SYMMETRIC ARGUMENTS IN THE DEDEKIND SUM

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1. INTRODUCTION

R. Dedekind [1] derived the following formula for the logarithm of the eta-function. Let $\eta(z) = e^{\pi iz/12}\prod_{m=1}^{\infty}(1-e^{2\pi imz})$. And let $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad-bc=1$ and $Vz = (az+b)/(cz+d)$. Then, for $\mathrm{Im}(z) > 0$ and $c > 0$,

$$\log\eta(Vz) = \log\eta(z) + \frac{1}{2}\log(cz+d) + \frac{\pi i(a+d)}{12c} - \frac{1}{4}\pi i - \pi is(d,c),$$

where

$$s(d,c) = \sum_{j=1}^{c}\left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right),$$

with

$$((x)) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x-[x]-\frac{1}{2}, & \text{otherwise.} \end{cases}$$

The sum appearing in Dedekind's formula, $s(d,c)$, is called the Dedekind sum. The sum has been studied extensively by many authors. See Rademacher and Grosswald [3] for a bibliography. The most important result about Dedekind sums, proved by Dedekind in his paper, is the reciprocity law. There are many different proofs in the literature, including four in [3].

Theorem 1 (Reciprocity Law): *If* $(h,k)=1$ *and* $h,k>0$, *then*

$$s(k,h) + s(h,k) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} + \frac{1}{kh}\right). \tag{1.1}$$

Our purpose in this paper is to examine the pairs of integers $\{h,k\}$ for which $s(h,k) = s(k,h)$. We will call $\{h,k\}$ a symmetric pair if this property holds. We show that $\{h,k\}$ is a symmetric pair if and only if $h = F_{2n+1}$ and $k = F_{2n+3}$ for $n \in \mathbb{N}$ and F_m is the m^{th} Fibonacci number.

2. SYMMETRIC PAIRS

We need the following facts about Dedekind sums. The properties are elementary and proofs can be found in [3]. Throughout the paper we will assume that h and k are relatively prime.

Property 1: *The denominator of* $s(h,k)$ *is a divisor of* $2k(3,k)$.

Property 2: *The only integer value taken by* $s(h,k)$ *is zero. This occurs if and only if* $h^2+1\equiv 0 \pmod{k}$.

The next theorem gives a necessary condition for $\{h,k\}$ to be a symmetric pair.

Theorem 2: *If* $(h,k)=1$ *and* $\{h,k\}$ *is a symmetric pair, then* $s(h,k)=0$.

Proof: Let D be the denominator of $s(h,k)$, and thus of $s(k,h)$. Then $D \mid 6k$ and $D \mid 6h$ by Property 1. From this and the fact that $(h,k)=1$, we deduce that $D=1,2,3$ or 6. If $D=1$, then we are done by Property 2. Suppose that $D=2,3$ or 6. Then $12s(h,k)\in\mathbb{Z}$. Let us rewrite the reciprocity law (1.1) as

$$12hks(k,h)+12hks(h,k)=-3hk+h^2+k^2+1. \tag{2.1}$$

Since $6hs(k,h)\in\mathbb{Z}$ by Property 1, (2.1) becomes

$$Ak+2Bk=-3hk+h^2+k^2+1$$

for some $A,B\in\mathbb{Z}$. Thus $h^2+1\equiv 0 \pmod{k}$ and, from Property 2, we conclude that $s(h,k)=0$. □

Since we now know that for a symmetric pair $\{h,k\}$ we must have $s(h,k)=0$. From (2.1), any such h and k must solve the Diophantine equation

$$h^2-3hk+k^2=-1. \tag{2.2}$$

Theorem 3: *The positive integral solutions to (2.2) are* $h=F_{2n+1}, k=F_{2n+3}$ *for* $n\in\{0,1,2,\dots\}$ *where* F_m *is the* m^{th} *Fibonacci number.*

Proof: Under the change of variable $\overline{k}=2k-3h$, the equation (2.2) becomes

$$\overline{k}^2-5h^2=-4. \tag{2.3}$$

Now [2], Theorem 7, implies that $\overline{k}=L_{2n+1}, h=F_{2n+1}$, where L_n denotes the n^{th} Lucas number. We conclude the proof with the observation that

$$k=\frac{\overline{k}+3h}{2}=\frac{L_{2n+1}+3F_{2n+1}}{2}=F_{2n+3}. \qquad \square$$

Theorem 2 and Theorem 3 imply the following characterization.

Theorem 4: *The pair* $\{h,k\}$ *is a symmetric pair if and only if* $h=F_{2n+1}$ *and* $k=F_{2n+3}$ *for* $n\in\mathbb{N}$ *and* F_m *is the* m^{th} *Fibonacci number.*

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