, respectively. Finally add (14) to its β-complement and use the Binet form (2) for the n^{th} general Fibonacci number to get the generalized Fibonacci convolution theorem which is given in (1).

3. APPLICATIONS OF THE THEOREM

Some direct applications are now considered.

For $m = 0$ in the general form of the theorem (1), one gets that

$$\begin{aligned} \sum_{k=0}^{n} G_{b-a-k} &= c_0^{(0)}(a) G_{b-a+2} - c_0^{(0)}(a+n+1) G_{b-a-n+1} \\ &= G_{b-a+2} - G_{b-a-n+1}, \end{aligned} \tag{15}$$

upon use of the first line of (12) to determine the coefficients. This is a known result and it is readily established by rewriting (2) in the following form, $G_{b-a+2-k} - G_{b-a+1-k} = G_{b-a-k}$, since the left-hand side then telescopes when summing over the index k.

For $m = 1$ in (1), get

$$\begin{aligned} \sum_{k=0}^{n} (a+k) G_{b-a-k} &= \sum_{l=0}^{1} l! \left[c_1^{(l)}(a) G_{b-a+2+l} - c_1^{(l)}(a+n+1) G_{b-a-n+1+l} \right] \\ &= c_1^{(0)}(a) G_{b-a+2} + c_1^{(1)}(a) G_{b-a+3} \\ &\quad - \left[c_1^{(0)}(a+n+1) G_{b-a-n+1} + c_1^{(1)}(a+n+1) G_{b-a-n+2} \right] \\ &= a G_{b-a+2} + G_{b-a+3} - [(a+n+1) G_{b-a-n+1} + G_{b-a-n+2}] \end{aligned} \tag{16}$$

upon use of the second line of (12).

Next, set $a = n$ and $b = 2n$ in (1) to get

$$\sum_{k=0}^{n} (n+k)^m G_{n-k} = \sum_{l=0}^{m} l! \left[c_m^{(l)}(n) G_{n+2+l} - c_m^{(l)}(2n+1) G_{l+1} \right]. \tag{17}$$

For $m = 1$, this then gives, upon using the second line of (12)),

$$\begin{aligned}\sum_{k=0}^{n}(n+k)G_{n-k} &= \sum_{l=0}^{1} l!\left[c_1^{(l)}(n)G_{n+2+l} - c_1^{(l)}(2n+1)G_{l+1}\right]\\ &= c_1^{(0)}(n)G_{n+2} + c_1^{(1)}(n)G_{n+3} - \left[c_1^{(0)}(2n+1)G_1 + c_1^{(1)}(2n+1)G_2\right]\\ &= nG_{n+2} + G_{n+3} - [(2n+1)G_1 + G_2]\\ &= nG_{n+2} + G_{n+3} - [2(n+1)G_1 + G_0]\,;\end{aligned} \tag{18}$$

(2) was finally used to get the last line of this expression. This result also follows directly from (16) upon setting $a = n$ and $b = 2n$.

Now, setting $a = 0$ and $b = n$ in (1) gives the usual number-theoretic convolution of the m^{th} powers of the consecutive integer sequence $\{n^m\}_{n=-\infty}^{+\infty}$ with the general Fibonacci sequence $\{G_n\}_{n=-\infty}^{+\infty}$:

$$\sum_{k=0}^{n} k^m G_{n-k} = \sum_{l=0}^{m} l!\left[c_m^{(l)}(0)G_{n+2+l} - c_m^{(l)}(n+1)G_{l+1}\right]. \tag{19}$$

see [15] for the usual definition. For $m = 1$, for example, one recovers the convolution result given in [21]. Indeed, putting $a = 0$ and $b = n$ in (16) yields the convolution of first powers of the integers with the general Fibonacci numbers,

$$\begin{aligned}\sum_{k=0}^{n} kG_{n-k} &= G_{n+3} - [(n+l)G_1 + G_2]\\ &= G_{n+3} - [(n+2)G_1 + G_0]\end{aligned} \tag{20}$$

upon use of $G_2 = G_1 + G_0$, from (2).

Similarly, with $m = 2$ in (19), one finds that

$$\begin{aligned}\sum_{k=0}^{n} k^2 G_{n-k} &= c_2^{(0)}(0)G_{n+2} + c_2^{(1)}(0)G_{n+3} + 2c_2^{(2)}(0)G_{n+4}\\ &\quad - \left[c_2^{(0)}(n+1)G_1 + c_2^{(1)}(n+1)G_2 + 2c_2^{(2)}(n+1)G_3\right]\\ &= G_{n+3} + 2G_{n+4} - \left[(n+1)^2G_1 + (2(n+1)+1)G_2 + 2G_3\right]\\ &= G_{n+6} - \left[(n^2+4n+8)G_1 + (2n+5)G_0\right]\end{aligned} \tag{21}$$

with the help of the third line of (12), and of (2).

Finally, proceeding in the same manner, one has the following regular convolution of the third powers with the general Fibonacci sequence, from (19):

$$\begin{aligned}\sum_{k=0}^{n} k^3 G_{n-k} &= \sum_{l=0}^{3} l! \left[c_3^{(l)}(0)G_{n+2+l} - c_3^{(l)}(n+1)G_{l+1}\right] \\ &= 12G_{n+4} + 7G_{n+3} - \left[(n^3+6n^2+24n+50)G_1 + (3n^2+15n+31)G_0\right].\end{aligned} \tag{22}$$

The details are omitted because they follow directly from the last line of (12), and from (2), just as in the previous examples.

Generally speaking, the ease with which the results (15) to (22) can be obtained is quite remarkable.

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