ON THE FIBONACCI NUMBERS OF TREES

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ABSTRACT

For a graph G, *Fibonacci Number* of G is defined as the number of subsets of $V(G)$ in which no two vertices are adjacent in G. In this paper, we first investigate the orderings of two classes of trees by their Fibonacci numbers. Using these orderings, we determine the unique tree with the second, and respectively the third smallest Fibonacci number among all trees with n vertices.

1. INTRODUCTION

Let $G = (V(G), E(G))$ denote a graph with $V(G)$ as the set of vertices and $E(G)$ as the set of edges. We denote, respectively, by $n(G)$ and $q(G)$ the number of vertices and the number of edges of G. All graphs considered here are finite and simple. Undefined notations and terminology will conform to those in [2].

For a graph G and $u \in V(G)$, we denote by $N_G(u)$ the set of all neighbors of u in G and by d_u the degree of the vertex u. Let G and H be two graphs. We denote by $G \cup H$ the disjoint union of G and H and by mH the disjoint union of m copies of H. Let C_n and P_n denote, respectively, the cycle and path with n vertices. By S_n we denote the star with n vertices and by $P_{n,m}$ the graph obtained from S_{n+1} and P_m by identifying the center of S_{n+1} with a vertex of degree 1 of P_m. By $S_{n,m}$ we denote the graph obtained from S_{n+2} and S_{m+1} by identifying a vertex of degree 1 of S_{n+2} with the center of S_{m+1}.

For a graph G, its *Fibonacci Number*, simply denoted by $f(G)$, is defined as the number of subsets of $V(G)$ in which no two vertices are adjacent in G, i.e., in graph-theoretical terminology, the number of independent sets of G, including the empty set. For example, for the graph $C_4 = v_1v_2v_3v_4$, all this kind of subsets of $V(C_4)$ are as follows: ϕ, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$, and so, $f(C_4) = 7$. The concept of the Fibonacci number for a graph was introduced in [5], and discussed later in [1]. This number for a molecular graph was extensively studied in a monograph [4]. There, the chemical use was demonstrated, and the number was called σ-index, or Merrifield and Simmons index. The authors of [3] gave its other properties and applications. There have been some literature studying the Fibonacci number, or σ-index of a graph, see [4,6] and the references therein for details.

Let F_n and L_n denote the n-th Fibonacci number and Lucas Number, respectively. It is not difficult to see that for $n \geq 1$ and $m \geq 3$, we have that $f(P_n) = F_{n+2}$ and $f(C_m) = L_m$. Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then the Binet form of F_n and L_n are $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ and $L_n = \alpha^n + \beta^n$ for all $n \geq 0$.

Let T be a tree, that is, T is a connected graph without any cycles. From [1,3,5], we can find that

Lemma 1: Let T be a tree. Then $F_{n+2} \leq f(T) \leq 2^{n-1}+1$ and $f(T) = F_{n+2}$ if and only if $T \cong P_n$ and $f(T) = 2^{n-1}+1$ if and only if $T \cong S_n$.

Let $v_1v_2v_3 \cdots v_n$ be a path, and let $P_{n,v_k,G}$ and T_{n_1,n_2,n_3} denote two graphs shown in Figures 1.

Figure 1. Graphs $P_{n,v_k,G}$ and T_{n_1,n_2,n_3}

The authors of [1] investigated the upper and lower bounds for the Fibonacci number of a maximal outer-planar graph. In this paper, we first investigate the orderings of two classes of trees $P_{n,v_k,G}$ and T_{n_1,n_2,n_3} by their Fibonacci numbers. Using these orderings, we determine the unique tree with the second, and respectively the third smallest Fibonacci number among all trees with n vertices. From [3] we know that these results may have potential use in combinatorial chemistry.

The following lemmas can be found from [3,5].

Lemma 2 ([3,5]): Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then $f(G) = \prod_{i=1}^{k} f(G_i)$.

Lemma 3 ([3,5]): For a graph G with $v \in V(G)$, we have

$$f(G) = f(G-v) + f(G-[v]),$$

where $[v] = N_G(v) \cup \{v\}$.

Lemma 4 ([3,5]): Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. If $V(G_1) = V(G_2)$ and $E(G_1) \subset E(G_2)$, then $f(G_1) > f(G_2)$.

2. ORDERINGS OF TWO CLASSES OF TREES BY FIBONACCI NUMBERS

Let H and H' be two graphs. Then $H \succeq H'$ means $f(H) \geq f(H')$ and $H \succ H'$ means $f(H) > f(H')$.

Theorem 1: Let $n = 4m+i$, $i \in \{1,2,3,4\}$ and $m \geq 2$. Then

$$P_{n,v_2,G} \succ P_{n,v_4,G} \succ \cdots \succ P_{n,v_{2m+2\rho},G} \succ P_{n,v_{2m+1},G} \succ \cdots \succ P_{n,v_3,G} \succ P_{n,v_1,G},$$

where $\rho = 0$ if $i = 1$ or 2 and $\rho = 1$ if $i = 3$ or 4.

Proof: Suppose that $f(G - v_k) = A$ and $f(G - [v_k]) = B$. Then by Lemmas 1, 2 and 3,

$$f(P_{n,v_k,G}) = AF_{k+1}F_{n-k+2} + BF_kF_{n-k+1}. \tag{1}$$

¿From the Binet form of F_n and L_n, by calculating we have

$$F_aF_b = \frac{1}{5}\left(L_{a+b} - (-1)^a L_{b-a}\right). \tag{2}$$

So, by (1) and (2) we get that

$$f(P_{n,v_k,G}) = AL_{n+3} + BL_{n+1} + (-1)^k L_{n-2k+1}(A-B). \tag{3}$$

Note that each independent set of $G-[v_k]$ is an independent set of $G-v_k$; however the other way around is nor true. So, $A > B$. Since $P_{n,v_k,G} \cong P_{n,v_{n-k+1},G}$, by (3) we have that $k \le (n+1)/2$ and

$$P_{n,v_2,G} \succ P_{n,v_4,G} \succ \cdots \succ P_{n,v_{2m},G} \succ P_{n,v_{2m+1},G} \succ \cdots \succ P_{n,v_3,G} \succ P_{n,v_1,G}$$

for $i \in \{1,2\}$ and

$$P_{n,v_2,G} \succ P_{n,v_4,G} \succ \cdots \succ P_{n,v_{2m+2},G} \succ P_{n,v_{2m+1},G} \succ \cdots \succ P_{n,v_3,G} \succ P_{n,v_1,G}$$

for $i \in \{3,4\}$. This completes the proof. □

Let $G \cong P_2$ or $G \cong P_3$. Then from Theorem 1, we have

$$T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4}$$

and

$$T_{2,1,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}.$$

Note that $T_{2,1,n-4} \cong T_{1,2,n-4}$. So, it follows that $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$.

For $3 \le n_1 \le n_2 \le n_3$ and T_{n_1,n_2,n_3}, we can obtain the followings:

(i) $T_{3,1,n-5} \succ T_{3,a,n-a-4} \succ T_{3,2,n-6}$ for $a \ge 3$,

(ii) $T_{4,1,n-6} \succ T_{4,a,n-a-5} \succ T_{4,2,n-7}$ for $a \ge 3$,

(iii) $T_{b,1,n-b-2} \succ T_{b,a,n-a-b-1} \succ T_{b,2,n-b-3}$ for $a \ge 3$ and $b \ge 5$.

¿From (i) to (iii), one can see that for $(n_1,n_2) \notin \{(1,1),(1,3),(2,2),(2,4)\}$.

$$T_{1,1,n-3} \succ T_{1,3,n-5} \succ T_{n_1,n_2,n_3} \succ T_{2,4,n-7} \succ T_{2,2,n-5}.$$

Furthermore, we have

Theorem 2: Let $n_1+n_2+n_3 = n-1$ and $n_1 \ge n_2 \ge n_3$. Then

(i) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m-1,n-2m-1} \succ T_{1,2m-2,n-2m} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m-2,n-2m-1} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m+1$,

(ii) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m-1,n-2m-1} \succ T_{1,2m,n-2m-2} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m-2,n-2m-1} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m+2$,

(iii) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m-1,n-2m-1} \succ T_{1,2m,n-2m-2} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m,n-2m-3} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m+3$,

(iv) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ \cdots \succ T_{1,2m+1,n-2m-3} \succ T_{1,2m,n-2m-2} \succ \cdots \succ T_{1,4,n-6} \succ T_{1,2,n-4} \succ T_{2,3,n-6} \succ \cdots \succ T_{2,2m-1,n-2m-2} \succ T_{2,2m,n-2m-3} \succ \cdots \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $n = 4m+4$,

(v) $T_{1,1,n-3} \succ T_{1,3,n-5} \succ T_{n_1,n_2,n_3} \succ T_{2,4,n-7} \succ T_{2,2,n-5}$ for $(n_1, n_2) \notin \{(1,1), (1,3), (2,2), (2,4)\}$. □

3. SMALLER FIBONACCI NUMBERS OF TREES

Suppose that $Q_{r,G}$ is the graph shown in Figure 2.

Figure 2. Graph $Q_{r,G}$

Lemma 5: Let $r \geq 1$ and G be a tree with m vertices. For graph $Q_{r,G}$, we have

$$Q_{r,G} \succeq P_{r,m}$$

and the equality holds if and only if $Q_{r,G} \cong P_{r,m}$.

Proof: We prove the lemma by induction on r. Clearly, the lemma is true if $r = 1$.

Suppose that the lemma holds for $r = k-1 \geq 1$. When $r = k$, by Lemmas 2 and 3 we have

$$f(Q_{k,G}) = f(Q_{k-1,G}) + 2^{k-1} f(G - v) \tag{4}$$

and

$$f(P_{k,m}) = f(P_{k-1,m}) + 2^{k-1} f(P_{m-1}). \tag{5}$$

By the induction hypothesis, $f(Q_{k-1,G}) \geq f(P_{k-1,m})$ and the equality holds if and only if $Q_{k-1,G} \cong P_{k-1,m}$. On the other hand, we may assume that $G - v = \bigcup_i^l H_i$ such that each H_i is a tree and $\sum_{i=1}^l n(H_i) = n(G-v) = m-1$. By Lemmas 1, 2 and 4, $f(G-v) \geq \prod_{i=1}^l f_{n(H_i)} \geq f(P_{m-1})$ and the equality holds if and only if $G - v \cong P_{m-1}$. So, from (4) and (5), the lemma is true. □

Let T be a tree with n vertices. Then, $T \cong S_n$ if $n = 1, 2, 3$. By calculating, we have

(i) for $n = 4$, $S_3 \succ P_3$,

(ii) for $n = 5$, $S_4 \succ T_{1,1,2} \succ P_5$,

(iii) for $n = 6$, $S_6 \succ S_{3,1} \succ S_{2,2} \succ T_{1,1,3} \succ T_{1,2,2,} \succ P_6$,

(iv) for $n = 7$ and $T \notin \{P_7, T_{1,2,3}, T_{2,2,2}\}$, $T \succ T_{1,2,3} \succ T_{2,2,2}$.

(v) for $n = 8, 9, 10$ and $T \notin \{P_n, T_{2,2,n-5}, T_{2,4,n-7}\}$, $T \succ T_{2,4,n-7} \succeq T_{2,2,n-5}$ and the equality holds if and only if $n = 9$.

Theorem 3: Let T be a tree with n vertices.

(i) If $T \ncong P_n$ and $n \geq 7$, then

$$f(T) \geq 4F_{n-1} + F_{n-3},$$

the equality holds if and only if $T \cong T_{2,2,n-5}$.

(ii) If $T \notin \{P_n, T_{2,2,n-5}\}$ and $n \geq 10$, then

$$f(T) \geq 2F_n + 8F_{n-5},$$

the equality holds if and only if $T \cong T_{2,4,n-7}$.

Proof: By induction on n. By the above argument, it is easy to check that (i) and (ii) of the theorem hold for trees T with $n(T) = 10$.

Suppose that $n(T) \geq 11$ and (i) and (ii) of the theorem are true for all T' with $n(T') < n$. For a tree T with $n(T) = n$, we distinguish the following cases:

Case 1: There exist an $r \geq 2$ and a tree G such that $T \cong Q_{r,G}$. By Lemma 5, $Q_{r,G} \succeq P_{r,n-r}$. By Lemmas 1, 2 and 3,

$$f(P_{r,n-r}) = 2^r F_{n-r+1} + F_{n-r}.$$

So, one can see that $P_{r,n-r} \succ P_{r-1,n-r+1}$ for $r \geq 2$. Thus we have $Q_{r,G} \succeq P_{r,n-r} \succeq T_{1,1,n-3}$. By (ii) of Theorem 2, we know that (i) and (ii) of the theorem are true.

Case 2: For each path uvw in T with $d_u = 1$, we have that $d_v = 2$ and $d_w \geq 2$. If T has only one vertex of degree 3, by Theorem 2 we have that (i) and (ii) of the theorem hold; otherwise, T contains at least one vertex of degree larger than 3 or two vertices of degree 3. From Lemma 3,

$$f(T) = f(T-u) + f(T-u-v), \tag{6}$$

$$f(T_{2,2,n-5}) = f(T_{2,2,n-6}) + f(T_{2,2,n-7}) \tag{7}$$

and

$$f(T_{2,4,n-7}) = f(T_{2,4,n-8}) + f(T_{2,4,n-9}). \tag{8}$$

It is not difficult to see that $T-u$ is a tree with $n-1$ vertices and it contains at least one vertex of degree larger than 3 or two vertices of degree 3; whereas $T-u-v$ is a tree of $n-2$ vertices and $T-u-v \ncong P_{n-2}$. By the induction hypothesis, $T-u \succ T_{2,2,n-6}$ and $T-u-v \succeq T_{2,2,n-7}$. So, from (6) and (7) (i) of the theorem follows.

On the other hand, if $d_w = 2$, then $T-u-v$ contains at least one vertex of degree larger than 3 or two vertices of degree 3. By the induction hypothesis as well as (6) and (8), (ii) of the theorem also holds. If $d_w \geq 3$ and $T-u-v \ncong T_{2,2,n-7}$, by the induction hypothesis we have $T-u \succ T_{2,4,n-8}$ and $T-u-v \succeq T_{2,4,n-9}$. Thus, from (6) and (8), (ii) of the theorem holds. If $d_w \geq 3$ and $T-u-v \cong T_{2,2,n-7}$, by $n \geq 11$ we know that T is one of the graphs T_1 and T_2 shown in Figure 3 (Otherwise there exists a path uvw in T such that $d_v = 1$ and $d_u = d_w = 2$).

Figure 3. Graphs T_1 and T_2

¿From Lemmas 1, 2 and 3, we have

$$f(T_1) = 2F_{n-1} + 9F_{n-4} + 9F_{n-7}$$

and

$$f(T_2) = 2F_{n-1} + 2F_{n-3} + 10F_{n-5} + 9F_{n-8}.$$

By $F_n = F_{n-1} + F_{n-2}$, it is not hard to obtain that

$$f(T_1) - f(T_2) = 4F_{n-9} + F_{n-11}$$

and

$$f(T_2) - f(T_{2,4,n-7}) = 3F_{n-8} + 2F_{n-10}.$$

Thus (ii) of the theorem holds. This completes the proof. $\square$

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