

GENERALIZATIONS OF SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS

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1. INTRODUCTION

The generalized Fibonacci and Lucas numbers are defined by

$$U_n(p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n(p, q) = \alpha^n + \beta^n, \quad (1)$$

where $\alpha = \frac{1}{2}(p + \sqrt{p^2 - 4q})$ and $\beta = \frac{1}{2}(p - \sqrt{p^2 - 4q})$. Clearly, $U_n(p, q)$ and $V_n(p, q)$ are the usual Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ when $p = 1$ and $q = -1$.

Definition 1.1: Let $d \geq 0$. For any $n \geq 0$, we define

$$s_d(n; p, q; k) = \sum_{j_1 + j_2 + \dots + j_d = n} \prod_{i=1}^d U_{k \cdot j_i}(p, q).$$

For the Fibonacci numbers, Zhang [2] found the following identities:

$$s_2(n; 1, -1; 1) = \frac{1}{5}((n-1)F_n + 2nF_{n-1}), \quad n \geq 1, \quad (2)$$

$$s_3(n; 1, -1; 1) = \frac{1}{50}((5n^2 - 9n - 2)F_{n-1} + (5n^2 - 3n - 2)F_{n-2}), \quad n \geq 2, \quad (3)$$

and when $n \geq 3$,

$$s_4(n; 1, -1; 1) = \frac{1}{150}((4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3}). \quad (4)$$

Recently, Zhao and Wang [3] extended these identities to the case of $\{U_n(p, q)\}$ and $\{V_n(p, q)\}$; for $n \geq 1$

$$s_2(n; p, q; k) = \frac{U_k(p, q)}{V_k^2(p, q) - 4q^k} ((n-1)U_{nk}(p, q)V_k(p, q) - 2nq^k U_{(n-1)k}(p, q)), \quad (5)$$

for $n \geq 2$,

$$\begin{aligned} s_3(n; p, q; k) = & \frac{U_k^2(p, q)}{2(V_k^2(p, q) - 4q^k)^2} ((n-1)(n-2)V_k^2(p, q)U_{nk}(p, q) \\ & - q^k V_k(p, q)(4n^2 - 6n - 4)U_{(n-1)k}(p, q) \\ & + (4n^2 - 28n + 28(n-3)V_k(p, q) + 80)U_{(n-2)k}(p, q)), \end{aligned} \quad (6)$$

and when $n \geq 3$,

$$\begin{aligned}
 s_4(n; p, q; k) &= \frac{U_k^3(p, q)}{6(V_k^2(p, q) - 4q^k)^3} (V_k^3(p, q)(n - 1)(n - 2)(n - 3)U_{nk}(p, q) \\
 &\quad - 6q^k V_k^2(p, q)(n - 2)(n - 3)(n + 1)U_{(n-1)k}(p, q) \\
 &\quad + 12q^{2k} V_k(p, q)(n - 3)(n^2 + n - 1)U_{(n-2)k}(p, q) \\
 &\quad - 8q^{3k} n(n^2 - 4)U_{(n-3)k}(p, q)).
 \end{aligned}
 \tag{7}$$

In this paper, we extend the above conclusions. We establish an identity for the case $s_d(n; p, q; k)$ for any $d \geq 2$.

2. MAIN RESULTS

We denote by $G_k(x; p, q)$ the generating function of $\{U_{n \cdot k}(p, q)\}$, that is, $G_k(x; p, q) = \sum_{n \geq 0} U_{n \cdot k}(p, q)x^n$, where k is a positive integer. Clearly, by Definition 1 and the geometric formula,

$$G_k(x; p, q) = \frac{xU_k(p, q)}{1 - V_k(p, q)x + q^k x^2}.$$

We define $F_k(x) = F_k(x; p, q) = \frac{G_k(x; p, q)}{x}$. Then

$$F_k(x) = \sum_{n \geq 1} U_{n \cdot k}(p, q)x^{n-1} = \frac{U_k(p, q)}{1 - V_k(p, q)x + q^k x^2}.
 \tag{8}$$

Definition 2.1: Let $a(0, d) = 4^d$ for any $d \geq 0$, and $a(\ell, 0) = 0$ for any $\ell \geq 1$. We define $a(\ell, d)$ for $\ell, d \geq 1$ by $a(\ell, d) = 4(\ell + 1) \cdot a(\ell, d - 1) + \ell \cdot a(\ell - 1, d - 1)$. Using this definition we quickly generate the numbers $a(\ell, d)$; the first few of these numbers are given in Table 1.

$d \setminus \ell$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	4	1	0	0	0	0	0
2	16	12	2	0	0	0	0
3	64	112	48	6	0	0	0
4	256	960	800	240	24	0	0
5	1024	7936	11520	6240	1440	120	0
6	4096	64512	154112	134400	53760	10080	720

Table 1: Values of $a(\ell, d)$ where $0 \leq \ell, d \leq 6$.

We can also use Definition 2.1 to find an explicit formula for $a(\ell, d)$.

Lemma 2.2: For any $\ell, d \geq 0$,

$$a(\ell, d) = 4^{d-\ell} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (\ell + 1 - j)^d.$$

Proof: By Definition 2.1 it is easy to see that the lemma holds for $\ell = 0$ or $d = 0$. Using induction on d and ℓ we get that

$$\begin{aligned} & 4(\ell + 1) \cdot a(\ell, d - 1) + \ell \cdot a(\ell - 1, d - 1) \\ &= (\ell + 1)4^{d-\ell} \sum_{j=0}^d (-1)^j \binom{\ell}{j} (\ell + 1 - j)^{d-1} + \ell \cdot 4^{d-\ell} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell-1}{j} (\ell - j)^{d-1} \\ &= 4^{d-\ell} \left[(\ell + 1) \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (\ell + 1 - j)^{d-1} + \ell \sum_{j=1}^{\ell} (-1)^{j-1} \binom{\ell-1}{j-1} (\ell + 1 - j)^{d-1} \right] \\ &= 4^{d-\ell} \left[(\ell + 1)^d + \sum_{j=1}^{\ell} (-1)^j \binom{\ell}{j} (\ell + 1 - j)^d \right] = a(\ell, d + 1), \end{aligned}$$

as requested.

Definition 2.3: Let $b(1, d) = (-2)^{d-1}$ for any $d \geq 1$, and $b(\ell, 1) = 0$ for any $\ell \geq 2$. We define $b(\ell, d)$ for $\ell, d \geq 2$ by $b(\ell, d) = b(\ell - 1, d - 1) - 2\ell \cdot b(\ell, d - 1)$.

Using this definition we quickly generate the numbers $b(\ell, d)$; the first few of these numbers are given in Table 2.

$d \backslash \ell$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	-2	1	0	0	0	0	0
3	4	-6	1	0	0	0	0
4	-8	28	-12	1	0	0	0
5	16	-120	100	-20	1	0	0
6	-32	496	-720	260	-30	1	0

Table 2: Values of $b(\ell, d)$ where $0 \leq \ell, d \leq 6$.

We can also use Definition 2.3 to find an explicit formula for the numbers $b(\ell, d)$.

Lemma 2.4: For any $\ell, d \geq 1$,

$$b(\ell, d) = \frac{(-1)^{d-1} 2^{d-\ell}}{(\ell - 1)!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell-1}{j} (j + 1)^{d-1}.$$

Proof: By Definition 2.3 it is easy to see that the lemma holds for $\ell = 1$ or $d = 1$. Using induction on d and ℓ we get that

$$\begin{aligned}
 & b(\ell - 1, d - 1) - 2\ell \cdot b(\ell, d - 1) \\
 &= \frac{(-1)^{d-2} 2^{d-\ell}}{(\ell - 2)!} \sum_{j=0}^{\ell-2} (-1)^j \binom{\ell - 2}{j} (j + 1)^{d-2} - \frac{2\ell(-1)^{d-2} 2^{d-\ell-1}}{(\ell - 1)!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-2} \\
 &= \frac{(-1)^{d-1} 2^{d-\ell}}{(\ell - 1)!} \left[\ell \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-2} - (\ell - 1) \sum_{j=0}^{\ell-2} (-1)^j \binom{\ell - 2}{j} (j + 1)^{d-2} \right] \\
 &= \frac{(-1)^{d-1} 2^{d-\ell}}{(\ell - 1)!} \left[(-1)^{d-1} \ell^{d-1} + \sum_{j=0}^{\ell-2} (-1)^j \left(\ell \binom{\ell - 1}{j} - (\ell - 1) \binom{\ell - 2}{j} \right) (j + 1)^{d-2} \right] \\
 &= \frac{(-1)^{d-1} 2^{d-\ell}}{(\ell - 1)!} \left[(-1)^{d-1} \ell^{d-1} + \sum_{j=0}^{\ell-2} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-1} \right] \\
 &= \frac{(-1)^{d-1} 2^{d-\ell}}{(\ell - 1)!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-1} = b(\ell, d),
 \end{aligned}$$

as requested.

Now we introduce a relation that plays the crucial role in the proof of the main result of this paper.

Proposition 2.5: *Let $d \geq 1$. The generating function $F_k(x; p, q)$ satisfies the following equation:*

$$\begin{aligned}
 & \sum_{j=0}^d \left[(4q^k)^{d-j} \left(\sum_{i=0}^j (-1)^i \binom{j}{i} (j + 1 - i)^d \right) \left(\frac{V_k^2(p, q) - 4q^k}{U_k(p, q)} \right)^j F_k^{j+1}(x; p, q) \right] \\
 &= \sum_{j=1}^d \left[\frac{(-1)^{d-1} (2q^k)^{d-j}}{(j - 1)!} \left(\sum_{i=0}^{j-1} (-1)^i \binom{j - 1}{i} (i + 1)^{d-1} \right) (V_k(p, q) - 2q^k x)^j F_k^{(j)}(x; p, q) \right],
 \end{aligned}$$

where $F_k^{(j)}(x; p, q)$ is the j^{th} derivative with respect to x of $F_k(x; p, q)$.

Proof: We define $A = \frac{V_k^2(p, q) - 4q^k}{U_k(p, q)}$ and $B = V_k(p, q) - 2q^k x$. Let us prove this theorem by induction on d . Noticing that

$$F_k^{(1)}(x; p, q) = \frac{(V_k(p, q) - 2q^k x)F_k(x; p, q)}{1 - V_k(p, q)x + q^k x^2}, \quad (9)$$

we get

$$4q^k F_k(x; p, q) + A \cdot F_k^2(x; p, q) = B \cdot F_k^{(1)}(x; p, q),$$

therefore, the theorem holds for $d = 1$. Now we suppose that the theorem holds for d , that is,

$$\begin{aligned} \sum_{j=0}^d a(j, d)q^{(d-j)k} \left(\frac{V_k^2(p, q) - 4q^k}{U_k(p, q)} \right)^j F_k^{j+1}(x; p, q) \\ = \sum_{j=1}^d b(j, d)q^{(d-j)k} (V_k(p, q) - 2q^k x)^j F_k^{(j)}(x; p, q). \end{aligned}$$

Therefore, derivative this equation with respect to x we have that

$$\begin{aligned} \sum_{j=0}^d (j+1)a(j, d)q^{(d-j)k} A^j F_k^j(x; p, q) F_k^{(1)}(x; p, q) \\ = \sum_{j=1}^d b(j, d)q^{(d-j)k} B^j F_k^{(j+1)}(x; p, q) - \sum_{j=1}^d 2jq^k b(j, d)q^{(d-j)k} B^{j-1} F_k^{(j)}(x; p, q). \end{aligned}$$

If multiplying by B and using Equation 9 then we get that

$$\begin{aligned} \sum_{j=0}^d (j+1)a(j, d)q^{(d-j)k} A^{j+1} F_k^{j+2}(x; p, q) + \sum_{j=0}^d 4(j+1)a(j, d)q^{(d+1-j)k} A^j F_k^{j+1}(x; p, q) \\ = \sum_{j=2}^{d+1} b(j-1, d)q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q) - \sum_{j=1}^d 2jb(j, d)q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q), \end{aligned}$$

equivalently,

$$\begin{aligned} \sum_{j=0}^{d+1} (ja(j-1, d) + 4(j+1)a(j, d))q^{(d+1-j)k} A^j F_k^{j+1}(x; p, q) \\ = \sum_{j=1}^{d+1} (b(j-1, d) - 2jb(j, d))q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q), \end{aligned}$$

Therefore, using Definition 2.1 and Definition 2.3 we have that

$$\sum_{j=0}^{d+1} a(j, d+1)q^{(d+1-j)k} A^j F_k^{j+1}(x; p, q) = \sum_{j=1}^{d+1} b(j, d+1)q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q).$$

Hence, using Lemma 2.2 and Lemma 2.4 we get the desired result. \square

By the above proposition, we have the main result of this paper.

Theorem 2.6: *Let $d \geq 1$. For any $n \geq d$,*

$$\begin{aligned} & \sum_{j=0}^d \left[(4q^k)^{d-j} \left(\sum_{i=0}^j (-1)^i \binom{j}{i} (j+1-i)^d \right) \left(\frac{V_k^2(p, q) - 4q^k}{U_k(p, q)} \right)^j s_{j+1}(n+j-d; p, q; k) \right] \\ &= \sum_{j=1}^d \left[\frac{(-1)^{d-1} (2q^k)^{d-j}}{(j-1)!} \left(\sum_{i=0}^{j-1} (-1)^i \binom{j-1}{i} (i+1)^{d-1} \right) \right. \\ & \quad \left. \left(\sum_{s=0}^j v_{d,j,s}(n) U_{(n+j-d-s)k}(p, q) \binom{j}{s} \right) \right], \end{aligned}$$

where $v_{d,j,s}(n) = (-2q^k)^s V_k^{j-s}(p, q) \prod_{i=1}^j (n+j-d-s-i)$.

Proof: If comparing the coefficients of $x^{n-(d+1)}$ on both sides of Proposition 2.5 we get the desired result. \square

Theorem 2.6 provides a finite algorithm for finding $s_d(n; p, q; k)$ in terms of $U_{nk}(p, q)$ and $V_{nk}(p, q)$, since we have to consider all $s_j(n; p, q; k)$ for $j = 1, 2, \dots, d$. The algorithm has been implemented in Maple, and yields explicit results for $1 \leq d \leq 6$. Below we present several explicit calculations.

Corollary 2.7: (see Zhao and Wang [3, Equation 9]) *For any $n \geq 1$,*

$$s_2(n; p, q; k) = \frac{U_k(p, q)}{V_k^2(p, q) - 4q^k} \left((n-1)V_k(p, q)U_{nk}(p, q) - 2nq^k U_{(n-1)k}(p, q) \right).$$

Proof: Theorem 2.6 for $d = 2$ yields

$$\begin{aligned} & 4q^k s_1(n-1; p, q; k) + \frac{V_k^2(p, q) - 4q^k}{U_k(p, q)} s_2(n; p, q; k) \\ &= (n-1)V_k(p, q)U_{nk}(p, q) - 2(n-2)q^k U_{(n-1)k}(p, q). \end{aligned}$$

Using the fact that $s_1(n; p, q; k) = U_{nk}(p, q)$ we get the desired result. \square

Corollary 2.8: (see Zhao and Wang [3, Equation 10]) *For any $n \geq 2$,*

$$\begin{aligned} s_3(n; p, q; k) &= \frac{U_k^2(p, q)}{2(V_k^2(p, q) - 4q^k)^2} ((n-1)(n-2)V_k^2(p, q)U_{nk}(p, q) \\ &\quad - 2q^k(n-2)(2n+1)V_k(p, q)U_{(n-1)k}(p, q) \\ &\quad + 4q^{2k}(n-2)(n+2)U_{(n-2)k}(p, q)). \end{aligned}$$

Proof: Theorem 2.6 for $d = 3$ yields

$$\begin{aligned} &16q^{2k}s_1(n-2; p, q; k) + 12q^k \frac{V_k^2(p, q) - 4q^k}{U_k(p, q)} s_2(n-1; p, q; k) + \frac{2(V_k^2(p, q) - 4q^k)^2}{U_k^2(p, q)} s_3(n; p, q; k) \\ &= (n-1)(n-2)V_k^2(p, q)U_{nk}(p, q) - 2(n-2)(2n-5)q^k V_k(p, q)U_{(n-1)k}(p, q) \\ &\quad + 4q^{2k}(n-3)^2 U_{(n-2)k}(p, q). \end{aligned}$$

Using Corollary 2.7 with the fact that $s_1(n; p, q; k) = U_{nk}(p, q)$ we get the desired result. \square

Similarly, if applying Theorem 2.6 for d with using the formulas of $s_j(n; p, q; k)$ for $j = 1, 2, \dots, d-1$, then we get the following result (in the case $d = 4$ see [3, Equation 11]).

Corollary 2.9: *We have*

(i) *For any $n \geq 3$,*

$$\begin{aligned} &s_4(n; p, q; k) \\ &= \frac{U_k^3(p, q)}{6(V_k^2(p, q) - 4q^k)^3} (V_k^3(p, q)(n-1)(n-2)(n-3)U_{nk}(p, q) \\ &\quad - 6q^k V_k^2(p, q)(n-2)(n-3)(n+1)U_{(n-1)k}(p, q) \\ &\quad + 12q^{2k} V_k(p, q)(n-3)(n^2 + n - 1)U_{(n-2)k}(p, q) \\ &\quad - 8q^{3k} n(n^2 - 4)U_{(n-3)k}(p, q)). \end{aligned}$$

(ii) For any $n \geq 4$,

$$\begin{aligned}
 & s_5(n; p, q; k) \\
 &= \frac{U_k^4(p, q)}{4!(V_k^2(p, q) - 4q^k)^4} (V_k^4(p, q)(n-1)(n-2)(n-3)(n-4)U_{nk}(p, q) \\
 &\quad - 4q^k V_k^3(p, q)(n-2)(n-3)(n-4)(2n+3)U_{(n-1)k}(p, q) \\
 &\quad + 12q^{2k} V_k^2(p, q)(n-3)(n-4)(2n^2 + 4n - 1)U_{(n-2)k}(p, q) \\
 &\quad - 8q^{3k} V_k(p, q)(n-4)(2n+1)(2n^2 + 2n - 9)U_{(n-3)k}(p, q) \\
 &\quad + 16q^{4k}(n-3)(n-1)(n+1)(n+3)U_{(n-4)k}(p, q)).
 \end{aligned}$$

(iii) For any $n \geq 5$,

$$\begin{aligned}
 & s_6(n; p, q; k) \\
 &= \frac{U_k^5(p, q)}{5!(V_k^2(p, q) - 4q^k)^5} (V_k^5(p, q)(n-1)(n-2)(n-3)(n-4)(n-5)U_{nk}(p, q) \\
 &\quad - 10q^k V_k^4(p, q)(n-2)(n-3)(n-4)(n-5)(n+2)U_{(n-1)k}(p, q) \\
 &\quad + 20q^{2k} V_k^3(p, q)(n-3)(n-4)(n-5)(2n^2 + 6n + 1)U_{(n-2)k}(p, q) \\
 &\quad - 40q^{3k} V_k^2(p, q)(n-4)(n-5)(n+1)(2n^2 + 4n - 9)U_{(n-3)k}(p, q) \\
 &\quad + 80q^{4k} V_k(p, q)(n-5)(n^4 + 2n^3 - 10n^2 - 11n + 9)U_{(n-4)k}(p, q) \\
 &\quad - 32q^{5k} n(n-4)(n-2)(n+2)(n+4)U_{(n-5)k}(p, q)).
 \end{aligned}$$

From these results, it is very easy to obtain Equations 2-4. If $k = 1$ and $p = -q = 1$, then by using Corollary 2.9 together with the recurrence $F_n = F_{n-1} + F_{n-2}$ we arrive to

$$\begin{aligned}
 & \sum_{a+b+c+d+e=n} F_a F_b F_c F_d F_e \\
 &= \frac{1}{4! \cdot 5^4} (3(n-1)(8n^3 - 5n^2 - 27n + 50)F_n - 20n(5n^2 - 17)F_{n-1})
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{a+b+c+d+e+f=n} F_a F_b F_c F_d F_e F_f \\
 &= \frac{1}{5! \cdot 5^4} ((n-1)(5n^4 - 70n^3 - 65n^2 + 490n + 264)F_n + 2n(5n^4 + 5n^2 - 226)F_{n-1}).
 \end{aligned}$$

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