ON PELL PARTITIONS

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1. INTRODUCTION

The Pell sequence, denoted $\{P_n\}$, is defined for $n \geq 0$ by:

$$P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2} \quad for \quad n \geq 2. \tag{1}$$

Let U_n denote the number of partitions of the natural number n, all of whose parts belong to $\{P_n\}$. In this note, we present a recursive algorithm for computing U_n. The techniques used here are also applicable to other second order linear recurrences that have the property of being *super-increasing*, that is, where each term exceeds the sum of all its predecessors. In [1], the second author solved the corresponding problem for the Fibonacci sequence.

2. MAIN RESULTS

Remarks: It is easily seen from (1) that $\{P_n\}$ is strictly increasing. Furthermore, we have:

Theorem 1: If $n \geq 1$, then

$$\sum_{j=1}^{n} P_j = \frac{1}{2}(P_{n+1} + P_n - 1). \tag{2}$$

Proof: Use (1) and induction on n. □

Next, we will show that every natural number has a unique *greedy* representation as a sum of Pell numbers. Let k be the unique index such that $P_k \leq m < P_{k+1}$. Now (1) implies $P_k \leq m < 2P_k + P_{k-1}$. In the greedy algorithm, we subtract from m the largest possible multiple of P_k, that is, we write:

$$m = tP_k + (m - tP_k)$$

where the multiplier $t \in \{1, 2\}$. We then iterate the process on the remainder and continue until we obtain a zero remainder. This yields a representation of m as a sum of Pell numbers. At each iteration, the values of the index and of the multiplier are uniquely determined. Therefore, the greedy Pell representation of m is unique.

For example, let $m = 151$. Since $P_6 = 70 < 151 < 169 = P_7$, we write $151 = 2(70) + 11$. Since $P_3 = 5 < 11 < 12 = P_4$, we write $11 = 2(5) + 1$. Since $1 = P_1$, we have $151 = 2(70) + 2(5) + 1$.

Theorem 2: If $m \in N$, then m has a unique representation:

$$m = \sum_{i=1}^{k} c_i P_i$$

where each $c_i \in \{0,1,2\}$, $c_k \neq 0$, and if $c_i = 2$, then $i \geq 2$ and $c_{i-1} = 0$.

Proof: The statement is trivially true if $m = P_k$ for some $k \geq 1$ or if $m = 2P_k$ for some $k \geq 2$. In particular, therefore, it is true when $m \in \{1,2\}$. Otherwise, we let k be the unique integer such that $P_k < m < P_{k+1}$, that is, $P_k < m < 2P_k + P_{k-1}$, and use induction on k.

Case 1: If $P_k < m < 2P_k$, then $0 < m - P_k < P_k$, so by induction hypothesis, we have

$$m - P_k = \sum_{i=1}^{s} c_i P_i$$

where $1 \leq s \leq k-1$, $c_s \neq 0$, $\forall c_i \in \{0,1,2\}$, and if $c_i = 2$, then $i \geq 2$ and $c_{i-1} = 0$. Therefore

$$m = P_k + \sum_{i=1}^{s} c_i P_i = \sum_{i=1}^{k} c_i P_i$$

where $c_k = 1$ and if $s \leq k-2$, then $c_{s+1} = c_{s+2} = \cdots = c_{k-1} = 0$.

Case 2: If $2P_k < m < 2P_k + P_{k-1}$, then $0 < m - 2P_k < P_{k-1}$, so by induction hypothesis, we have

$$m - 2P_k = \sum_{i=1}^{s} c_i P_i$$

where $1 \leq s \leq k-1$, $c_s \neq 0, \forall c_i \in \{0,1,2\}$, and if $c_i = 2$, then $i \geq 2$ and $c_{i-1} = 0$. Therefore

$$m = 2P_k + \sum_{i=1}^{s} c_i P_i = \sum_{i=1}^{k} c_i P_i$$

where $c_k = 2$ and $c_{s+1} = c_{s+2} = \cdots = c_{k-1} = 0$. Since k is uniquely determined, it follows that the greedy representation of m as a sum of Pell numbers is also unique. $\square$

Remarks: If $m > 1$, then by repeated use of (1), one may generate additional Pell representations of m that satisfy some, but not all of the conditions of the conclusion of Theorem 2. For example,

$$30 = 29 + 1 = P_5 + P_1$$

but also

$$30 = 2(12) + 5 + 1 = 2P_4 + P_3 + P_1.$$

The first of these two Pell representations of 30 is greedy; the second is not.

If $\{u_n\}$ is a strictly increasing sequence of natural numbers, let

$$g(z) = \prod_{n \geq 1} (1 - z^{u_n}). \tag{3}$$

The product in (9) converges absolutely to an analytic function without zeroes on compact subsets of the unit disc. Let $g(z)$ have the Maclaurin series representation:

$$g(z) = \sum_{n\geq 0} a_n z^n. \tag{4}$$

Here $a_0 = 1.$ If $n \geq 1,$ then a_n is the difference between the number of partitions of n into evenly many distinct parts from $\{u_n\}$ and the number of partitions of n into oddly many distinct parts from $\{u_n\}$. If we let

$$f(z) = 1/g(z) \tag{5}$$

then $f(z)$ is also an analytic function without zeroes on compact subsets of the unit disc. We have:

$$f(z) = \prod_{n\geq 1}(1 - z^{u_n})^{-1} = \sum_{n\geq 0} U_n z^n \tag{6}$$

with $U_0 = 0,$ where U_n denotes the number of partitions of n into parts from $\{u_n\}$. Since $f(z)g(z) = 1,$ we obtain the recurrence relation:

$$\sum_{k=0}^{n} a_{n-k} U_k = 0 \tag{7}$$

for $n \geq 1$. This provides a convenient way to compute the $U_n,$ once the a_n are known. The following theorem is helpful, not only for the Pell sequence, but for any sequence of natural numbers that is *super-increasing*.

Theorem 3: Let $\{u_n\}$ be a strictly increasing sequence of natural numbers. If z is a complex variable such that $|z| < 1,$ let

$$g(z) = \prod_{n\geq 1}(1 - z^{u_n}) = \sum_{n\geq 0} a_n z^n.$$

Suppose that

$$u_n > \sum_{j=1}^{n-1} u_j \quad \forall n \geq 2.$$

Then $\forall n \geq 0$, we have

$$a_n \in \{-1, 0, 1\}.$$

Proof: If $m \geq 1,$ let

$$g_m(z) = \prod_{k=1}^{m}(1 - z^{u_k}) = \sum_{n\geq 0} a_{m,n} z^n.$$

An elementary argument shows that

$$\lim_{m\to\infty} g_m(z) = g(z)$$

so that

$$\lim_{m\to\infty} a_{m,n} = a_n \quad \forall n.$$

Therefore it suffices to prove that $a_{m,n} \in \{-1, 0, 1\} \quad \forall m, n$. This will be done by induction on m. Now

$$g_1(z) = 1 - z^{u_1}$$

so the statement holds for $m = 1$. Also

$$g_{m+1}(z) = (1 - z^{u_{m+1}})g_m(z).$$

Since $u_{m+1} > \sum_{j=1}^{m} u_j$ by hypothesis, it follows that $a_{m+1,n} = a_{m,n} \forall n \leq \sum_{j=1}^{m} u_j$. The conclusion now follows by applying the induction hypothesis. $\square$

The following theorem shows the relation between the greedy Pell representation of n and the coefficient a_n:

Theorem 4: Let

$$g(z) = \prod_{n \geq 1} (1 - z^{P_n}) = \sum_{n \geq 0} a_n z^n.$$

Let n have the greedy Pell representation:

$$n = \sum_{i=1}^{r} c_i P_i$$

where each $c_i \in \{0, 1, 2\}$ and $c_r \neq 0$. If there exists i such that $c_i = 2$, then $a_n = 0$. Otherwise, $a_n = (-1)^t$ where t is the number of indices, i, such that $c_i = 1$.

Proof: First, we verify that by (1) and Theorem 1, $\{P_n\}$ is super-increasing, that is,

$$P_{n+1} > \sum_{j=1}^{n} P_j \quad \forall n \geq 1.$$

If 2 occurs as a digit, then since $\{P_n\}$ is super-increasing, there can be no representation of n as a sum of distinct Pell numbers, so $a_n = 0$. If the greedy Pell representation of n has $t\, 1's$ and no $2's$, then n is the unique sum of t Pell numbers, so $a_n = 1$ if t is even and $a_n = -1$ if t is odd, that is, $a_n = (-1)^t$. $\square$

We conclude by listing some numerical results in Table 1 below. For each n such that $0 \leq n \leq 50,$ we list the greedy Pell representation of n, followed by a_n and U_n.

n	$Pell(n)$	a_n	U_n	n	$Pell(n)$	a_n	U_n
0	0	1	1	26	2010	0	63
1	1	-1	1	27	2011	0	68
2	10	-1	2	28	2020	0	74
3	11	1	2	29	10000	-1	81
4	20	0	3	30	10001	1	88
5	100	-1	4	31	10010	1	95
6	101	1	5	32	10011	-1	103
7	110	1	6	33	10020	0	110
8	111	-1	7	34	10100	1	120
9	120	0	8	35	10101	-1	128
10	200	0	10	36	10110	-1	139
11	201	0	11	37	10111	1	148
12	1000	-1	14	38	10120	0	159
13	1001	1	15	39	10200	0	170
14	1010	1	18	40	10201	0	182
15	1011	-1	20	41	11000	1	195
16	1020	0	23	42	11001	-1	208
17	1100	1	26	43	11010	-1	221
18	1101	-1	29	44	11011	1	236
19	1110	-1	32	45	11020	0	250
20	1111	1	36	46	11100	-1	267
21	1120	0	39	47	11101	1	282
22	1200	0	44	48	11110	1	300
23	1201	0	47	49	11111	-1	317
24	2000	0	53	50	11120	0	336
25	2001	0	57				

Table 1: Pell Partitions

REFERENCES

[1] N. Robbins. "Fibonacci partitions." *The Fibonacci Quarterly* **34** (1996): 306-313.

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