FRACTAL DIMENSION OF ARITHMETICAL STRUCTURES OF GENERALIZED BINOMIAL COEFFICIENTS MODULO A PRIME

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(Submitted July 2003-Final Revision June 2004)

ABSTRACT

Given a sequence (u_n) of positive integers generated by $u_1 = 1, u_2 = a, u_n = au_{n-1} + bu_{n-2} (n \geq 3)$, define the generalized factorial by $[n]! = u_1 u_2 \cdots u_n$ and the generalized binomial coefficient by $C(i,j) = [i+j]!/([i]![j]!)$. Assume that the prime p does not divide b. Let $r = \min\{n : p|u_n\}$. **Theorem 1 (Asymptotic abundance of residues):** $\#\{(i,j)|0 \leq i,j < rp^k \text{ and } C(i,j) \equiv \rho (\text{mod } p)\} \sim \frac{r(r+1)}{2(p-1)} \binom{p+1}{2}^k$ as $k \to \infty$ for $\rho = 1, \ldots, p-1$. **Theorem 2 (Fractal dimension):** Let $s_k = rp^k$. The Hausdorff dimension of $\cap_k \cup_{i,j<s_k} \{[i/s_k, (i+1)/s_k) \times [j/s_k, (j+1)/s_k) : p \not| C(i,j)\}$ is $\log \binom{p+1}{2} / \log p$.

1. INTRODUCTION

A classical theorem of E. Lucas [15] expresses the binomial coefficient $\binom{N}{m}$ modulo a prime p in terms of the binomial coefficients of the base-p digits of N and m: If $N = \sum N_j p^j$ and $m = \sum m_j p^j$ where $0 \leq N_j, m_j < p$, then

$$\binom{N}{m} \equiv \prod \binom{N_j}{m_j} \pmod{p}.$$

Alternatively, letting

$$B(m,n) := \binom{m+n}{m} = \frac{(m+n)!}{m!n!},$$

we have

$$B(m,n) \equiv B(m \div p, n \div p) B(m \bmod p, n \bmod p) \pmod{p}$$

where $m \div p$ is the integer quotient of m by p, and $m \bmod p$ is the remainder. As noted in [18], this implies that, modulo p, the matrix $[B(m,n) \bmod p]$ with $0 \leq m,n < p^k$ is equivalent to $\boldsymbol{B}^{\otimes k}$, the k-fold tensor (or, Kronecker) product of $\boldsymbol{B} = [B(i,j) \bmod p]$ where $0 \leq i,j < p$. Note that matrix indices start at index pair $(0,0)$. This is an algebraic and "square" representation of the oft-noted self-similarity structure of Pascal's "triangle"; see, e.g., [19], [2], [7], [8], [14], [22], and [1]. For example, if $p = 3$, then the matrix $[B(m,n) \bmod p]$ for $0 \leq m,n < 9$ is given as follows:

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1\\ 1&2&0&1&2&0&1&2&0\\ 1&0&0&1&0&0&1&0&0\\ 1&1&1&2&2&2&0&0&0\\ 1&2&0&2&1&0&0&0&0\\ 1&0&0&2&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&2&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} \equiv \begin{bmatrix} 1\boldsymbol{B} & 1\boldsymbol{B} & 1\boldsymbol{B}\\ 1\boldsymbol{B} & 2\boldsymbol{B} & 0\boldsymbol{B}\\ 1\boldsymbol{B} & 0\boldsymbol{B} & 0\boldsymbol{B} \end{bmatrix} \equiv \boldsymbol{B}\otimes\boldsymbol{B} \pmod p,$$

where

$$\boldsymbol{B} = \begin{bmatrix} 1&1&1\\ 1&2&3\\ 1&3&6 \end{bmatrix} \equiv \begin{bmatrix} 1&1&1\\ 1&2&0\\ 1&0&0 \end{bmatrix} \pmod p.$$

The nonzero residues of the matrix $\boldsymbol{B}^{\otimes k}$ may be associated with the subset B_k of $[0,1)\times[0,1)$ formed by taking the union of those squares $[m/p^k,(m+1)/p^k)\times[n/p^k,(n+1)/p^k)$ for which $p\not| B(m,n)$ $(0\le m,n<p^k)$. Then $B:=\cap B_k$ is the union of $N=p+(p-1)+\cdots+1=\binom{p+1}{2}$ self-similar sets. Its "self-similarity dimension" (see Mandelbrot [16], [17, p. 37]), also called the "box-counting dimension" [4, p. 20], is $D=\log N/\log(1/r)$ where $r=1/p$ is the scaling ratio. This result was noted by Wolfram [22] in 1984. Using a different geometric construction, Flath and Peele [5] solved the more difficult problem of determining that the Hausdorff dimension of B is also $\log\binom{p+1}{2}/\log p$. The *Hausdorff dimension* $\dim_H(B)$ of a subset B of $\mathbb{R}^2$ is defined as follows. See, e.g., [4, p. 22]. First, for $s\ge 0$, define the Hausdorff measure

$$\mathcal{H}^s(B)=\sup_{\delta>0}\inf_{\{U_i\}}\sum|U_i|^s$$

where $|U_i|$ is the diameter of U_i and the infimum is taken over all countable covers $\{U_i\}$ of B with every $|U_i|\le\delta$. Then

$$\dim_H(B)=\inf\{s:\mathcal{H}^s(B)=0\}=\sup\{s:\mathcal{H}^s(B)=\infty\}.$$

The purpose of this paper is to provide proofs of similar fractal dimension results and density results (previously announced in [11]) for a large class of generalized binomial coefficients. The matrix of generalized binomial coefficients modulo a prime turns out to be formed of basic building blocks arrayed in a pattern that results from superimposing binomial self-similarity upon a doubly periodic "tiling." The proof relates the enumeration of these building blocks to a Markov chain, and invokes Perron-Frobenius theory to obtain the box-counting-type fractal dimension result. The more challenging Hausdorff dimension result is achieved by employing the mass distribution principle of fractal geometry. Multifractal results have been published elsewhere [9].

2. GENERALIZED BINOMIAL COEFFICIENTS

Generalized binomial coefficients corresponding to a given sequence (u_n) are defined analogously to $B(m,n)$ by replacing $n!$ by the product of u_1 through u_n,

$$[n]! := \prod_{j=1}^{n} u_j,$$

and then defining

$$C(m,n) := \frac{[m+n]!}{[m]![n]!}$$

(assuming any zero factors in the numerator and denominator are first paired and then cancelled).

In this paper we assume that the sequence is defined by a second-order recurrence relation as follows:

$$u_0 = 0; u_1 = 1; u_n = au_{n-1} + bu_{n-2} \text{ for } n = 2, 3, 4, \ldots$$

where a and b are integers.

When $a = 2$ and $b = -1$, then $u_n = n$ and the generalized binomial coefficients become the ordinary binomial coefficients: $C(m,n) = B(m,n)$. When $a = 1 + q$ and $b = -q$, then $u_n = 1 + q + q^2 + \cdots + q^{n-1}$ and the generalized binomial coefficients are the Gauss q-binomial coefficients. When $a = 1$ and $b = 1$, then $u_n = F_n$, the n^{th} Fibonacci number, and the generalized binomial coefficients become the fibonomial coefficients.

3. WELLS'S THEOREM AND THE PATTERN OF THE RESIDUES

Wells [20] [21] has proved a generalization of the Lucas theorem for these generalized binomial coefficients. For the purposes of our fractal dimension calculations, we use one of the alternative versions given in [10]. To state it, we need to introduce the following definitions and notations.

Definition 1: Let r denote the rank of apparition of p; thus, $r := \min\{n \in \mathbb{N} : u_n \equiv 0 \pmod{p}\}$. Let t denote the (least) period of $\langle u_n \bmod p\rangle$, if it exists. Let $s := t/r$.

Notation: If $r < \infty$, then for each nonnegative integer n, let

$$\begin{aligned} n_0 &:= n \bmod r, \\ n' &:= n \div r, \\ n^* &:= n \bmod t, \\ n'' &:= n^* \div r = n' \bmod s. \end{aligned}$$

Definition 2: For $i, j \geq 0$ and for $0 \leq k, l < r$, let $A_{i,j}(k,l)$ denote the solution of the modulo-p recurrence relation

$$A_{i,j}(k,l) \equiv u_{ir+k+1} A_{i,j}(k,l-1) + bu_{jr+l-1} A_{i,j}(k-1,l)$$

for $0 \leq k, l < r$ together with the boundary conditions

$$A_{i,j}(k,-1) \equiv 0 \pmod{p} \quad \text{for} \quad 1 \leq k < r$$

and

$$A_{i,j}(-1,l) \equiv 0 \pmod p \quad \text{for} \quad 1 \le l < r$$

and

$$A_{i,j}(0,0) \equiv 1 \pmod p.$$

Definition 3: For $i, j \ge 0$ and $0 \le k, l < r$, define

$$H_{i,j}(k,l) := u_{r+1}^{rij} A_{i,j}(k,l).$$

As noted in [10], $H_{i,j} \equiv H_{i \bmod s, j \bmod s} \pmod p$, so $H_{m',n'}(m_0,n_0) \equiv H_{m'',n''}(m_0,n_0) \pmod p$. Also $H_{m'',n''}(m_0,n_0) \equiv 0 \pmod p$ if $m_0 + n_0 > r$.

Here is the generalization of Lucas's theorem from [10] that we shall use.

Proposition 1: If $p \not| b$, then, for $m, n \ge 0$,

$$C(m,n) \equiv B(m',n') H_{m'',n''}(m_0,n_0) \pmod p.$$

This result simplifies nicely when $s = 1$. Then $m'' = n'' = 0$, and $H_{0,0}(m_0,n_0) \equiv C(m_0,n_0) \pmod p$ for $0 \le m_0, n_0 < r$. Thus, in this case, as in the Pascal triangle case, the pattern of residues exhibits self-similarity upon scaling by p.

Corollary: *If $p \not| b$ and $s = 1$, then, for $m, n \ge 0$,*

$$C(m,n) \equiv B(m',n') C(m_0,n_0) \pmod p,$$

or, letting $\boldsymbol{B}$ denote the matrix $[B(i,j)]$ with $0 \le i, j < p$ and $\boldsymbol{C}^{(k)} = [C(m,n)]$ with $0 \le m, n < rp^k$, we have

$$\boldsymbol{C}^{(k)} \equiv \boldsymbol{B}^{\otimes k} \otimes \boldsymbol{C}^{(0)} \pmod p.$$

The following examples are borrowed from [10].

Example 1: q-binomial coefficients. Take $u_n = \sum_{k=0}^{n-1} q^k$ to obtain the q-binomial coefficients. For a numerical example, take $q = 2$ and $p = 5$. Then $u_1 = 1, u_2 = 3, u_3 = 7, u_4 = 15, u_5 = 31, \ldots$, whence $r = 4$, and

$$\boldsymbol{C}^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 7 & 15 \\ 1 & 7 & 35 & 155 \\ 1 & 15 & 155 & 1395 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \pmod 5,$$

so, for $k = 0, 1, 2, \ldots,$

$$\boldsymbol{C}^{(k+1)} \equiv \boldsymbol{B} \otimes \boldsymbol{C}^{(k)} \equiv \begin{bmatrix} 1\boldsymbol{C}^{(k)} & 1\boldsymbol{C}^{(k)} & 1\boldsymbol{C}^{(k)} & 1\boldsymbol{C}^{(k)} & 1\boldsymbol{C}^{(k)} \\ 1\boldsymbol{C}^{(k)} & 2\boldsymbol{C}^{(k)} & 3\boldsymbol{C}^{(k)} & 4\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} \\ 1\boldsymbol{C}^{(k)} & 3\boldsymbol{C}^{(k)} & 1\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} \\ 1\boldsymbol{C}^{(k)} & 4\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} \\ 1\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} & 0\boldsymbol{C}^{(k)} \end{bmatrix} \pmod 5.$$

Example 2: Fibonomial coefficients modulo p. Let $a = b = 1$ so that $u_n = F_n$, and consider the case $p = 3$. Then $r = 4$, $t = 8$, and $s = 2$. By Definition 3,

$$\boldsymbol{H}_{0,0} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \boldsymbol{H}_{0,1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix};$$

$$\boldsymbol{H}_{1,0} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \boldsymbol{H}_{1,1} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

The structure of the matrix of fibonomial coefficients modulo 3, in accordance with Proposition 1, is given in Table 1.

$$\begin{bmatrix}
1\boldsymbol{H}_{0,0} & 1\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & 1\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & 1\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & 1\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & \cdots \\
1\boldsymbol{H}_{1,0} & 2\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 1\boldsymbol{H}_{1,1} & 2\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 1\boldsymbol{H}_{1,0} & 2\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & \cdots \\
1\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 1\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & \cdots \\
1\boldsymbol{H}_{1,0} & 1\boldsymbol{H}_{1,1} & 1\boldsymbol{H}_{1,0} & 2\boldsymbol{H}_{1,1} & 2\boldsymbol{H}_{1,0} & 2\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & \cdots \\
1\boldsymbol{H}_{0,0} & 2\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 2\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & \cdots \\
1\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 2\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & \cdots \\
1\boldsymbol{H}_{0,0} & 1\boldsymbol{H}_{0,1} & 1\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & \cdots \\
1\boldsymbol{H}_{1,0} & 2\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & 0\boldsymbol{H}_{1,1} & 0\boldsymbol{H}_{1,0} & \cdots \\
1\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & 0\boldsymbol{H}_{0,1} & 0\boldsymbol{H}_{0,0} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots
\end{bmatrix}$$

Table 1. Submatrices of the fibonomial coefficients mod 3

Proposition 1 and the example show that the infinite matrix $[C(i,j) \bmod p]$ may be partitioned into $r \times r$ submatrices which form basic, natural "tiling units." The pattern of the residues is obtained by superimposing the self-similar array of binomial coefficients modulo p upon the doubly periodic "tiling" of the plane by "hidden" $r \times r$ $\boldsymbol{H}$ matrices. The binomial structure is self-similar upon scaling by the factor p. The $r \times r$ tiling structure has period s both horizontally and vertically, and so the period is t at the element level. When $s = 1$, there are $p - 1$ different nonzero $r \times r$ submatrices, one for each nonzero residue value of $B(m', n')$ mod p times $\boldsymbol{C}^{(0)}$. In the general case, there are also $s \cdot s$ different $H_{m'',n''}$-matrices. In fact, there are $(p-1)s^2$ different nonzero "tiles," by the following proposition of [10, p. 234].

Proposition 2: *Assume $p \not| b$. The number of different nonzero $r \times r$ submatrices of the infinite matrix $[C(i,j) \bmod p]$ is $(p-1)s^2$. Furthermore, the mapping $(\rho, \mu, \nu) \mapsto \rho\boldsymbol{H}_{\mu,\nu}$ is one to one from $\{1, \ldots, p-1\} \times \{0, \ldots, s-1\} \times \{0, \ldots, s-1\}$ into the set of $r \times r$ matrices* mod p.

In the case of the the fibonomial coefficients modulo 3, the matrix exhibited in Table 1 shows these seven submatrices:

$$1\boldsymbol{H}_{0,0}, 1\boldsymbol{H}_{0,1}, 1\boldsymbol{H}_{1,0}, 1\boldsymbol{H}_{1,1}, 2\boldsymbol{H}_{0,1}, 2\boldsymbol{H}_{1,0}, 2\boldsymbol{H}_{1,1}.$$

The places of the missing $2\boldsymbol{H}_{0,0}$ are farther out—at $(5,11),(11,5),(5,13),(13,5)\ldots$ in Table 1.

4. SCALING-UP RECURSION FORMULA

Define

$$C_{\alpha,\beta}(m,n) \equiv B(m',n')H_{\alpha+m'',\beta+n''}(m_0,n_0) \pmod p.$$

By Proposition 1, if $p \not| b$, then $C_{0,0}(m,n) \equiv C(m,n) \pmod p$.

Proposition 3: *Assume $p \not| b$. If $m = m_k p^{k-1} r + m^{(k)}$ and $n = n_k p^{k-1} r + n^{(k)}$ where $0 \le m^{(k)}, n^{(k)} < rp^{k-1}$, then*

$$C_{\alpha,\beta}(m,n) \equiv B(m_k,n_k)C_{\alpha+m_k,\beta+n_k}(m^{(k)},n^{(k)}) \pmod p.$$

Proof: Here $m' := m \div r = m_k p^{k-1} + (m^{(k)})'$, so, by Lucas's Theorem, $B(m',n') \equiv B(m_k,n_k)B(m^{(k)\prime},n^{(k)\prime}) \pmod p$. Also $m'' := m' \bmod s \equiv m_k + m^{(k)\prime} \pmod s$, because $p^{k-1} \equiv 1 \pmod s$, a consequence of $s|p-1$ ([10, p. 229]), so by s-periodicity, $H_{\alpha+m'',\beta+n''} \equiv H_{\alpha+m_k+m^{(k)\prime},\beta+n_k+n^{(k)\prime}} \pmod p$. Invoke the definitions of $C_{\alpha,\beta}(m,n)$ and $C_{\alpha+m_k,\beta+n_k}(m^{(k)},n^{(k)})$ to complete the proof. □

5. ASYMPTOTIC ABUNDANCE OF RESIDUES

Define the matrices

$$\mathbb{C}^{(k)}_{\alpha,\beta} := [C_{\alpha,\beta}(m,n)] \qquad (0 \le m,n < rp^k),$$

and let

$$f^{(k)}_{\alpha,\beta}(\rho,\mu,\nu) := \#\{(i,j) : 0 \le i,j < p^k,$$

$$C_{\alpha,\beta}(ir+i_0, jr+j_0) \equiv \rho H_{\mu,\nu}(i_0,j_0) \pmod p \text{ for } 0 \le i_0, j_0 < r\}$$

and

$$f^{(k)}(\rho,\mu,\nu) := f^{(k)}_{0,0}(\rho,\mu,\nu).$$

The quantity $f^{(k)}(\rho,\mu,\nu)$ is our focus for now. It is the number of $\rho\boldsymbol{H}_{\mu\nu}$ tiles in the initial $rp^k \times rp^k$ square of $C(i,j)$ values.

Lemma 1:

$$f^{(k)}_{\alpha,\beta}(\rho,\mu,\nu) = f^{(k)}(\rho,\mu-\alpha,\nu-\beta).$$

Proof: Use the definitions of $f^{(k)}_{\alpha,\beta}$, $f^{(k)}$, and $C_{\alpha,\beta}$, together with the fact that the mapping $(\rho,\mu,\nu) \mapsto \rho\boldsymbol{H}_{\mu,\nu}$ is one to one on $\{1,\ldots,p-1\} \times \{0,\ldots,s-1\} \times \{0,\ldots,s-1\}$. □

Lemma 2: *For $1 \le \rho < p$ and $0 \le \mu,\nu < s$,*

$$f^{(k)}(\rho,\mu,\nu) = \sum_{\substack{0 \le i,j < p \\ i+j < p}} f^{(k-1)}(B(i,j)^{-1}\rho, \mu-i, \nu-j).$$

Here $B(i,j)^{-1}$ and $B(i,j)^{-1}\rho$ are calculated modulo p.

Proof: By Proposition 3, $\mathbb{C}^{(k)}_{0,0}$ is a $p \times p$ block matrix whose (i,j) block is $B(i,j)\mathbb{C}^{(k-1)}_{i,j}$ $(0 \le i,j < p)$. For $i+j<p$, the number of $\rho H_{\mu,\nu}$ tiles in the (i,j) block is $f^{(k-1)}_{i,j}(B(i,j)^{-1}\rho,\mu,\nu)$. (For $i+j \ge p$, we have $B(i,j) \equiv 0 \pmod p$.) The proof is completed by applying Lemma 1. $\square$

Let $\mathcal{J} := \{1,\dots,p-1\}\times\{0,\dots,s-1\}\times\{0,\dots,s-1\}$. For $I=(\rho,\mu,\nu)$, $J=(\tilde\rho,\tilde\mu,\tilde\nu)\in\mathcal{J}$, define

$$\begin{aligned} Q^I_J := \#\{(i,j) : 0 \le i,j < p, i+j<p, B(i,j)^{-1}\tilde\rho \equiv \rho \pmod p,\\ \tilde\mu - i \equiv \mu \pmod s, \quad \tilde\nu - j \equiv \nu \pmod s\}\\ = \#\{(i,j) : \quad 0 \le i,j < p, i+j<p, B(i,j)\rho \equiv \tilde\rho \pmod p,\\ \mu + i \equiv \tilde\mu \pmod s, \quad \nu + j \equiv \tilde\nu \pmod s\} \end{aligned}$$

and the matrix

$$\mathbb{Q} := [Q^I_J].$$

Lemma 3:

$$\sum_{J\in\mathcal{J}} Q^I_J = \frac{p(p+1)}{2} \quad \textit{and} \quad \sum_{I\in\mathcal{J}} Q^I_J = \frac{p(p+1)}{2}.$$

Proof:

$$\sum_{J\in\mathcal{J}} Q^I_J = \#\{(i,j) : 0 \le i,j<p, i+j<p\} = \frac{p(p+1)}{2}.$$

Likewise for $\sum_{I\in\mathcal{J}} Q^I_J$. $\square$

Accordingly, let

$$P^I_J = \frac{2}{p(p+1)} Q^I_J.$$

Then

$$\mathbb{P} := [P^I_J]$$

is a doubly stochastic matrix.

Lemma 4: *Regarding $f^{(k)}$ as a row vector with indices $I=(\rho,\mu,\nu)$, we have*

$$f^{(k)} = f^{(0)}\mathbb{Q}^k.$$

Proof: By Lemma 2,

$$f^{(k)}(\tilde\rho,\tilde\mu,\tilde\nu) = \sum_{\substack{0\le i,j<p\\ i+j<p}} f^{(k-1)}(B(i,j)^{-1}\tilde\rho, \tilde\mu - i, \tilde\nu - j)$$

$$= \sum_{(\rho,\mu,\nu)\in\mathcal{J}} f^{(k-1)}(\rho,\mu,\nu) Q^{(\rho,\mu,\nu)}_{(\tilde\rho,\tilde\mu,\tilde\nu)} = \sum_{I\in\mathcal{J}} f^{(k-1)}(I) Q^I_J,$$

so $f^{(k)} = f^{(k-1)}\mathbb{Q}$, whence $f^{(k)} = f^{(0)}\mathbb{Q}^k$. $\square$

Note: $f^{(0)}(\rho,\mu,\nu) = 1$ if $(\rho,\mu,\nu) = (1,0,0)$ and otherwise equals 0.

A nonnegative matrix is *primitive* if some power of it has all positive entries. A Markov chain is *regular* if its transition probability matrix is primitive.

Lemma 5: *Every entry of* $\mathbb{Q}^3$ *is positive, and so is every entry of* $\mathbb{P}^3$. *Consequently, the finite Markov chain having* $\mathbb{P}$ *as its transition matrix is regular.*

Proof: Note that $(\mathbb{Q}^3)^I_J \geq Q^I_K Q^K_L Q^L_J$ and $Q^{(\rho_1,\mu_1,\nu_1)}_{(\rho_2,\mu_2,\nu_2)} > 0$ if and only if there exists (i,j) with $0 \leq i,j < p$ and $i+j < p$ such that $B(i,j)\rho_1 \equiv \rho_2 \pmod p$, $\mu_1 + i \equiv \mu_2 \pmod s$, and $\nu_1 + j \equiv \nu_2 \pmod s$. Let $I = (\rho,\mu,\nu)$ and $J = (\tilde{\rho},\tilde{\mu},\tilde{\nu})$. For the first factor, let $i_1 = 1, j_1 = (\rho_1\rho^{-1} - 1) \bmod p, \rho_1 = \tilde{\rho}, \mu_1 = (\mu + i_1) \bmod s, \nu_1 = (\nu + j_1) \bmod s$, and $K = (\rho_1,\mu_1,\nu_1)$. Then $Q^I_K > 0$. Second, let $i_2 = (\tilde{\mu} - \mu_1) \bmod s, j_2 = 0, \rho_2 = \rho_1, \mu_2 = \tilde{\mu}, \nu_2 = \nu_1$, and $L = (\rho_2,\mu_2,\nu_2)$. Then $Q^K_L > 0$. Finally, let $i_3 = 0$ and $j_3 = (\tilde{\nu} - \nu_2) \bmod s$ to show $Q^L_J > 0$. Therefore, $(\mathbb{Q}^3)^I_J > 0$. $\square$

The example of the fibonomials modulo 3 shows that 3 is the least power that will work in this lemma.

Theorem 1: *For* $1 \leq \rho < p$ *and* $0 \leq \mu,\nu < s$,

$$f^{(n)}(\rho,\mu,\nu) \sim \frac{1}{(p-1)s^2}\left[\frac{p(p+1)}{2}\right]^n \quad \textit{as} \quad n \to \infty$$

and

$$\lim_{n\to\infty} \frac{\log f^{(n)}(\rho,\mu,\nu)}{\log p^n} = \frac{\log p(p+1)/2}{\log p}.$$

For $\rho = 0$ *we have that*

$$\sum_{\mu,\nu} f^{(n)}(\rho,\mu,\nu) = p^{2n} - [p(p+1)/2]^n = p^{2n}[1 - \{(p+1)/(2p)\}^n]$$

is the number of zero tiles in $\mathbb{C}^{(n)}$.

Proof: By Lemma 3, the stationary vector of the matrix $\mathbb{P}$ is $\frac{1}{(p-1)s^2}(1,\ldots,1)$. Since $\mathbb{P}$ is the transition matrix of a regular Markov chain, by Lemma 4, then $f^{(0)}\mathbb{P}^n$ converges to this stationary vector as $n \to \infty$, by Perron-Frobenius theory (see, e.g., [12, p. 125]). Finally, according to Kummer's theorem [13], the number of pairs (i,j) with $0 \leq i,j < p^n$ for which p does not divide the binomial coefficient $B(i,j)$ is the same as the number of pairs of n-digit p-ary numbers $(\sum_{k=0}^{n-1} i_k p^k, \sum_{k=0}^{n-1} j_k p^k)$ for which there are no carries when added in base-p arithmetic. This is the same as the number of digit pairs (i_k, j_k) with $i_k + j_k < p$, which is $[p(p+1)/2]^n$. Therefore the number of nonzero tiles $B(i,j)\rho H_{\mu,\nu}$ in $\mathbb{C}^{(n)}$ is precisely $[p(p+1)/2]^n$, and the number of zero tiles is $p^{2n} - [p(p+1)/2]^n$. $\square$

Corollary 1: *Let*

$$R^{(n)}(\rho) := \#\{(i,j) : 0 \leq i,j < rp^n, C(i,j) \equiv \rho \pmod p\},$$

the number of $C(i,j)$'s in the initial $rp^n \times rp^n$ square congruent to ρ modulo p. Then the asymptotic abundance of the residue ρ, where $1 \le \rho < p$, is given by

$$R^{(n)}(\rho) \sim \frac{r(r+1)}{2(p-1)} \left[\frac{p(p+1)}{2}\right]^n,$$

and so the logarithmic density, or box-counting dimension, of the set of generalized binomial coefficients that are congruent to ρ is

$$\lim_{n\to\infty} \frac{\log R^{(n)}(\rho)}{\log p^n} = \frac{\log[p(p+1)/2]}{\log p}.$$

Proof: Let

$$g(\rho,\mu,\nu) := \#\{(i,j) : 0 \le i,j < r, H_{\mu,\nu}(i,j) \equiv \rho \pmod p\},$$

the number of entries in the $r \times r$ matrix $H_{\mu,\nu}$ that are congruent to ρ modulo p. Then

$$R^{(n)}(\rho) = \sum_{1\le\tilde\rho<p} \sum_{0\le\mu,\nu<s} f^{(n)}(\rho\tilde\rho^{-1},\mu,\nu) g(\tilde\rho,\mu,\nu),$$

so

$$R^{(n)}(\rho) \sim \sum_{1\le\tilde\rho<p} \sum_{0\le\mu,\nu<s} \frac{[p(p+1)/2]^n}{(p-1)s^2} g(\tilde\rho,\mu,\nu)$$

$$= \frac{[p(p+1)/2]^n}{(p-1)s^2} \sum_{0\le\mu,\nu<s} \sum_{1\le\tilde\rho<p} g(\tilde\rho,\mu,\nu)$$

$$= \frac{[p(p+1)/2]^n}{(p-1)s^2} \cdot s^2 \cdot \frac{r(r+1)}{2}. \quad \square$$

6. HAUSDORFF DIMENSION OF $C(m,n) \mod p$

A "fractal set" corresponding to the pattern of all nonzero residues of the generalized binomial coefficients modulo a prime p is constructed as a subset of the square $[0,1) \times [0,1)$ by "tremas" as follows. We combine all the nonzero residues because the construction below for a fixed residue will not always yield nested sets. Flath and Peele [5] give an alternative, rescaled lattice construction.

For each k let $\mathcal{G}_k$ denote the class of sets

$$G^{(k)}_{m,n} = \bigcup_{\substack{0\le i,j<r\\ i+j<r}} \left[\frac{mr+i}{rp^k}, \frac{mr+i+1}{rp^k}\right) \times \left[\frac{nr+j}{rp^k}, \frac{nr+j+1}{rp^k}\right)$$

with $0 \le m,n < p^k$ and $p \not| B(m,n)$, and let G_k be their union. Proposition 1 and Lucas's theorem imply that $G^{(k)}_{m,n}$ is contained in some set in $\mathcal{G}_{k-1}$ and contains a finite number of disjoint sets of $\mathcal{G}_{k+1}$, and $G_{k+1} \subset G_k$. Accordingly our fractal set is

$$G := \bigcap_{k\in\mathbb{N}} G_k.$$

Figure 1 shows a density plot of a pre-fractal image of the fibonomial residues modulo 3; the nonwhite squares are components of G_3.

FIGURE 1. Fibonomial coefficients mod 3

Theorem 2: *If p is a prime that does not divide b, then the fractal set G constructed above has Hausdorff dimension*

$$\dim_H(G) = \frac{\log \binom{p+1}{2}}{\log p}.$$

Proof: The proof uses (1) the fact [3, p. 43] that

$$\dim_H G \le \dim_B G,$$

where $\dim_B G$ is the box-counting dimension of G, and (2) the mass distribution principle [3, p. 55]: if μ is a measure on a set F and for some s there are numbers $c > 0, \delta > 0$ such that

$$\mu(U) \le c|U|^s$$

for all sets U with $|U| \le \delta$ (where $|U|$ is the diameter of U), then the Hausdorff measure $\mathcal{H}^s(F) \ge \mu(F)/c$ and $s \le \dim_H(F)$. The box-counting dimension of G may be calculated [3, p. 41] by the formula

$$\dim_B G = \lim_{k\to\infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k},$$

where $N_\delta(G)$ is the smallest number of δ-mesh squares that intersect the set G, provided that the sequence (δ_k) decreases to zero and $\delta_{k+1} \ge \eta\delta_k$ for some positive constant η. Let us choose $\delta_k = 1/(rp^k)$ (and $\eta = 1/p$). Then

$$N_{\delta_k}(G) = \frac{r(r+1)}{2}\left[\frac{p(p+1)}{2}\right]^k,$$

so

$$\begin{aligned}\dim_B(G) &= \lim_{k\to\infty} \frac{\log N_{\delta_k}(G)}{-\log \delta_k} \\ &= \lim_{k\to\infty} \frac{\log\left(\frac{r(r+1)}{2}\left[\frac{p(p+1)}{2}\right]^k\right)}{-\log[1/(rp^k)]} \\ &= \lim_{k\to\infty} \frac{\log\frac{r(r+1)}{2} + k\log\frac{p(p+1)}{2}}{\log r + k\log p} \\ &= \frac{\log\binom{p+1}{2}}{\log p}.\end{aligned}$$

Now let μ be the "natural measure" defined by repeated subdivision [3, pp. 13–14] that assigns weight $\binom{p+1}{2}^{-k}$ to each set in $\mathcal{G}_k$ and weight 0 to the complement of G_k: At stage $k+1$, the weight of each $G^{(k)}_{m,n}$ is evenly divided among the $\binom{p+1}{2}$ sets in $\mathcal{G}_{k+1}$ contained therein. We shall see that there exist $c > 0$ and $\delta > 0$ such that

$$\mu(U) \le c|U|^d \quad \text{where} \quad d := \frac{\log\binom{p+1}{2}}{\log p}$$

for all sets U with diameter $|U| \le \delta$. Let $\delta \in (0,1)$. Suppose $|U| \le \delta$. Let k be the integer such that $1/p^{k+1} \le |U| < 1/p^k$. Note that then $1/p^k \le p|U|$ and U meets at most four of the sets in $\mathcal{G}_k$ (because U is contained in a square of side $|U|$ with sides parallel the coordinate axes, and this containing square can intersect no more than four $G^{(k)}_{m,n}$'s). Therefore,

$$\begin{aligned}\mu(U) &\le 4\frac{1}{\binom{p+1}{2}^k}\\ &= \frac{4}{(p^d)^k}\left[\text{because } d = \frac{\log\binom{p+1}{2}}{\log p}\right]\\ &= 4\left(\frac{1}{p^k}\right)^d\\ &\le 4(p|U|)^d,\end{aligned}$$

so $\mu(U) \le c|U|^d$ for all sets U with $|U| \le \delta$ where $c = 4p^d$. By the mass distribution principle, $d \le \dim_H G$. But from before, $\dim_B G = d$, and we know $\dim_H G \le \dim_B G$, so we must have $\dim_H G = d = \log\binom{p+1}{2}/\log p$. $\square$

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AMS Classification Numbers: 11B65, 11B39, 28A78

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