FIBONACCI POLYTOPES AND THEIR APPLICATIONS

Fred J. Rispoli
Department of Mathematics, Dowling College, Oakdale, NY 11769
(Submitted November 2002-Final Revision April 2004)

ABSTRACT

A Fibonacci d-polytope of order k is defined as the convex hull of $\{0,1\}$-vectors with d entries and no consecutive k ones, where $k \leq d$. We show that these vertices can be partitioned into k subsets such that the convex hull of the subsets give the equivalent of Fibonacci $(d-i)$-polytopes, for $i = 1, \ldots, k$, which yields a "Fibonacci like" recursive formula to enumerate the vertices. Surprisingly, the polytopes are totally unimodular and require a small number of inequalities to describe them. These facts are used to enumerate compositions of a positive integer with bounded summands, and to find various compositions.

INTRODUCTION

The *Fibonacci d-polytope of order k*, denoted by $FP_d(k)$, is the convex hull of the set of $\{0,1\}$-vectors having d entries and no consecutive k ones. For example, $FP_3(2)$ is the convex hull of $\{000, 001, 100, 010, 101\}$ (see Figure 1.) Notice that $FP_3(2)$ contains a face which is "combinatorially equivalent" to $FP_2(2)$ (the triangle) and another face that is equivalent to $FP_1(2)$ (the line segment), as indicated by the bold edges. An illustration of $FP_3(3)$ is also given. Observe that $\{000, 100, 110, 010\}$, $\{001, 101\}$, and $\{011\}$ is a partition of the vertices of $FP_3(3)$ such that the convex hull of each subset gives the equivalent of Fibonacci polytopes of order 3 of dimension 2, 1 and 0 respectively.

Fibonacci 3-polytope of order 2

Fibonacci 3-polytope of order 3

Figure 1 Fibonacci 3-polytopes of order 2 and 3.

Here we investigate the Fibonacci d-polytopes of order k and discuss some interesting properties and applications. For example, the vertices of every $FP_d(k)$ can be partitioned into k subsets $F_1, \ldots, F_k$ such that the convex hull of F_i generates a Fibonacci polytope of order k, of dimension $d-1,\ d-2, \ldots, d-k$, respectively. The partition is obtained by considering special faces of $FP_d(k)$ and implies that the number of vertices of $FP_d(k)$, denoted by a_d, follows the "Fibonacci-like" recurrence relation given by $a_d = 2^d$, for $d < k$, $a_d = 2^d - 1$, for $d = k$, and $a_d = a_{d-1} + a_{d-2} + \cdots + a_{d-k}$, for $d > k$. When $k = 2$, we see that the number of vertices of $FP_d(2)$ satisfies $a_d = a_{d-1} + a_{d-2}$, where $a_1 = 2$ and $a_2 = 3$. Hence, the number of vertices of $FP_d(2)$ is given by the famous Fibonacci numbers, denoted by F_d. From Binet's formula we know that

$$F_d = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^d - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^d$$

(for proof, see [5]). Therefore, the number of vertices of $FP_d(k)$ grows exponentially in d. However, the Fibonacci polytopes may be defined using roughly $3d$ linear inequalities. In particular, $FP_d(k)$ is defined using variables with lower and upper bounds, together with a $(d-k+1) \times d$ matrix A, where A is totally unimodular. We conclude with applications of Fibonacci polytopes to study compositions of a positive integer.

CONVEX POLYTOPES

A subset of points $V \subseteq \boldsymbol{R}^d$ is called *convex* if for every $\boldsymbol{x}, \boldsymbol{y} \in V$, the line segment $\{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} : 0 \leq \lambda \leq 1\}$ is contained in V. For $V = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subseteq \boldsymbol{R}^d$, the *affine hull* of V is $\{\lambda_1\boldsymbol{x}_1 + \cdots + \lambda_n\boldsymbol{x}_n : \lambda_j \in \boldsymbol{R} \text{ and } \sum_{j=1}^n \lambda_j = 1\}$, and the *convex hull* of V, denoted by conv(V), is defined by conv(V)$= \{\lambda_1\boldsymbol{x}_1 + \cdots + \lambda_n\boldsymbol{x}_n : \lambda_j \in \boldsymbol{R}, \lambda_j \geq 0 \text{ and } \sum_{j=1}^n \lambda_j = 1\}$. The convex hull of a finite set of points in some $\boldsymbol{R}^d$, for $d \geq 1$, is called a *polytope*, and a polytope of dimension d is called a *d-polytope*. The intersection of finitely many closed halfspaces in some $\boldsymbol{R}^d$ is called a *polyhedron*. It is known that every polytope is the intersection of a finite set of closed halfspaces. Furthermore, P is a polytope if and only if P is a bounded polyhedron. Thus every polytope may be defined either as a convex hull of points, or as the intersection of halfspaces usually defined in terms of linear inequalities. For more information on polytopes and the above facts, see [4] or [9].

Let $P \subseteq \boldsymbol{R}^d$ be a polytope. A point $\boldsymbol{x} \in P$ is called a *vertex* of P if $\boldsymbol{y}, \boldsymbol{z} \in P$ and $\boldsymbol{x} = \lambda\boldsymbol{y} + (1-\lambda)\boldsymbol{z}$, where $0 < \lambda < 1$, implies that $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$. Two vertices $\boldsymbol{x} \neq \boldsymbol{y}$ of P are *adjacent* if every point on the segment $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}$, where $0 \leq \lambda \leq 1$, has a unique representation as a convex combination of vertices of P. The *vertex-edge graph of* P, denoted by $G(P)$, is the graph whose vertices represent vertices of P, and $G(P)$ contains edge $\{\boldsymbol{x}, \boldsymbol{y}\}$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent on P. Two polytopes P_1 and P_2 are called *combinatorially equivalent*, denoted by $P_1 \equiv P_2$, if $G(P_1)$ and $G(P_2)$ are isomorphic graphs.

Let Q_d denote the *d-cube* defined as the convex hull of the 2^d $\{0,1\}$-vectors having d entries. Notice that Q_d may also be defined via inequalities as $Q_d = \{\boldsymbol{x} \in \boldsymbol{R}^d : 0 \leq x_i \leq 1$, for all $i\}$. A *truncated d-cube*, denoted by $\overline{Q}_d$, is the convex hull of the set of $2^d - 1$ $\{0,1\}$-vectors having d entries excluding the vector with all ones. One can prove that for all $d \geq 2, \overline{Q}_d = \{\boldsymbol{x} \in \boldsymbol{R}^d : 0 \leq x_i \leq 1$, for all i, and $x_1 + \cdots + x_d \leq d-1\}$, details are omitted. Observe that if $d < k$, then $FP_d(k)$ is the d-cube Q_d, and if $d = k$, then $FP_d(k)$ is $\overline{Q}_d$ (e.g. $\overline{Q}_3 = FP_3(3)$, which is given in Figure 1.)

THE VERTICES OF FIBONACCI POLYTOPES

A linear inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leq a_0$ is called *valid* for a polytope P if it is satisfied by all $\boldsymbol{x} \in P$. A *face* of P is any set of the form $P \cap \{\boldsymbol{x} \in \boldsymbol{R}^d : \boldsymbol{a} \cdot \boldsymbol{x} = a_0\}$, where $\boldsymbol{a} \cdot \boldsymbol{x} \leq a_0$ is valid for

P. The dimension of a face is the dimension of its affine hull. Thus, a vertex of P is a face of dimension 0. If P is a d-polytope, then a face of dimension $d-1$ is called a *facet* of P. For example, $FP_3(3)$ has 7 facets. Let $V_d(k)$ be the set of $\{0,1\}$-vectors having d entries with no consecutive k ones, for $d \geq 1$ and $k \geq 2$. Then, by definition, $FP_d(k) = \text{conv}\{V_d(k)\}$.

Theorem 1: (a) *Every element in $V_d(k)$ is a vertex of $FP_d(k)$.*

(b) *The dimension of $FP_d(k)$ is d.*

(c) *For $2 \leq k \leq d$, the vertices of $FP_d(k)$ can be partitioned into k subsets $F_1, \cdots, F_k$, such that conv$(F_i) \equiv FP_{d-i}(k)$, for $i = 1, \ldots, k$.*

(d) *If a_d is the number of vertices of $FP_d(k)$, then a_d satisfies the recurrence relation $a_d = a_{d-1} + a_{d-2} + \cdots + a_{d-k}$, where $a_d = 2^d$, for $d < k$, and $a_d = 2^d - 1$, for $d = k$.*

Proof: (a) It is left as an exercise for the reader to show that no element of $V_d(k)$ can be expressed as a convex combination of other elements of $V_d(k)$.

(b) The proof of this is immediate since $V_d(k)$ contains the d unit vectors and $FP_d(k)$ is contained in $\boldsymbol{R}^d$.

(c) First, observe that the elements $\boldsymbol{x} \in V_d(k)$ ending in 0 are the same as the elements in $V_{d-1}(k)$ when $x_d = 0$ is removed from $\boldsymbol{x}$. If $F_1 = \{\boldsymbol{x} \in V_d(k) : x_d = 0\}$, then the face of $FP_d(k)$ defined by conv$(F_1) = \{\boldsymbol{x} \in FP_d(k) : x_d = 0\}$ is combinatorially equivalent to $FP_{d-1}(k)$. Similarly, if $F_2 = \{\boldsymbol{x} \in V_d(k) : x_{d-1} = 0 \text{ and } x_d = 1\}$, then conv$(F_2) \equiv FP_{d-2}(k)$. Repeating this we will obtain $F_k = \{\boldsymbol{x} \in V_d(k) : x_{d-k} = 0 \text{ and } x_d = x_{d-1} = \cdots = x_{d-k+1} = 1\}$, which satisfies conv$(F_k) \equiv FP_{d-k}(k)$. Since every element of $V_d(k)$ ends in exactly one of $0, 01, 011, \ldots,$ or $01\ldots1$, we have the desired partition. Note that when $d = k$, conv(F_k) is the 0-dimensional face given by the vertex $01\ldots1$.

(d) This follows from (c) and the fact that when $d < k$, all vertices of Q_d are elements in $V_d(k)$, and for $d = k$ all vertices of $\overline{Q}_d$ are elements in $V_d(k)$. □

To illustrate the above theorem, we first examine the family of polytopes $FP_d(8)$ and then discuss $FP_4(3)$. The terms $a_1, \ldots, a_{10}$ given below count the vertices of $FP_d(8)$ for $d = 1, 2, \ldots, 10$. Notice that both a_9 and a_{10} are the sum of the previous 8 terms. The term a_{25} which counts the vertices of $FP_{25}(8)$ is also given.

d	1	2	3	4	5	6	7	8	9	10	...	25
a_d	2	4	8	16	32	64	128	255	509	1,016	...	32,316,160

Next we invite the reader to construct the graph $G(FP_4(3))$. This graph contains 13 vertices. Moreover, the vertices $V_4(3)$ of $FP_4(3)$ can be partitioned into $F_1 = \{\boldsymbol{x} \in V_4(3) : x_4 = 0\}, F_2 = \{\boldsymbol{x} \in V_4(3) : x_3 = 0 \text{ and } x_4 = 1\}$ and $F_3 = \{\boldsymbol{x} \in V_4(3) : x_2 = 0, x_3 = 1 \text{ and } x_4 = 1\}$, where conv$(F_1) \equiv \overline{Q}_3$, conv$(F_2) \equiv Q_2$, and conv$(F_3) \equiv Q_1$. Additional adjacencies can be checked by using either the definition or the property that if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ in only one coordinate, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent on $FP_d(k)$. This is a well known characterization for adjacency on Q_d, and gives a sufficient (but not necessary) condition for adjacency on $FP_d(k)$.

THE FACETS OF FIBONACCI POLYTOPES

Finding a set of linear inequalities that define a polytope with $\{0,1\}$-valued vertices can sometimes be a hard problem because an exponentially large number of inequalities may be necessary. This happens, for example, with the famous traveling salesman polytope (e.g., see [2] or [6]). On the other hand, many polytopes require a relatively small set of inequalities in their description. Hence, both the number of vertices and the number of facets of a polytope are important parameters throughout the literature on convex polytopes.

A matrix A is called *totally unimodular* if each subdeterminant of A is 0 or ± 1. Surprisingly, this is a very important property with respect to describing a polytope with integer valued extreme points using linear inequalities. This is because polytopes described by a totally unimodular matrix usually require a relatively small number of inequalities in their description. For it is known that if an $m \times d$ matrix A is totally unimodular, then for all integral vectors $\boldsymbol{a}, \boldsymbol{b}$, with m entries, and all integral vectors $\boldsymbol{l}, \boldsymbol{u}$ with d entries, the polyhedron $\{\boldsymbol{x} \in \boldsymbol{R}^d : l \leq \boldsymbol{x} \leq \boldsymbol{u} \text{ and } \boldsymbol{a} \leq A\boldsymbol{x} \leq \boldsymbol{b}\}$ has only integral vertices (see [8]). Now let $A_d(k)$ be the $(d-k+1) \times d$ matrix where row i has k consecutive ones in columns i to $i+k-1$, and zeros elsewhere, for $i = 1, 2, \ldots, d-k+1$, and let $\mathbf{1}$ be the $(d-k+1) \times 1$ column vector consisting of all entries equal to 1. For example, $A_5(3)$ is given below. It is easy to check that $A_5(3)$ is totally unimodular.

$$A_5(3) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Lemma: *For all k and d satisfying $2 \leq k \leq d$, the matrix $A_d(k)$ is totally unimodular.*

Proof: A $\{0,1\}$-matrix A is called an *interval matrix* if in each column, the 1's appear consecutively. It is known that interval matrices are totally unimodular (see [8].) Clearly, $A_d(k)$ is an interval matrix, and hence, it is totally unimodular. $\square$

Theorem 2: *If $2 \leq k \leq d$, then*

$$FP_d(k) = \{\boldsymbol{x} \in \boldsymbol{R}^d : 0 \leq x_i \leq 1, \textit{for all } i, \textit{ and } A_d(k)\boldsymbol{x} \leq (k-1)\mathbf{1}\}.$$

Proof: Let $P = \{\boldsymbol{x} \in \boldsymbol{R}^d : 0 \leq x_i \leq 1, \text{ for all } i; \text{ and } A_d(k)\boldsymbol{x} \leq (k-1)\mathbf{1}\}$. We first show that $FP_d(k) \subseteq P$. Let $\boldsymbol{y} \in F_d(k)$. Since $FP_d(k)$ is the convex hull of $V_d(k)$, there exists extreme points $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^p$ and $\lambda_j \geq 0$ such that $\boldsymbol{y} = \sum_{j=1}^{p} \lambda_j \boldsymbol{x}^j$ and $\sum_{j=1}^{p} \lambda_j = 1$. Since no $\boldsymbol{x}^j$ has k or more consecutive ones, the $\boldsymbol{x}^j$ must all satisfy $x_i^j + x_{i+1}^j + \cdots + x_{i+k-1}^j \leq k-1$, for $j = 1, \ldots, p$, and $i = 1, \ldots, d-k+1$. Therefore,

$$\begin{aligned} y_i + y_{i+1} + \cdots + y_{i+k-1} &= \sum_{j=1}^{p} \lambda_j x_i^j + \sum_{j=1}^{p} \lambda_j x_{i+1}^j + \cdots + \sum_{j=1}^{p} \lambda_j x_{i+k-1}^j \\ &= \sum_{j=1}^{p} \lambda_j (x_i^j + x_{i+1}^j + \cdots + x_{i+k-1}^j) \\ &\leq \sum_{j=1}^{p} \lambda_j (k-1) = k-1. \end{aligned}$$

Since $y_i \geq 0$, for all i, the given inequalities are all valid for $FP_d(k)$, and hence $FP_d(k) \subseteq P$.

Suppose that, to obtain a contradiction, P is not contained in $FP_d(k)$. Then there exists a point $\boldsymbol{x} \in P$ such that $\boldsymbol{x} \notin F_d(k)$. Since P is a polytope, it is the convex hull of some set of vertices, say S. Moreover, $A_d(k)$ totally unimodular, implies that the vertices of P must be integer valued, and the constraints $0 \le x_i \le 1$, for all i, implies that these vertices must be $\{0,1\}$-vectors. Thus $\boldsymbol{x} = \sum_{j=1}^{p} \lambda_j \boldsymbol{z}^j$, where $\boldsymbol{z}^j \in S$ and $\sum_{j=1}^{p} \lambda_j = 1$. Since $\boldsymbol{x} \notin F_d(k)$, there must be some , say $\boldsymbol{z}^h$, such that $\boldsymbol{z}^h \notin V_d(k)$. However, any $\{0,1\}$-vector that is not in $V_d(k)$ must have k or more consecutive ones. Thus, $\boldsymbol{z}^h$ has k or more consecutive ones. But this is a contradiction since $\boldsymbol{z}^h \in P$. $\square$

A system of inequalities and equations defining a polytope is called *minimal* if no inequality can be made into an equation without reducing the size of the solution set, and no inequality or equation can be omitted without enlarging the solution set. In a minimal defining system each inequality induces a distinct facet and each facet corresponds to a distinct inequality (see [2]). Notice that the inequalities of Theorem 2 are minimal for $3 \le k \le d$. For if we omit any inequality, we can find $\boldsymbol{x} \notin FP_d(k)$, satisfying the remaining inequalities. To demonstrate, consider $FP_5(3)$. If we omit $x_1+x_2+x_3 \le 2$, then $\boldsymbol{x} = 11100$ satisfies all remaining inequalities, but $\boldsymbol{x} \notin FP_5(3)$. Similarly, $\boldsymbol{y} = 20000$ satisfies all inequalities, except $x_1 \le 1$, but $\boldsymbol{y} \notin FP_5(3)$, and $\boldsymbol{z} = (-1)0000$ satisfies all inequalities, except $0 \le x_1$, where $\boldsymbol{z} \notin FP_5(3)$. Moreover, no inequality of Theorem 2 can be made into an equation without reducing the solution set (e.g. changing $x_1 + x_2 + x_3 \le 2$ into an equation "cuts off " $10000 \in FP_5(3)$.) These examples can be generalized to prove that the inequalities of Theorem 2 all induce facets and leads to Theorem 3.

Theorem 3: *For $k = 2$, the number of facets of $FP_d(k)$ is $2d - k + 1$, and for $3 \le k \le d$, the number of facets is $3d - k + 1$.*

Proof: For $3 \le k \le d$, we have demonstrated that the following facets are both necessary and sufficient for $FP_d(k) : \{\boldsymbol{x} \in FP_d(k) : x_i = 0\}$, for $i = 1, \dots, d, \{\boldsymbol{x} \in FP_d(k) : x_i = 1\}$, for $i = 1, \dots, d$, and $\{\boldsymbol{x} \in FP_d(k) : x_i + x_{i+1} + \cdots + x_{i+k-1} = k - 1\}$, for $i = 1, \dots, d - k + 1$. Hence, the total number of facets is $3d - k + 1$.

When $k = 2$, the inequalities $x_i + x_{i+1} + \cdots + x_{i+k-1} \le k - 1$, imply $x_i \le 1$. So $x_i \le 1$ is unnecessary, and the number of facets is $2d - k + 1$. $\square$

RELATED COMBINATORIAL PROBLEMS

There are several possible combinatorial interpretations of the vertices of $FP_d(k)$. Here we use the notion of a *composition* of a positive integer d, which is a representation of d as a sum of positive integer summands where the order is relevant. Some distinct compositions of 5 using summands 1, 2 and 3 are:

$$1+1+1+1+1 \quad 1+1+2+1 \quad 1+2+1+1 \quad 2+1+1+1 \quad 2+2+1 \quad 3+1+1.$$

Each composition of 5 using summands 1, 2 and 3 may be represented as a unique vertex on $FP_4(3)$. The connection is to use a coordinate for each of the 4 possible addition signs where a 1 indicates that an addition has been executed. So $1+1+1+1+1$ corresponds to the vertex (0,0,0,0). We think of $1+1+2+1$ as $1+1+(1+1)+1$ which corresponds to (0,0,1,0). The composition $3+1+1$ corresponds to (1,1,0,0). Since $FP_4(3)$ has 13 vertices, we may deduce that there are 13 compositions of 5 using summands 1, 2 and 3. This leads us to Theorem 4; the formal proof is left for the reader.

Theorem 4: (a) *For $2 \le k \le d$, there is a one-to-one correspondence between the compositions of $d + 1$ using summands $1, 2, \dots, k$ and the vertices of $FP_d(k)$.*

(b) *If a_d is the number of compositions of $d+1$ into positive integer summands less than k, then a_d satisfies the recurrence relation $a_d = a_{d-1} + a_{d-2} + \cdots + a_{d-k}$, where $a_d = 2^d$, for $d < k$, and $a_d = 2^d - 1$, for $d = k$.*

Theorem 4 allows us to study compositions using $FP_d(k)$ in two ways. Part (b) provides a recurrence relation which can be used to count compositions with bounded summands. Others who have derived methods for counting certain constrained compositions include [1] and [3].

We can also use $FP_d(k)$ to help find certain compositions arising in practical applications. For suppose that we must form clusters among say 26 computer records in a file, possibly accessible to a relational database. The records are stored as rows in a table and are identified as $r_1, r_2, \ldots, r_{26}$ (e.g, see Chapter 2 of [7]). Each cluster can have between 1 and 8 records, and the clusters must contain records with consecutive labels. For example, $\{r_1, r_2, \ldots, r_8\}$, $\{r_9, r_{10}, \ldots, r_{16}\}$, $\{r_{17}, r_{18}, \ldots, r_{24}\}$, $\{r_{25}, r_{26}\}$ is one such cluster. Clustering records allows an operating system to transfer a "block" of data instead of just a single record. Due to advantages arising from efficient memory retrieval, there is a benefit denoted by c_i, derived from placing r_i and r_{i+1} in the same cluster, for $i = 1, 2, \ldots, 25$. We assume that there are no other significant benefits. Now, if $\boldsymbol{c} = (c_i)$ is the vector given below, how should the clusters be formed so that the total benefit is maximum?

$$\begin{aligned} c = (&598\ 294\ 211\ 173\ 247\ 371\ 259\ 738\ 794\ 211\ 813\ 516\ 590 \\ &350\ 315\ 51\ 856\ 25\ 249\ 859\ 792\ 579\ 593\ 798\ 113) \end{aligned}$$

The clustering problem asks for a composition of 26 whose summands are at most 8, and also maximizes the total benefit. By Theorem 4 and a calculation given above, we know that there are 32,316,160 possible compositions. However, the best composition is found by solving *maximize* $\{\boldsymbol{c} \cdot \boldsymbol{x} : \boldsymbol{x} \in FP_{25}(8)\}$. This requires using 25 variables x_i satisfying $0 \le x_i \le 1$, and 18 inequalities given by $x_i + x_{i+1} + \cdots + x_{i+7} \le 7$, for $i = 1, \ldots, 18$. So a total of 68 inequalities are needed to describe $FP_{25}(8)$.

Using $FP_{25}(8)$ the problem can be solved with an algorithm such as the simplex method. In terms of the polytope $FP_{25}(8)$, the simplex method would start at the vertex $\boldsymbol{x} = 00\ldots0$, and move to an adjacent vertex that increases the total benefit as large as possible. The algorithm repeats this process, moving along an edge of $FP_{25}(8)$ each iteration, until a vertex corresponding to an optimal solution has been found. Hence the algorithm creates a walk along vertices of $FP_{25}(8)$. Linear programming software such as Solver, a subroutine of Microsoft Excel, can be used to implement the simplex method. Using Solver with an IBM 300GL PC, the problem is solved in less than 1 second. The optimal solution $\boldsymbol{x}^*$ is given below.

$$x^* = (1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1).$$

Notice that $\boldsymbol{x}^*$ corresponds to the composition $4+6+8+8$. So we place the first 4 records in cluster 1, the next 6 in cluster 2, the next 8 in cluster 3, and the last 8 in cluster 4. The total benefit obtained is $\boldsymbol{c} \cdot \boldsymbol{x}^* = 10,986$.

The inequalities defining $FP_d(k)$ may also be supplemented to model additional constraints. The composition $4+6+8+8$ is a clustering into 4 parts, but suppose that we desired the best clustering with 5 parts. In general, to find a composition of $d+1$, with summands at most k, into exactly p parts, we use the inequalities for $FP_d(k)$, together with the equation $\sum_{i=1}^{d} x_i = d+1-p$. The coefficients of this equation are all ones. Using properties of determinants, we could show that adding a row of ones to the matrix $A_d(k)$ used in Theorem

2 will yield another totally unimodular matrix. So this important property is preserved in a description of the new polytope. Adding the additional constraint to our example and resolving, we obtain the composition $4+6+6+2+8$, which gives total benefit $\boldsymbol{c}\cdot\boldsymbol{x}^* = 10,935$. We conclude by inviting the reader to find other constraints related to compositions that can be used in conjunction with $FP_d(k)$ which will result in a totally unimodular matrix.

ACKNOWLEDGMENT

The author would like to thank Steven Cosares and Julius Dichter for their valuable input and an anonymous referee for many useful suggestions.

REFERENCES

[1] P. Chinn and S. Heubach. "Integer Sequences Related to Compositions without 2's." *Journal of Integer Sequences* **6** (2003).
[2] W. Cook, W. Cunningham, W. Pulleyblank, and A. Schrijver. *Combinatorial Optimization.* Wiley, New York 1998.
[3] R.P. Grimaldi. "Compositions without the summand 1." *Congr. Numer.* **152** (2001): 33-43.
[4] B. Grünbaum. *Convex Polytopes*, Second Addition. Springer-Verlag, New York 2003.
[5] R.J. Hendel. "Approaches to the formula for the n^{th} Fibonacci number." *College Mathematic Journal* **25** (1994): 139-142.
[6] G. Nemhauser and L. Wolsey. *Integer and Combinatorial Optimization.* Wiley, New York 1988.
[7] P. Rob and C. Coronel. *Database Systems: Design, Implementation and Management.* Boyd & Fraser, Danvers 1995.
[8] A. Schrijver. *Theory of Linear and Integer Programming.* Wiley, New York 1986.
[9] G. Ziegler. *Lectures on Polytopes.* Springer-Verlag, New York 1995.

AMS Classification Numbers: 52B05, 90C57

✠ ✠ ✠