ON THE PARITY OF THE PARTITION FUNCTION

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Let $p(n)$ denote the unrestricted partition function. Some time ago, O. Kolberg [3] proved that $p(n)$ changes parity infinitely often. In this note, we offer an alternate proof of this fact.

Lemma 1: Let $N = \{0, 1, 2, \ldots\}$. Let r be a function from N to N such that $r(k) - r(k-1) \to \infty$ as $k \to \infty$. Consider the following functions of x, expressed as series with coefficients in GF(2):

$$f(x) = \sum_{n=0}^{\infty} a(n) x^n$$

where for each integer $n \geq 0$, we have

$$a(n) = \begin{cases} 1 & \text{if n = r(k)} \quad for \quad some \quad k \in N \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \sum_{n=0}^{\infty} b(n) x^n$$

where $b(0) = 1$ and $b(n) = 0$ if $n > m$ for some fixed m;

$$h(x) = f(x)g(x) = \sum_{n=0}^{\infty} c(n) x^n$$

Then $c(n) = 1$ for infinitely many n.

Remark: There are many examples of functions from N to N that satisfy the conditions of Lemma 1. For example, let $r(k) = k^m$ where the integer $m \geq 2$.

Proof: Suppose that $c(n) = 0 \quad \forall n > t$ for some t. By hypothesis, there exists k such that $r(k) - r(k-1) > t$. Choose $n = r(k)$. Now

$$c(n) = \sum_{j=0}^{n} a(n-j) b(j) = a(r(k)) b(0) + \sum_{j \geq 1} a(r(k-j)) b(r(k) - r(k-j))$$

But $\forall j \geq 1$ we have $r(k) - r(k-j) \geq r(k) - r(k-1) > t$ by hypothesis. Therefore $b(r(k) - r(k-j)) = 0 \quad \forall j \geq 1$ by hypothesis, which in turn implies $c(n) = a(r(k)) = 1$, contrary to hypothesis. We are done. □

In the proof of Theorem 1, which follows below, we will make use of the well-known identities:

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{k=1}^{\infty} (-1)^k (x^{k(3k-1)/2} + x^{k(3k+1)/2}) \tag{1}$$

$$\prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n) x^n.$$

Remarks: Identity (1) is due to Euler. (See [1], p. 17, (1.3.1) or [2], p. 284, Theorem 353). Identity (2) may be found in [2], p. 274, (19.3.1). Both identities assume that x is a complex variable such that $|x| < 1$.

Theorem 1: $p(n)$ changes parity infinitely often.

Proof: Assuming that x is a complex variable such that $|x| < 1$, let

$$P(x) = \prod_{n=1}^{\infty}(1 + x^n) \ .$$

Now (1) implies

$$P(x) \equiv 1 + \sum_{k=1}^{\infty}(x^{k(3k-1)/2} + x^{k(3k+1)/2}) \pmod 2.$$

Also, (2) implies

$$Q(x) \equiv \prod_{n=1}^{\infty}(1 + x^n)^{-1} \equiv \sum_{n=0}^{\infty} p(n)x^n \equiv \sum_{n=0}^{\infty} b(n)x^n \pmod 2$$

where

$$b(n) = \begin{cases} 1 \text{ if } p(n) \text{ is odd} \\ 0 \text{ if } p(n) \text{ is even.} \end{cases}$$

Moreover, we have

$$h(x) = P(x)Q(x) = 1.$$

Note that $P(x)$ satisfies the hypothesis of Lemma 1 with regard to $f(x)$. If $p(n)$ is odd for only finitely many n, then $Q(x)$ also satisfies the hypothesis of Lemma 1 with regard to $g(x)$. But we have a contradiction, since $h(x) = 1$. Therefore $p(n)$ is odd for infinitely many n.

Now suppose that $p(n)$ is even for only finitely many n, that is,

$$Q(x) = 1 + x + x^3 + x^4 + x^5 + x^6 + x^7 + x^{12} + \cdots + x^s + x^r + x^{r+1} + x^{r+2} + \cdots$$

where $s \leq r - 1$, that is

$$Q(x) \equiv 1 + x + x^3 + x^4 + x^5 + x^6 + x^7 + x^{12} + \cdots + x^s + \frac{x^r}{1+x} \pmod 2.$$

Then we have

$$(1 + x)Q(x) \equiv (1 + x)(1 + x + x^3 + x^4 + x^5 + x^6 + x^7 + x^{12} + \cdots + x^s) + x^r \pmod 2.$$

Now $(1 + x)Q(x)$ satisfies the hypothesis of Lemma 1 with respect to $g(x)$. But

$$h(x) = P(x)(1 + x)Q(x) = 1 + x$$

which contradicts Lemma 1. Therefore $p(n)$ is even for infinitely many n. □

REFERENCES

[1] George E. Andrews. *The Theory of Partitions*, (1976) Cambridge University Press.

[2] G. H. Hardy & E. M. Wright . *An Introduction to the Theory of Numbers*, 4^{th} Ed. (1960) Oxford University Press.

[3] O. Kolberg. "Note on the Parity of the Partition Function." *Math. Scand.* **7** (1959): 377-378.

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