SOME IDENTITIES FOR BERNOULLI AND EULER POLYNOMIALS

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1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. The Bernoulli polynomials $B_n(x)$ $(n \in \mathbb{N})$ and the Euler polynomials $E_n(x)$ $(n \in \mathbb{N})$ are defined by means of

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad \text{and} \quad \frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}.$$

Those $B_n = B_n(0)$ and $E_n = 2^n E_n(1/2)$ are called the Bernoulli numbers and the Euler numbers respectively. ¿From the definitions we can easily deduce the following well known properties:

$$B_n(1-x) = (-1)^n B_n(x) \text{ and } B_n(x+1) - B_n(x) = nx^{n-1};$$
$$E_n(1-x) = (-1)^n E_n(x) \text{ and } E_n(x+1) + E_n(x) = 2x^n.$$

In 1995 M. Kaneko [1] found that B_{2n} can be computed in terms of those B_i with $n \le i < 2n$, namely he proved the formula

$$\sum_{i=0}^{n} \binom{n+1}{i} (n+i+1) B_{n+i} = 0 \quad \text{for } n = 1, 2, 3, \dots .$$

In 2001 H. Momiyama [2] extended the above result as follows: If $m, n \in \mathbb{N}$ and $m + n > 0$, then

$$(-1)^m \sum_{i=0}^{m} \binom{m+1}{i} (n+i+1) B_{n+i} + (-1)^n \sum_{j=0}^{n} \binom{n+1}{j} (m+j+1) B_{m+j} = 0. \tag{1}$$

In this paper we aim to make further extensions by a new method.

Now we state our main results.

Theorem 1: *Let $\{f_k(x)\}_{k=0}^{\infty}$ be a sequence of polynomials given by*

$$\sum_{k=0}^{\infty} f_k(x) \frac{z^k}{k!} = e^{(x-1/2)z} F(z) \tag{2}$$

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where $F(z)$ is a formal power series. Let $m, n \in \mathbb{N}$. If F is even, i.e. $F(-z) = F(z)$, then

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} f_{n+i}(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} f_{m+j}(-x); \tag{3}$$

if F is odd, i.e. $F(-z) = -F(z)$, then

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} f_{n+i}(x) = -(-1)^n \sum_{j=0}^{n} \binom{n}{j} f_{m+j}(-x). \tag{4}$$

This general theorem will be proved in Section 2. Now we give a consequence of it.
Corollary 1: *Let $F(z)$ be an even or odd formal power series, and let $f_k(x)$ $(k \in \mathbb{N})$ be given by (2). Let $m, n \in \mathbb{N}$ and $\varepsilon = 1$ or -1 according to whether $F(z)$ is even or odd. Then*

$$(-1)^m \sum_{i=0}^{m+1} \binom{m+1}{i} (n+i+1) f_{n+i}(x) = -\varepsilon(-1)^n \sum_{j=0}^{n+1} \binom{n+1}{j} (m+j+1) f_{m+j}(-x). \tag{5}$$

Proof: Clearly $-zF(-z) = -\varepsilon z F(z)$ and

$$e^{(x-1/2)z} z F(z) = z \sum_{k=0}^{\infty} f_k(x) \frac{z^k}{k!} = \sum_{k=1}^{\infty} f_k^*(x) \frac{z^k}{k!}$$

where $f_k^*(x) = k f_{k-1}(x)$. In view of Theorem 1, we have

$$(-1)^{m+1} \sum_{i=0}^{m+1} \binom{m+1}{i} f^*_{n+1+i}(x) = -\varepsilon(-1)^{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} f^*_{m+1+j}(-x)$$

which is equivalent to (5). □

Observe that

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = e^{(x-1/2)z} \frac{z}{e^{z/2} - e^{-z/2}} \text{ and } \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = e^{(x-1/2)z} \frac{2}{e^{z/2} + e^{-z/2}}.$$

Also,

$$\begin{aligned} &B_{m+n+1}(x) + (-1)^{m+n} B_{m+n+1}(-x) \\ =&B_{m+n+1}(x) - B_{m+n+1}(1-(-x)) = -(m+n+1)x^{m+n} \end{aligned}$$

and

$$\begin{aligned} &E_{m+n+1}(x) + (-1)^{m+n} E_{m+n+1}(-x) \\ =&E_{m+n+1}(x) - E_{m+n+1}(1+x) = 2E_{m+n+1}(x) - 2x^{m+n+1}. \end{aligned}$$

So Theorem 1 and Corollary 1 imply the following result.

Theorem 2: *Let $m, n \in \mathbb{N}$. Then*

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{n+i}(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_{m+j}(-x) \tag{6}$$

and

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} E_{n+i}(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} E_{m+j}(-x); \tag{7}$$

also

$$\begin{aligned}
&(-1)^m \sum_{i=0}^{m} \binom{m+1}{i} (n+i+1) B_{n+i}(x) \\
&+ (-1)^n \sum_{j=0}^{n} \binom{n+1}{j} (m+j+1) B_{m+j}(-x) \\
=&(-1)^m (m+n+2)(m+n+1) x^{m+n}
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&(-1)^m \sum_{i=0}^{m} \binom{m+1}{i} (n+i+1) E_{n+i}(x) \\
&+ (-1)^n \sum_{j=0}^{n} \binom{n+1}{j} (m+j+1) E_{m+j}(-x) \\
=&(-1)^m 2(m+n+2) \left(x^{m+n+1} - E_{m+n+1}(x) \right).
\end{aligned} \tag{9}$$

Clearly (8) in the case $x = 0$ yields Momiyama's formula (1), and (9) provides a recurrent formula for Euler polynomials.

Putting $x = 0$ in (6) and $x = 1/2$ in (7) we then get

Corollary 2: *For $m, n \in \mathbb{N}$, we have*

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_{m+j} \tag{10}$$

and

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} \frac{E_{n+i}}{2^{n+i}} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} E_{m+j} \left(-\frac{1}{2} \right). \tag{11}$$

2. PROOF OF THEOREM 1

Suppose that $F(-z) = \varepsilon F(z)$ for all z where $\varepsilon \in \{1, -1\}$. Consider the generating function

$$G(x,y,z) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left((-1)^m \sum_{i=0}^{m}\binom{m}{i} f_{n+i}(x)\right)\frac{y^m}{m!}\cdot\frac{z^n}{n!}.$$

What we have to show is the identity $G(x, y, z) = \varepsilon G(-x, z, y)$. Changing the order of summation, we obtain

$$\begin{aligned}
G(x,y,z) &= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}(-1)^m\binom{m}{i} f_{n+i}(x)\frac{y^m}{m!}\cdot\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} f_{n+i}(x)\frac{z^n}{n!}\sum_{m=i}^{\infty}(-1)^m\binom{m}{i}\frac{y^m}{m!}\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} f_{n+i}(x)\frac{z^n}{n!}\cdot\frac{(-y)^i}{i!}e^{-y}\\
&= e^{-y}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{f_k(x)}{k!}\binom{k}{i} z^{k-i}(-y)^i\\
&= e^{-y}\sum_{k=0}^{\infty} f_k(x)\frac{(z-y)^k}{k!}\\
&= e^{-y}e^{(x-1/2)(z-y)}F(z-y)\\
&= e^{x(z-y)-(y+z)/2}F(z-y).
\end{aligned}$$

From this, we have

$$G(-x,z,y) = e^{-x(y-z)-(z+y)/2}F(y-z) = e^{x(z-y)-(y+z)/2}\varepsilon F(z-y) = \varepsilon G(x,y,z),$$

as desired.

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Added in Proof. The main results of this paper were further extended in [3] by the second author.

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