A REPRESENTATION OF REGULAR SUBSEQUENCES OF RECURRENT SEQUENCES

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ABSTRACT

This paper is devoted to analysis of subsequences of general recurrent sequences. Let $\{u_n\}_{n=0}^{\infty}$ be a recurrent sequence of order m. We consider subsequences $v_n = u_{sn+j}$ where s is a natural number, j is a non-negative integer. Hereinafter we call these sequences 'regular subsequences'. Our goal is to calculate the general representation for these subsequences and to create the simple algorithm for its calculation which does not use the roots of the corresponding characteristic equation. Also, the general formula of representation for 'regular' subsequences is presented. An algorithm for calculation of 'regular' subsequences is given. We propose also an algorithm for calculating elements of the sequence equal to the sum of n^{th} powers of the roots of characteristic equation, e.g. for elements , $U_{n+5k} = L_k U_{n+4k} + \frac{1}{2}(L_{2k} - (L_k)^2)U_{n+3k} + \frac{1}{2}(C_5)^k(L_{2k} - (L_k)^2)U_{n+2k} - (C_5)^k L_{-k} U_{n+k} + (C_5)^k U_n$. This paper is a generalization of the result by Prof. F. T. Howard.

INTRODUCTION

A recurrent sequence of order m can be defined as a sequence

$$u_0, u_1, ..., u_n, ...$$

with $m-1$ arbitrary real (complex) numbers and for $k \geq 0$:

$$u_{m+k} = C_1 u_{m+k-1} + C_2 u_{m+k-2} + ... + C_m u_k. \tag{1}$$

The numbers $C_1, ..., C_m$ are arbitrary real numbers and $C_m \neq 0$. For the sequences (1) depending on $2m$ parameters we use the notation:

$$W_n(u_0, u_1, ..., u_{m-1}, C_1, C_2, ..., C_m).$$

The corresponding characteristic equation for the sequence (1) is defined by the formula:

$$x^m - C_1 x^{m-1} - ... - C_m = 0. \tag{2}$$

Let $x_1, ..., x_m$ be the complex roots (including multiple roots) of equation (2).

Definition: Let $\{L_n\}_{n=0}^{\infty}$, $L_n = x_1^n + x_2^n + ... + x_m^n$, be the recurrent sequence of order m. This sequence depends on parameters $(C_1, C_2, ..., C_m)$. We call this sequence *L-sequence for parameters* $(C_1, C_2, ..., C_m)$.

Let us prove that $\{L_n\}_{n=0}^{\infty}$ is a sequence from the set $W_n(u_0, u_1, ..., u_{m-1}, C_1, C_2, ..., C_m)$. Let us consider sequences of the following form:

$$v_k^{(1)} = \{(x_1)^k\}_{k=0}^{\infty}, v_k^{(2)} = \{(x_2)^k\}_{k=0}^{\infty}, ..., v_k^{(m)} = \{(x_m)^k\}_{k=0}^{\infty}.$$

It is clear that these sequences are the recurrent sequences from the set $W_n(u_0, u_1, ..., u_{m-1}, C_1, C_2, ..., C_m)$, since $x_1, ..., x_m$ are the roots of the characteristic equation. Hence, the sum of these sequences, i.e. the sequence $\{L_n\}_{n=0}^{\infty}$, is the recurrent sequence from the set $W_n(u_0, u_1, ..., u_{m-1}, C_1, C_2, ..., C_m)$.

Thus, for each set of parameters $(C_1, C_2, ..., C_m)$, there exists the unique L-sequence. In particular, the L-sequence with parameters $C_1 = C_2 = 1; m = 2$ coincides with the well-known Lucas sequence.

Definition: Let $\{u_n\}_{n=0}^{\infty}$ be a recurrent sequence from the set $W_n(u_0, u_1, ..., u_{m-1}, C_1, C_2, ..., C_m)$. We say that a subsequence $\{v_n\}_{n=0}^{\infty}$ of sequence $\{u_n\}_{n=0}^{\infty}$ is called *regular subsequence* of $\{u_n\}_{n=0}^{\infty}$ if there exists a natural number s and a nonnegative integer j such that for every nonnegative integer k the following formula holds:

$$v_k = u_{sk+j}. \tag{3}$$

We prove below that a regular subsequence of order m is a recurrent sequence of order m. That is, there exist real numbers $U_1, .., U_m$ such that for every nonnegative integer k the following relations take place:

$$v_{k+m} = U_1 v_{k+m-1} + U_2 v_{k+m-2} + ... + U_m v_k. \tag{4}$$

The main result of this paper is an algorithm for calculating these coefficients $U_1, .., U_m$.

It is known that the calculation of roots of a linear equation is a cumbersome problem. Basing on properties of L-sequences we propose an alternative algorithm for calculating coefficients $U_1, .., U_m$. This algorithm significantly reduces the complexity of calculations.

In particular, for $m = 3$:

$$U_{n+3k} = L_k U_{n+2k} - (C_3)^k L_{-k} U_{n+k} + (C_3)^k U_n.$$

This result was proved by Prof. F. T. Howard (see [4]).

For $m = 4$:

$$U_{n+4k} = L_k U_{n+3k} + \frac{1}{2}(L_{2k}-$$
$$-(L_k)^2)U_{n+2k} + (-C_4)^k L_{-k} U_{n+k} - (-C_4)^k U_n$$

For $m = 5$:

$$U_{n+5k} = L_k U_{n+4k} + \frac{1}{2}(L_{2k} - (L_k)^2)U_{n+3k}+$$

$$+\frac{1}{2}(C_5)^k(L_{2k} - (L_k)^2)U_{n+2k} - (C_5)^k L_{-k} U_{n+k} + (C_5)^k U_n$$

Etc.

SOME PROPERTIES OF L-SEQUENCES

Lemma 1: For any integer $1 \le i < n$ the following formula is valid:

$$L_{i+1} = C_1 L_i + C_2 L_{i-1} + ... + C_i L_1 + (i+1)C_{i+1}. \tag{5}$$

In particular,

$$\begin{aligned}
&L_0 = m;\\
&L_1 = C_1;\\
&L_2 = C_1 L_2 + 2C_2;\\
&...\\
&L_n = C_1 L_{n-1} + C_2 L_{n-2} + ... + nC_n;
\end{aligned}$$

Thus, one can calculate all elements of L-sequence directly from parameters $(C_1, C_2, ..., C_m)$. It is not necessary to know values of roots $x_1, ..., x_m$ of characteristic equation.

Proof: First, let us represent the numbers $(C_1, C_2, ..., C_m)$ using $x_1, ..., x_m$.
Consider next transformations:

$$x^m - C_1 x^{m-1} - ... - C_m x^0 = (x - x_1) \cdot (x - x_2) \cdot ... \cdot (x - x_m) =$$
$$= x^m - (x_1 + x_2 + ... + x_m)x^{m-1} + (x_1 x_2 + x_1 x_3 + ...+$$
$$+x_{m-1}x_m)x^{m-2} - (x_1 x_2 x_3 + x_1 x_2 x_4 + ...+$$
$$x_{m-2}x_{m-1}x_n)x^{m-3} + ... - (-1)^{m+1}(x_1 x_2 x_3 ....x_m)x^0$$

Hence, the following relations take place:

$$\begin{aligned}
&C_1 = x_1 + x_2 + ... + x_m\\
&C_2 = -(x_1 x_2 + x_1 x_3 + ... + x_{m-1}x_n) =\\
&\qquad \sum_{j=1}^{m-1} \sum_{k=j+1}^{m} x_j x_k\\
&C_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + ... + x_{m-2}x_{m-1}x_n =\\
&\qquad \sum_{j=1}^{m-2} \sum_{k=j+1}^{m-1} \sum_{l=k+1}^{m} x_j x_k x_l\\
&..................................\\
&C_h = (-1)^{h+1} \cdot \sum_{p_1=1}^{m-h+1} \cdot \sum_{p_2=p_1+1}^{m-h+2} \times\\
&\qquad ... \cdot \sum_{p_h=p_{h-1}+1}^{m} x_{p_1} x_{p_2} ... x_{p_h}\\
&..................................\\
&C_m = (-1)^{m+1} x_1 x_2 x_3 ....x_m.
\end{aligned} \tag{6}$$

Now, let us calculate $C_1 L_t$:

$$C_1 L_t = (x_1 + x_2 + ... + x_m)(x_1^t + x_2^t + ... + x_m^t).$$

First, we open the left bracket:

$$= x_1(x_1^t + x_2^t + ... + x_m^t) + x_2(x_1^t + x_2^t + ... + x_m^t) + ...+$$

$$+x_m(x_1^t + x_2^t + ... + x_m^t).$$

Let us group together terms containing the highest power x_i^{t+1}, $i = 1, ..., m$. We have:

$$= (x_1^{t+1} + x_2^{t+1} + ... + x_m^{t+1}) + x_1(x_2^t + x_3^t + ... + x_m^t)+$$

$$+x_2(x_1^t + x_3^t + ... + x_m^t) + ... + x_m(x_1^t + x_2^t + ... + x_{m-1}^t).$$

The expression in the first bracket is equal to L_{t+1}. Let us regroup the remaining terms as follows:

$$= L_{t+1} + x_1^t(x_2 + x_3 + x_4 + ... + x_m) + x_2^t(x_1 + x_3 + x_4 + ... + x_m)+$$

$$+... + x_m^t(x_1 + x_2 + x_3 + ... + x_{m-1}).$$

Note, that $x_i^t = x_i^{t-1} x_i$, $1 \leq i \leq m$. Let us put x_i into the inner brackets. Hence, we obtain:

$$= L_{t+1} + x_1^{t-1}(x_1x_2 + x_1x_3 + x_1x_4 + ... + x_1x_m)+$$

$$+x_2^{t-1}(x_1x_2 + x_3x_2 + x_4x_2 + ... + x_mx_2) + ...+$$

$$+x_m^{t-1}(x_mx_1 + x_mx_2 + x_mx_3 + ... + x_mx_{m-1}).$$

Let us introduce the following notations.

By the symbol $\varphi_k(i)$, $1 \leq i \leq m$, we denote the sum of terms in the right hand side of relation (6) for coefficient C_i which do not contain element x_k. By the symbol $\varphi_k^*(i)$ we denote the sum of remaining terms, $\varphi_k^*(i) = C_i - \varphi_k(i)$.

According to these definitions we can rewrite the last identity in the following way:

$$C_1 L_t = L_{t+1} + x_1^t \varphi_1(1) + x_2^t \varphi_2(1) + ... + x_m^t \varphi_m(1),$$

or:

$$C_1 L_t = L_{t+1} + \sum_{k=1}^{m} x_k^t \varphi_k(1).$$

Let us indicate the following properties of functions φ_k and φ_k^*:

a) $x_k \varphi_k(i)$ is the sum of terms from the right hand side of relation (6) for coefficient C_i which contain x_k;

b) all terms of the sum $x_k \varphi_k(i)$ are different;

c) each term of the C_{i+1} representation by formula (6), which include x_k, exists in $x_k \varphi_k(i)$, and conversely.

d) the following relation is valid:

$$x_k \varphi_k(i) = -\varphi_k^*(i+1) \tag{7}$$

for every $1 \le k \le m$ and $1 \le i < m$.

According to the formula (7) we fulfill the following transformations:

$$\sum_{k=1}^{m} x_k^s \varphi_k(i) = -\sum_{k=1}^{m} x_k^{s-1} \varphi_k^*(i+1) = \sum_{k=1}^{m} x_k^{s-1}(-C_{i+1} + \varphi_k(i+1)) =$$

$$= -C_{i+1} \sum_{k=1}^{m} x_k^{s-1} + \sum_{k=1}^{m} x_k^{s-1} \varphi_k(i+1) = -C_{i+1} L_{s-1} + \sum_{k=1}^{m} x_k^{s-1} \varphi_k(i+1).$$

Hence, the following equality holds:

$$\sum_{k=1}^{m} x_k^s \varphi_k(l) = -C_{i+1} L_{s-1} + \sum_{k=1}^{m} x_k^{s-1} \varphi_k(i+1).$$

Continuing transformations we obtain:

$$C_1 L_t = L_{t+1} + \sum_{k=1}^{m} x_k^t \varphi_k(1) = L_{t+1} - C_2 L_{t-1} + \sum_{k=1}^{m} x_k^{t-1} \varphi_k(2) = ... =$$

$$= L_{t+1} - C_2 L_{t-1} - C_3 L_{t-2} - ... - C_t L_1 + \sum_{k=1}^{m} x_k \varphi_k(t) =$$

$$= L_{t+1} - C_2 L_{t-1} - C_3 L_{t-2} - ... - C_t L_1 - \sum_{k=1}^{m} \varphi_k^*(t+1).$$

I.e.

$$C_1 L_t = L_{t+1} - C_2 L_{t-1} - C_3 L_{t-2} - ... - C_t L_1 - \sum_{k=1}^{m} \varphi_k^*(t+1). \tag{8}$$

Now, let us calculate the sum $\sum_{i=1}^{m} \varphi_k^*(t+1)$.

Let us note that:

a) by definition, $\varphi_k^*(t+1)$ is the sum of terms from the right hand side of relation (6) for coefficient C_{t+1}. Each term contains x_k;

b) Each term from right hand side of equation (6) for coefficient C_{t+1} is included into the sum $\sum_{k=1}^{m} \varphi_k^*(t+1)$ exactly $t+1$ times.

Let us explain these properties more precisely. According to (6) we have:

$$C_{t+1} = (-1)^t \cdot \sum_{p_1=1}^{m-(t+1)+1} \cdot \sum_{p_2=p_1+1}^{m-(t+1)+2} \cdot ... \cdot \sum_{p_{(t+1)}=p_{(t+1)-1}+1}^{m} x_{p_1} x_{p_2} ... x_{p_{(t+1)}}.$$

Let us fix the set of numbers $(p_1, p_2, ..., p_{t+1})$, where $1 \leq p_1 < p_2 < ... < p_{t+1} \leq m$, and consider the corresponding term $x_{p_1} x_{p_2} ... x_{p_{t+1}}$. This term is included exactly one time into the sum $\varphi^*_{p_1}(t+1)$, exactly one time into the sum $\varphi^*_{p_2}(t+1)$,..., and exactly one time into the sum $\varphi^*_{p_{t+1}}(t+1)$, totally $t+1$ times. And this term is not included in sums $\varphi^*_k(t+1)$ such that $k \notin \{p_1, p_2, ..., p_m\}$.

Hence, running through all sets of numbers (p_1, p_2, p_{t+1}), $1 \leq p_s \leq m$, $1 \leq s \leq t+1$ we get the following relation:

$$\sum_{k=1}^{m} \varphi^*_k(t+1) = (t+1)C_{t+1}. \tag{9}$$

Substituting (9) into (8) we obtain the necessary relation (5).

Lemma 1 is proved.

THEOREM ABOUT REGULAR SUBSEQUENCE

Let us define u_n as recurrent sequence of order m $(C_m \neq 0)$:

$$u_n = W_n(u_0, ..., u_{m-1}, C_1, ..., C_m). \tag{10}$$

The representation of sequence is following:

$$u_{k+m} = C_1 u_{k+m-1} + C_2 u_{k+m-3} + ... + C_m u_k.$$

The characteristic equation for this sequence is:

$$x^m - C_1 x^{m-1} - ... - C_m = 0.$$

Let $x_1, ..., x_i$ be the different complex roots of this equation and $r_1, ..., r_i$ be their multiplicities $(r_1 + ... + r_i = m)$.

We use the following well-known lemma.

Lemma 2: Let us define m sequences:

$$\begin{array}{l} \{(x_1)^n\}_{n=1}^{\infty}, \{n(x_1)^n\}_{n=1}^{\infty}, ..., \{n^{r_1-1}(x_1)^n\}_{n=1}^{\infty}, \\ \{(x_2)^n\}_{n=1}^{\infty}, \{n(x_2)^n\}_{n=1}^{\infty}, ..., \{n^{r_2-1}(x_2)^n\}_{n=1}^{\infty}, ..., \\ \{(x_i)^n\}_{n=1}^{\infty}, \{n(x_i)^n\}_{n=1}^{\infty}, ..., \{n^{r_i-1}(x_i)^n\}_{n=1}^{\infty}. \end{array} \tag{11}$$

There exist complex numbers $(a_1, ..., a_m)$ such that the recurrent sequence u_n can be decomposed to a linear combination of numbers $(a_1, ..., a_m)$ and sequences (11). This linear combination is unique.

Now let us examine the regular subsequence

$$v_k = u_{sk+j} \tag{12}$$

of sequence $\{u_n\}$. Here s is a natural number, and j is a nonnegative integer.

Now, let us define $y_1, ..., y_m$ as roots of equation

$$y^m - C_1 y^{m-1} - ... - C_m = 0$$

including multiple roots, i.e.

$$y^m - C_1 y^{m-1} - ... - C_m = (y - y_1) \cdot (y - y_2) \cdot ... \cdot (y - y_m). \tag{13}$$

Let us define the numbers:

$$\begin{aligned}
&U_1 = (y_1)^s + (y_2)^s + ... + (y_m)^s \\
&U_2 = (-1) \cdot [(y_1 y_2)^s + (y_1 y_3)^s + ... + (y_1 y_m)^s + \\
&+ (y_2 y_3)^s + ... + (y_{m-1} y_m)^s] \\
&U_3 = (y_1 y_2 y_3)^s + (y_1 y_2 y_4)^s + ... + (y_1 y_2 y_m)^s + \\
&+ (y_1 y_3 y_4)^s + ... + (y_{m-2} y_{m-1} y_m)^s \\
&\cdots \\
&U_m = (-1)^{m+1} (y_1 y_2 ... y_m)^s.
\end{aligned} \tag{14}$$

Theorem 1 (Theorem about regular subsequences): Let $\{u_k\}$ and $\{v_k\}$ and numbers $U_1, U_2, ..., U_m$ be defined by (10), (12) and (14). Then the sequence $\{v_n\}$ is a recurrent sequence of order m_1, where m_1 is a natural number, $m_1 \le m$ and the next equality is valid:
$v_{m+k} = U_1 v_{m+k-1} + U_2 v_{m+k-2} + ... + U_m v_k$

Proof: According to Lemma 2, the sequence u_n is a linear combination of complex numbers $(a_1, ..., a_m)$ and sequences:

$$\begin{aligned}
&\{(x_1)^n\}_{n=1}^{\infty}, \{n(x_1)^n\}_{n=1}^{\infty}, ..., \{n^{r_1-1}(x_1)^n\}_{n=1}^{\infty}, \\
&\{(x_2)^n\}_{n=1}^{\infty}, \{n(x_2)^n\}_{n=1}^{\infty}, ..., \{n^{r_2-1}(x_2)^n\}_{n=1}^{\infty}, ..., \\
&\{(x_i)^n\}_{n=1}^{\infty}, \{n(x_i)^n\}_{n=1}^{\infty}, ..., \{n^{r_i-1}(x_i)^n\}_{n=1}^{\infty}.
\end{aligned}$$

By definition of the subsequence $\{v_k\}$, there exist complex numbers $b_1, ..., b_m$ such that the sequence $\{v_k\}$ can be represented as a linear combination of these numbers and sequences:

$$\begin{aligned}
&\{(x_1)^{sn+j}\}_{n=1}^{\infty}, \{n(x_1)^{sn+j}\}_{n=1}^{\infty}, ..., \{n^{r_1-1}(x_1)^{sn+j}\}_{n=1}^{\infty}, \\
&\{(x_2)^{sn+j}\}_{n=1}^{\infty}, \{n(x_2)^{sn+j}\}_{n=1}^{\infty}, ..., \{n^{r_2-1}(x_2)^{sn+j}\}_{n=1}^{\infty}, ..., \\
&\{(x_i)^{sn+j}\}_{n=1}^{\infty}, \{n(x_i)^{sn+j}\}_{n=1}^{\infty}, ..., \{n^{r_i-1}(x_i)^{sn+j}\}_{n=1}^{\infty}.
\end{aligned}$$

Hence, there exist complex numbers $(d_1, ..., d_m)$ such that the sequence v_k can be represented as a linear combination of numbers $(d_1, ..., d_m)$ and sequences:

$$\begin{aligned}
&\{(x_1)^{sn}\}_{n=1}^{\infty}, \{n(x_1)^{sn}\}_{n=1}^{\infty}, ..., \{n^{r_1-1}(x_1)^{sn}\}_{n=1}^{\infty}, \\
&\{(x_2)^{sn}\}_{n=1}^{\infty}, \{n(x_2)^{sn}\}_{n=1}^{\infty}, ..., \{n^{r_2-1}(x_2)^{sn}\}_{n=1}^{\infty}, ..., \\
&\{(x_i)^{sn}\}_{n=1}^{\infty}, \{n(x_i)^{sn}\}_{n=1}^{\infty}, ..., \{n^{r_i-1}(x_i)^{sn}\}_{n=1}^{\infty}.
\end{aligned}$$

Now, consider the next expression:

$$F(x) = (x - x_1^s)^{r_1} \cdot (x - x_2^s)^{r_2} \cdot ... \cdot (x - x_i^s)^{r_i}.$$

Let us make the following transformation:

$$\begin{aligned} &F(x) = (x - x_1^s)^{r_1} \cdot (x - x_2^s)^{r_2} \cdot ... \cdot (x - x_i^s)^{r_i} = \\ &= (x - y_1^s) \cdot (x - y_2^s) \cdot ... \cdot (x - y_m^s) = \\ &= x^m - (y_1^s + ... + y_m^s)x^{m-1} + ((y_1 y_2)^s + ...+ \\ &+ (y_{m-1} y_m)^s)x^{m-2} + ... + (-1)^{m+1}(y_1 y_2 ... y_m)^s = \\ &= x^m - U_1 x^{m-1} - ... - U_m x^0. \end{aligned} \tag{15}$$

Let us consider the following equation

$$F(x) = x^m - U_1 x^{m-1} - ... - U_m x^0 = 0.$$

Note that some numbers in the set $(x_1^s, x_2^s, ..., x_i^s)$ may not be different.

Now let us apply Lemma 2 to the recurrent sequences: $W_n(v_0, v_1, ..., v_{m-1}, U_1, U_2, ..., U_m)$. The following statements are evident:

the multiplicity of root x_1^s is greater or equal to r_1;

the multiplicity of root x_2^s is greater or equal to r_2;

...

the multiplicity of root x_i^s is greater or equal to r_i.

Hence, by Lemma 2 the sequence

$W_n(v_0, v_1, ..., v_{m-1}, U_1, U_2, ..., U_m)$ is a linear combination of sequences:

$$\begin{aligned} &\{(x_1)^{sn}\}_{n=1}^{\infty}, \{n(x_1)^{sn}\}_{n=1}^{\infty}, ..., \{n^{r_1-1}(x_1)^{sn}\}_{n=1}^{\infty}, \\ &\{(x_2)^{sn}\}_{n=1}^{\infty}, \{n(x_2)^{sn}\}_{n=1}^{\infty}, ..., \{n^{r_2-1}(x_2)^{sn}\}_{n=1}^{\infty}, \\ &... \\ &\{(x_i)^{sn}\}_{n=1}^{\infty}, \{n(x_i)^{sn}\}_{n=1}^{\infty}, ..., \{n^{r_i-1}(x_i)^{sn}\}_{n=1}^{\infty}. \end{aligned} \tag{16}$$

Note that in the case when some terms in the set $(x_1^s, x_2^s, ..., x_i^s)$ are not different the number of different sequences in (16) is less than m. In this case we have $m_1 < m$ (see statement of the theorem 1). In case when these terms are different the following equality holds $m_1 = m$.

So, we obtain that $\{v_n\}$ is a linear combination of sequences from (16). Hence, $\{v_n\}$ can be represented by the following formula:

$$v_{m+k} = U_1 v_{m+k-1} + U_2 v_{m+k-2} + ... + U_m v_k.$$

The Theorem 1 is proved.

One can mention that it is difficult to calculate U_i using roots of characteristic equation in the case $m > 3$. In this case we need to find roots of linear equation of order m, and this problem is rather cumbersome.

In the next paragraph we show how to calculate coefficients U_i using L-sequences.

Property 1: Let C_m and U_m be defined by (10) and (14). The next statement is valid.

If m and s are both even integer numbers then:

$$U_m = -(C_m)^s$$

otherwise

$$U_m = C_m^s.$$

Proof: By definition we have:

$$U_m = (-1)^{m+1}(y_1 y_2 ... y_m)^s =$$

$$= (-1)^{m+1}((-1)^{m+1}[(-1)^{m+1}(y_1 y_2 ... y_m)])^s) =$$
$$= (-1)^{m+1}((-1)^{m+1}C_m)^s) = (-1)^{sm+s+m+1}(C_m)^s.$$

If s and m are both even then $sm+s+m+1$ is odd; otherwise $sm+s+m+1$ is even.
Property 1 is proved.

Property 2: Let C_m and U_m be defined by (10) and (14). The next statement is valid.
If m is odd then

$$U_m = C_m^s,$$

and if m is even then

$$U_m = -(-C_m)^s.$$

Proof: Let us consider the next case: m is even and s is odd. In this case

$$U_m = -(-C_m)^s = -(-1)^s \cdot (C_m)^s = (C_m)^s.$$

In other cases this statement is valid according to the Property 1.
Property 2 is proved.

Remark: In [6] Prof. P.T. Young proved that for each integer k: $1 \le k \le m-1$ the following identity holds:

$$U_k = \sum_{i=1}^{m}(-1)^{k-1}D_s^{(k)}(C_1, C_2, \dots, C_m, 0),$$

where $D_s^{(k)}$ are the generalized Dickson polynomials defined by eq. (2.6) in [6]. Theorem 1 in [6] is an analogue of the Theorem 1 in this paper.

ADDITIONAL PROPERTIES OF L-SEQUENCES AND REGULAR SUBSEQUENCES

Now, we apply the result of the Theorem 1 to L-sequences.

Let's use the notations of roots of characteristic equation $y_1, y_2, ..., y_m$ and numbers $U_1, U_2, ..., U_m$ from equalities (14) and (13) of the previous paragraph. We have shown that:

$$(x - y_1^s) \cdot (x - y_2^s) \cdot ... \cdot (x - y_m^s) = x^m - U_1 x^{m-1} - ... - U_m x^0, \tag{17}$$

and

$$(x-y_1)\cdot(x-y_2)\cdot\ldots\cdot(x-y_m)=x^m-C_1x^{m-1}-\ldots-C_mx^0. \tag{18}$$

Let us denote by the symbol $\{L_n\}_{n=0}^{\infty}$ the L-sequence with parameters $(C_1,C_2,...,C_m)$. From (18) and definition of L-sequence it follows:

$$L_n=(y_1)^n+(y_2)^n+...+(y_m)^n.$$

Let us denote by the symbol $\{L_n^{(1)}\}_{n=0}^{\infty}$ the L-sequence with parameters $(U_1,U_2,...,U_m)$. From (17) it follows:

$$L_n^{(1)}=(y_1^s)^n+(y_2^s)^n+...+(y_m^s)^n=(y_1)^{sn}+(y_2)^{sn}+...+(y_m)^{sn}.$$

Hence, for every natural number k the following equality is valid:

$$L_k^{(1)}=L_{sk} \tag{19}$$

Using Lemma 1 let us prove the following statement.

Lemma 3: Let $\{L_n\}$ is the L-sequence with parameters $(C_1,C_2,...,C_m)$ and $\{U_n\}$ be defined from (14). The following relation is valid for every integer number $0\le s<m$:

$$L_{k(s+1)}=U_1L_{ks}+U_2L_{k(s-1)}+...+U_sL_k+(s+1)U_{s+1}. \tag{20}$$

Proof: To prove Lemma 3 it is sufficient to replace L_{sk} by $L_k^{(1)}$. This replacement is possible due to relation (19). Hence, Lemma 3 follows from the Lemma 1 since $\{L_{sk}\}_{n=0}^{\infty}$ is an L-sequence.

Let us suppose that terms of the sequence $\{L_n\}_{n=0}^{\infty}$ are given. Let us put numbers $s=0,1,...,m$ into (20) in a consecutive way. By this way we obtain the following equalities:

$$\begin{aligned}
&U_1=L_k\\
&U_2=\tfrac{1}{2}(L_{2k}-L_kU_1)=\tfrac{1}{2}(L_{2k}-L_k^2)\\
&U_3=\tfrac{1}{3}(L_{3k}-L_kU_2-L_{2k}U_1)=\tfrac{1}{3}(L_{3k}-2L_kL_{2k}+(L_k)^3\\
&\ldots\,.
\end{aligned} \tag{21}$$

So, Lemma 3 provides the method for calculating $U_1,...,U_m$ using only parameters $C_1,...,C_m$.

Note that we can rewrite (21) in the alternative form:

$$\begin{aligned}
&L_k=U_1\\
&L_{2k}=L_k^2+2U_2\\
&L_{3k}=2L_kL_{2k}-(L_k)^3+3U_3\\
&\ldots\,.
\end{aligned}$$

These equalities generalize the well-known relations of Lucas sequences.

Let us remind that the numbers $(U_1, U_2, ..., U_m)$ are defined by (14). Now let us define the adjoint numbers:

$$\begin{aligned}
&U_{-1} = (y_1)^{-s} + (y_2)^{-s} + ... + (y_m)^{-s};\\
&U_{-2} = (-1) \cdot [(y_1y_2)^{-s} + (y_1y_3)^{-s} + ... + (y_1y_m)^{-s} +\\
&+ (y_2y_3)^{-s} + ... + (y_{m-1}y_m)^{-s}];\\
&U_{-3} = (y_1y_2y_3)^{-s} + (y_1y_2y_4)^{-s} + ... + (y_1y_2y_m)^{-s} +\\
&+ (y_1y_3y_4)^{-s} + ... + (y_{m-2}y_{m-1}y_m)^{-s};\\
&...\\
&U_{-m} = (-1)^{m+1}(y_1y_2...y_m)^{-s}.
\end{aligned} \tag{22}$$

By analogy with (15) the following relation takes place for numbers $U_{-1}, U_{-2}, ..., U_{-m}$:

$$y^m - U_{-1}y^{m-1} - U_{-2}y^{m-2} - ... - U_{-m} = (y - x_1^{-i}) \cdot (y - x_2^{-i}) \cdot ... \cdot (y - x_m^{-i}).$$

Hence, the L-sequence for numbers $U_{-1}, U_{-2}, ..., U_{-m}$ is equal to $\{L_{-sk}\}$.

Thus, we can generalize Lemma 3 in the following form.

Lemma 4: Let L_n is the L-sequence with parameters $(C_1, C_2, ..., C_m)$ and numbers $U_{-1}, ..., U_{-m}$ be defined by (22). The following equality holds for every integer $1 \leq s < n$:

$$\begin{aligned}
&L_{-k(s+1)} = U_{-1}L_{-ks} + U_{-2}L_{-k(s-1)} +\\
&+ ... + U_{-s}L_{-k} + (s+1)U_{-(s+1)}.
\end{aligned} \tag{23}$$

Let us prove the following statement.

Lemma 5: Let $U_{-m}, U_{-m+1}, ..., U_{-1}, U_1, ..., U_m$ be defined by (14) and (22). Then the following formula is valid for $1 \leq i < m$:

$$U_{m-i} = (-1) \cdot U_m U_{-i}. \tag{24}$$

Proof: For proving of Lemma 5 let us make the following transformations using (14) and (22):

$$\begin{aligned}
&U_1 = (y_1)^s + (y_2)^s + ... + (y_m)^s =\\
&= (y_1y_2...y_m)^s[(y_1y_2...y_{m-1})^{-s} + (y_1y_2...y_{m-2}y_m)^{-s} +\\
&+ ... + (y_2y_3...y_m)^{-s}] =\\
&= U_mU_{-(m-1)}(-1)^m(-1)^{m+1} = (-1)U_mU_{-(m-1)}\\
&U_2 = (-1) \cdot [(y_1y_2)^s + (y_1y_3)^s + ... + (y_1y_m)^s +\\
&+ (y_2y_3)^s + ... + (y_{m-1}y_m)^s] =\\
&= (-1) \cdot (y_1y_2...y_m)^s[(y_1y_2...y_{m-2})^{-s} + (y_1y_2...y_{m-3}y_{m-1})^{-s} +\\
&+ (y_1y_2...y_{m-3}y_m)^{-s} + (y_1y_3y_4...y_{m-1})^{-s} + ... + (y_3y_3...y_m)^s] =\\
&= (-1)U_mU_{-(m-2)}(-1)^{m+1}(-1)^{m-1} = (-1)U_mU_{-(m-2)}\\
&......
\end{aligned}$$

$$U_{m-1} = (-1)^m[(y_1y_2...y_{m-1})^s + (y_1y_2...y_{m-2}y_m)^s + ...+$$
$$+ (y_2y_3...y_m)^s] = (-1)^m(y_1y_2...y_m)^s[(y_1)^{-s} + (y_2)^{-s}+$$
$$+ ... + (y_m)^{-s}] = (-1)^mU_mU_{-1}(-1)^{m+1} = (-1)U_mU_{-1}.$$

Hence, in all cases the statement of Lemma 5 takes place.

Lemma 5 is proved.

According to Property 2 number U_m can be trivially found using parameter C_m.

A SIMPLIFIED METHOD FOR CALCULATING THE COEFFICIENTS $U_1, ..., U_m$

Let us pass to the main task of this paragraph.

Consider the following algorithm.

Let sequence $\{u_n\}$ be defined by (1). Suppose that parameters $C_1, ..., C_m$ are already known. Let us define the recurrent sequence $\{v_n\}$ by (3)-(4). The goal of the algorithm is to find parameters $U_1, ..., U_m$.

Step 1. According to (5) we calculate terms of L-sequence $\{L_n\}$ for parameters $C_1, ..., C_m$.

Step 2. According to (20), (23), and (24) and Property 2 we calculate numbers $U_1, ..., U_m$.

In most cases we recommend to find $U_1, ..., U_{[\frac{m}{2}]}$ by (20), $U_{-1}, ..., U_{-[\frac{m}{2}]}$ - by (23), and the remaining numbers - by (24).

For example,

By equality (5) we have:

$$U_1 = L_s;$$

$$U_2 = \frac{1}{2}(L_{2s} - L_sU_1) = \frac{1}{2}(L_{2s} - (L_s)^2).$$

By Property 2 we have:

$U_m = (C_m)^s$ - in the case when m is odd;
$U_m = -(-C_m)^s$ - in the case when m is even.

By equality (24) we have:

$U_{m-1} = (-1)U_m \cdot U_{-1} = -(C_m)^sL_{-s}$ - in the case when m is odd;
$U_{m-1} = (-1)U_m \cdot U_{-1} = (-C_m)^sL_{-s}$ - in the case when m is even;

$$U_{m-2} = (-1) \cdot U_m \cdot U_{-2} = -\frac{1}{2}(C_m)^s(L_{-2s} - (L_{-s})^2)$$

- in the case when m is odd;

$$U_{m-2} = (-1) \cdot U_m \cdot U_{-2} = \frac{1}{2}(-C_m)^s(L_{-2s} - (L_{-s})^2)$$

- in the case when m is even;

Etc.
Let us examine the next cases:

Case 1. $m = 2$:
$U_1 = L_s$
$U_2 = U_m = -(-C_2)^s$.
Hence,

$$v_{k+2} = L_s v_{k+1} - (-C_2)^s v_k.$$

The last relation is the well-known formula (see, for example, [2]).

Case 2. $m = 3$:

$$U_1 = L_s; U_2 = U_{m-1} = -(C_3)^s L_{-s}; U_3 = U_m = (C_3)^s.$$

Hence:

$$v_{k+3} = L_s v_{k+2} - (C_3)^s L_{-s} v_{k+1} + (C_3)^s v_k.$$

The last formula was obtained by Prof. F. T. Howard in [4].

Case 3. $m = 4$:

$$\begin{aligned}
&U_1 = L_s;\\
&U_2 = \tfrac{1}{2}(L_{2s} - (L_s)^2);\\
&U_3 = U_{m-1} = (-C_4)^s L_{-s};\\
&U_4 = U_m = -(-C_4)^s.
\end{aligned}$$

Hence,

$$v_{k+4} = L_s v_{k+3} + \frac{1}{2}(L_{2s} - (L_s)^2) v_{k+2} + (-C_4)^s L_{-s} v_{k+1} - (-C_4)^s v_k.$$

The author believes this result is new.

Case 4. $m = 5$:

$$\begin{aligned}
&U_1 = L_s;\\
&U_2 = \tfrac{1}{2}(L_{2s} - (L_s)^2);\\
&U_3 = U_{m-2} = -\tfrac{1}{2}(C_5)^s (L_{-2s} - (L_{-s})^2);\\
&U_4 = U_{m-1} = -(C_5)^s L_{-s};\\
&U_5 = U_m = (C_5)^s.
\end{aligned}$$

Hence,

$$v_{k+5} = L_s v_{k+4} + \frac{1}{2}(L_{2s} - (L_s)^2) v_{k+3} -$$

$$-\frac{1}{2}(C_5)^s (L_{-2s} - (L_{-s})^2) v_{k+2} - (C_5)^s L_{-s} v_{k+1} + (C_5)^s v_k.$$

Remark: Let us examine the following example

$$u_{n+2} = 4u_n.$$

In this case the corresponding L-sequence, let denote it L_n, is a sequence of the second order

$$L_n = 2^n + (-2)^n.$$

The regular subsequence

$$v_n = L_{2n}$$

is a sequence of the first order:

$$v_n = 4^n.$$

We use here the order of terms of recurrent sequences like in Definition 1 in [4].

Finally, we examine the following example $m = 4$, $C_1 = 1$, $C_2 = 3$, $C_3 = 4$, $C_4 = 2$. Hence,

$$u_{n+4} = 3u_{n+3} + 5u_{n+2} + 7u_{n+1} + 2u_n. \tag{25}$$

Let $s = 5$ and j be a natural number.

In this case we have:

$$\begin{aligned}
&L_0 = m = 4;\\
&L_1 = C_1 = 1;\\
&L_2 = L_1C_1 + 2C_2 = 1 + 2 \cdot 3 = 7;\\
&L_3 = L_1C_2 + L_2C_1 + 3C_3 = 1 \cdot 3 + 7 \cdot 1 + 3 \cdot 4 = 22.
\end{aligned}$$

Hence, by (25) we can calculate the following values:

$$\begin{aligned}
&L_5 = 151;\\
&L_{10} = 23167;\\
&L_{15} = 3526402.
\end{aligned}$$

According to Lemma 3 we can calculate the following values

$$\begin{aligned}
&U_1 = L_5 = 151;\\
&U_2 = (L_{10} - L_5U_1)/2 = (23167 - 151 \cdot 151)/2 = 183;\\
&U_3 = (L_{15} - L_{10}U_1 - L_5U_2)/3 =\\
&(3526402 - 23167 \cdot 151 - 151 \cdot 183)/3 = 184.
\end{aligned}$$

According to Property 1 we obtain:

$$U_4 = 2^5 = 32.$$

Hence,

$$v_{k+4} = 151v_{k+3} + 183v_{k+2} + 184v_{k+1} + 32v_k.$$

Thus, we obtain the following recurrent formula

$$u_{k+20} = 151u_{k+15} + 183u_{k+10} + 184u_{k+5} + 32u_k.$$

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