A COMBINATORIAL INTERPRETATION OF IDENTITIES INVOLVING STIRLING NUMBERS AND THEIR GENERALIZATIONS

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ABSTRACT

Purely combinatorial methods are used to show that Stirling numbers, defined combinatorially, satisfy orthogonality relations. The proofs are extended to several generalizations of Stirling numbers.

1. INTRODUCTION

Stirling numbers are probably most simply defined as the coefficients in an expansion of non-negative integral powers of a variable in terms of factorial powers, or vice-versa:

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k, \qquad n \geq 0, \tag{1}$$

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \qquad n \geq 0, \tag{2}$$

where

$$\begin{aligned} &(x)_n = x(x-1)\dots(x-n+1), \qquad n \geq 1, \\ &(x)_0 = 1. \end{aligned}$$

The numbers $s(n,k)$ and $S(n,k)$ are, in the notation of Riordan [12], Stirling numbers of the first and second kind, respectively. The *signless* Stirling number of the first kind, $c(n,k)$, is defined by

$$c(n,k) = (-1)^{n-k}s(n,k). \tag{3}$$

By substituting (1) into the right-hand side of (2), (or vice-versa), changing the order of summation, and equating coefficients of x^l, (or $(x)_l$), we obtain the following identities:

$$\sum_{k=l}^{n} S(n,k)s(k,l) = \delta_{nl}, \tag{4}$$

$$\sum_{k=l}^{n} s(n,k)S(k,l) = \delta_{nl}, \tag{5}$$

where δ_{nl} is the Kronecker delta symbol.

Alternative combinatorial definitions of Stirling numbers may be found in, for example, Riordan [12, pages 71, 99]:

(i) the signless Stirling number of the first kind, $c(n,k)$, is the number of permutations of the set $\{1,2,\ldots,n\}$ with k cycles;

(ii) the Stirling number of the second kind, $S(n,k)$, is the number of partitions of the set $\{1,2,\ldots,n\}$ into k subsets.

The purpose of this article is, firstly, to provide a combinatorial derivation of the identities (4) and (5), starting from these combinatorial definitions. This is described in section 2.

A number of generalizations of (1) and (2) can be unified within the context of weighted Stirling pairs (see Hsu and Shiue [9], Yu [14] and Corcino [7]; for a review, see section 6 of Branson [2]). If the generalized factorial power $(x|a)_n$ is defined by

$$\begin{aligned} &(x|a)_n = x(x-a)(x-2a)\ldots(x-na+a), \qquad n \geq 1, \\ &(x|a)_0 = 1, \end{aligned}$$

then we may define a weighted Stirling pair, $S(n,k;a,b,c)$ and $S(n,k;b,a,-c)$, by

$$(x|a)_n = \sum_{k=0}^{n} S(n,k;a,b,c)(x-c|b)_k, \tag{6}$$

$$(x|b)_n = \sum_{k=0}^{n} S(n,k;b,a,-c)(x+c|a)_k. \tag{7}$$

Clearly, the standard Stirling numbers $s(n,k)$ and $S(n,k)$ are equal to the pair $S(n,k;1,0,0)$ and $S(n,k;0,1,0)$. We can, in fact, restrict our attention to weighted Stirling pairs with a or b equal to 1 because of the following relationship, which follows from (6) or (7):

$$S(n,k;a,b,c) = a^{n-k} S\left(n,k;1,\frac{b}{a},\frac{c}{a}\right) = b^{n-k} S\left(n,k;\frac{a}{b},1,\frac{c}{b}\right). \tag{8}$$

The functions $(x|a)_l$, $l = 0,1,\ldots,n$, form a basis for the space of polynomials of degree less than or equal to n. Therefore, by substituting (7) into the right-hand side of (6), changing the order of summation and equating coefficients of $(x|a)_l$, we obtain the orthogonality relation

$$\sum_{k=l}^{n} S(n,k;a,b,c)S(k,l;b,a,-c) = \delta_{nl}. \tag{9}$$

Putting $a=0, b=1, c=0$ or $a=1, b=0, c=0$ we recover (4) or (5) respectively. In section 3 we generalize the discussion of section 2 by giving a combinatorial proof of (9), starting from combinatorial definitions of the Stirling numbers, for several choices of the parameters a, b and c.

2. STIRLING NUMBERS OF THE FIRST AND SECOND KINDS

We first rephrase the definitions (i) and (ii) in section 1 by expressing them in terms of the number of ways we can partition a set into (non-empty) subsets, and then order the elements of the resulting subsets in certain specified ways. We talk of "partitioning a set into ordered subsets subject to such-and-such restriction".

Define P_{nk} to be the set of all partitions of the set $\{1, 2, \dots, n\}$ into k ordered subsets subject to the restriction that the smallest element of each subset appears as its first element. The number of such partitions is equal to the number of partitions into k cyclically ordered subsets, so, by (i) above, we see that $|P_{nk}|$, the number of elements in P_{nk}, is equal to $c(n, k)$.

Define Q_{nk} to be the set of all partitions of the set $\{1, 2, \dots, n\}$ into k ordered subsets subject to the restriction that the elements of each subset appear in increasing order of magnitude. (We call these *ascending* subsets.) The number of such partitions is equal to the number of partitions into k unordered subsets, so, by (ii) above, $|Q_{nk}|$ is equal to $S(n, k)$.

It follows immediately from the combinatorial definitions that

$$s(n, n) = 1, \qquad S(n, n) = 1, \qquad n \geq 0, \tag{10}$$

so that (4) and (5) are trivially satisfied when n is equal to l. It remains to establish these identities in cases of inequality.

Combinatorial proof of identity (4): We say that a partition belonging to the set P_{nl} possesses property A_u, $(u = 1, 2, \dots, n)$, if u is not the first element of the subset to which it belongs and if the element immediately to its left is less than u. (Note that no partition belonging to P_{nl} possesses property A_1.) For example, the partition $\{\{1, 5, 3, 6\}\{2, 4\}\}$, belonging to $P_{6,2}$, possesses properties A_4, A_5 and A_6. If n is greater than l then at least one subset belonging to each partition of P_{nl} has at least two elements, and so, by the definition of P_{nl}, each partition possesses property A_u for at least one u (namely, the second element in any subset that has more than one element). We conclude that the number of members of P_{nl} that possess none of the properties A_u is zero. Hence by the principle of inclusion and exclusion [12, page 51] we have, for n greater than l,

$$\begin{aligned} &|P_{nl}| - \sum_u N(A_u) + \sum_{u<v} N(A_u A_v) \\ &\qquad - \sum_{u<v<w} N(A_u A_v A_w) + \cdots + (-1)^{n-1} N(A_2 A_3 \dots A_n) = 0, \end{aligned} \tag{11}$$

where $N(A_u A_v \dots)$ is the number of elements of P_{nl} that possess property A_u and property A_v and property ....

We have already seen that $|P_{nl}|$, the first term in (11), is equal to $c(n, l)$. As a first step in evaluating the second term in (11), we choose a particular value of u, and split $\{1, 2, \dots, n\}$ into $n-1$ 'building blocks', namely $n-2$ singlets and an ascending doublet whose second element is u. We can now assemble these $n-1$ building blocks into l larger subsets whose smallest elements are at the left. The definition of P_{nk} shows that this can be done in $c(n-1, l)$ ways, and the result is an element of P_{nl} which necessarily possesses property A_u. For example, each of the sets of building blocks $\{\{1, 3\}\{2\}\{4\}\{5\}\{6\}\}$ and $\{\{1\}\{2, 3\}\{4\}\{5\}\{6\}\}$ can be assembled in $c(5, 2)$ ways into elements of $P_{6,2}$, such as $\{\{1, 3, 5\}\{2, 6, 4\}\}$ or $\{\{1, 6, 2, 3, 5\}\{4\}\}$ respectively, which possess property A_3. Conversely, given any element of P_{nl} possessing property A_u, we can uniquely decompose it into its $n-1$ building blocks by putting a pair of braces, $\}\{$, in front of every element except u. Since the doublet in any set of building blocks is ascending, it follows that the number of ways we can choose the sets of building blocks, taking account of all possible sets of blocks with u as the second element of an ascending doublet, and all possible values of u, is $|Q_{n,n-1}|$, which is equal to $S(n, n-1)$. Hence the second term in (11) is equal to $-S(n, n-1)c(n-1, l)$.

Similarly, when we consider the third term in (11), we choose particular values of u and v, with u less than v, and construct $n-2$ building blocks which are either $n-4$ singlets and two ascending doublets whose second elements are u and v, or $n-3$ singlets and an ascending triplet whose second and third elements are respectively u and v. Each set of these $n-2$ building blocks may be assembled in $c(n-2, l)$ ways into l subsets forming an element of P_{nl} which has both property A_u and property A_v. For example, each of the sets of building blocks $\{\{1,2\}\{3,4\}\{5\}\{6\}\}$ and $\{\{1,2,4\}\{3\}\{5\}\{6\}\}$ can be assembled in $c(4,2)$ ways into elements of $P_{6,2}$, such as $\{\{1,2,6,5\}\{3,4\}\}$ or $\{\{1,2,4,5,6\}\{3\}\}$ respectively, which possess properties A_2 and A_4. There is a one-one correspondence between a particular choice of building blocks, for some choice of u and v, and an element of $Q_{n,n-2}$. Hence the number of different sets of building blocks is $S(n, n-2)$, and the third term in (11) is $S(n, n-2)c(n-2, l)$.

We continue in the same fashion. If we wish to combine the j properties $A_{u_1}, A_{u_2}, \ldots, A_{u_j}$, where $2 \leq u_1 < u_2 < \cdots < u_j \leq n$, there are $n-j$ building blocks which are either singlets or ascending subsets whose non-leading elements are $u_1, u_2, \ldots, u_j$. These can be combined to give elements of P_{nl} with properties $A_{u_1}, A_{u_2}, \ldots, A_{u_j}$: one must simply arrange that, within every one of the l subsets formed from the building blocks, the block with the smallest left-hand element appears at the left. It follows that the number of ways of arranging the $n-j$ building blocks is $c(n-j, l)$. Conversely, given any element of P_{nl} possessing the properties $A_{u_1}, A_{u_2}, \ldots, A_{u_j}$, we can uniquely decompose it into its $n-j$ building blocks by putting a pair of braces in front of all elements except $u_1, u_2, \ldots, u_j$. The number of different sets of building blocks, totalled over all values of $u_1, u_2, \ldots, u_j$, is $S(n, n-j)$, so the $(j+1)$th term in (11) is $(-1)^j S(n, n-j)c(n-j, l)$.

An element of P_{nl} cannot possess simultaneously more than $n-l$ properties of type A_u, so the process terminates at that stage; there are then l building blocks which themselves form the l subsets of an element of P_{nl}. The subsequent terms in (11) are therefore equal to zero. Hence we can write equation (11) as

$$c(n,l) - S(n, n-1)c(n-1, l) + S(n, n-2)c(n-2, l) + \cdots + (-1)^{n-l} S(n, l)c(l, l) = 0.$$

Using (10) and (3) we see that this is equivalent to (4) in the case where n is not equal to l.

Combinatorial proof of identity (5): We say that a partition belonging to the set P_{nl} possesses property B_u, $(u = 1, 2, \ldots, n)$, if u is not the first element of the subset to which it belongs and if all the elements to its right in the subset (if any) are greater than u. (Again we note that no partition belonging to P_{nl} possesses property B_1.) For example, the partition $\{\{1,3,6,5\}\{2,4\}\}$, belonging to $P_{6,2}$, possesses properties B_3, B_4 and B_5. If n is greater than l then at least one subset belonging to each partition of P_{nl} has at least two elements, and, if the rightmost element of this subset is v, say, then that partition possesses property B_v. We conclude that the number of members of P_{nl} that possess none of the properties B_u is zero.

Hence, again using the principle of inclusion and exclusion, and with a similar notation to (11), we have

$$|P_{nl}| - \sum_{u} N(B_u) + \sum_{u<v} N(B_uB_v)$$

$$- \sum_{u<v<w} N(B_uB_vB_w) + \cdots + (-1)^{n-1} N(B_2B_3\ldots B_n) = 0. \tag{12}$$

We now assemble elements of P_{nl} using a different set of building blocks. To evaluate the second term in (12), we first, for a given u, split $\{1, 2, \ldots, n\}$ into $l+1$ building blocks, each of which has its smallest element at the left; one such smallest element must be chosen to be u. We can now assemble these $l+1$ building blocks into l larger subsets by joining the block beginning with u to the right-hand end of one of the other blocks whose first element is less than u. The result is an element of P_{nl} which necessarily possesses property B_u. For example, each of the sets of building blocks $\{\{1,6\}\{2,4\}\{3,5\}\}$ and $\{\{1,2\}\{3,5,4\}\{6\}\}$ can be assembled into elements of $P_{6,2}$, such as $\{\{1,6,3,5\}\{2,4\}\}$ or $\{\{1,2,3,5,4\}\{6\}\}$ respectively, which possess property B_3. Conversely, given any element of P_{nl} possessing property B_u, we can uniquely decompose it into its $l+1$ building blocks by putting a pair of braces, $\}\{$, immediately before u. Considering all possible sets of building blocks where u is at the left of one block, and all possible values of u, the total number of ways we can choose the sets of building blocks is $|P_{n,l+1}|$, which is equal to $c(n, l+1)$. In assembling each set of $l+1$ building blocks into l larger blocks, the leftmost elements of the joined blocks increase from left to right. It follows that a given set of blocks can be assembled in $|Q(l+1, l)|$ different ways. Hence the second term in (12) is equal to $-c(n, l+1)S(l+1, l)$.

For the third term in (12), we choose values of u and v, with u less than v, and we construct $l+2$ building blocks (each with its smallest element at the left), two of which start with the elements u and v. These may be assembled into l subsets forming an element of P_{nl} *either* by joining the blocks starting with u and v to the right-hand ends of two other blocks whose leftmost elements are less than u and v respectively, *or* by joining first the block beginning with u and then the block beginning with v to the right-hand end of a block whose first element is less than u. The resulting element of P_{nl} has both property B_u and property B_v. For example, each of the sets of building blocks $\{\{1\}\{2,5\}\{3\}\{4,6\}\}$ and $\{\{1\}\{2\}\{3,6,5\}\{4\}\}$ can be assembled into elements of $P_{6,2}$, such as $\{\{1,3\}\{2,5,4,6\}\}$ or $\{\{1,3,6,5,4\}\{2\}\}$ respectively, which possess properties B_3 and B_4. A total of $c(n, l+2)$ choices of building blocks may each be combined in $S(l+2, l)$ ways, so the third term in (12) is $c(n, l+2)S(l+2, l)$.

In general, when considering properties $B_{u_1}, B_{u_2}, \ldots, B_{u_j}$, where $2 \le u_1 < u_2 < \cdots < u_j \le n$, the building blocks are $l+j$ subsets, each with its smallest element at the left; j of these subsets have $u_1, u_2, \ldots, u_j$ as their smallest (and therefore leftmost) elements. We now take, in turn, the subsets headed by $u_1, u_2, \ldots, u_j$ and attach them to the right-hand end of one of the other l subsets in such a way that the leftmost elements of the blocks joined to form a larger subset increase from left to right. The outcome is a member of P_{nl} possessing all the properties $B_{u_1}, B_{u_2}, \ldots, B_{u_j}$. As before, any member of P_{nl} possessing all the properties $B_{u_1}, B_{u_2}, \ldots, B_{u_j}$ can be uniquely decomposed into its $l+j$ building blocks by putting a pair of braces in front of $u_1, u_2, \ldots, u_j$. Taking account of all values of $u_1, u_2, \ldots, u_j$, the building blocks can be chosen in $c(n, l+j)$ ways, and any choice of $l+j$ building blocks can be assembled in $S(l+j, l)$ ways. Hence the $(j+1)$th term in (12) is equal to $(-1)^j c(n, l+j)S(l+j, l)$.

No element of P_{nl} can possess simultaneously more than $n-l$ properties of type B_u so the procedure terminates when $l+j$ equals n (when the building blocks are all singlets). Recalling that $|P_{nl}|$ is equal to $c(n,l)$, we can therefore write equation (12) as

$$c(n,l) - c(n,l+1)S(l+1,l) + c(n,l+2)S(l+2,l) + \cdots + (-1)^{n-l}c(n,n)S(n,l) = 0.$$

Equations (10) and (3) show that, when n is greater than l, this is equivalent to (5).

For another combinatorial proof of (4) and (5), see reference [1].

3. WEIGHTED STIRLING PAIRS

Degenerate Stirling numbers, C-numbers and Lah numbers: Carlitz [4] defined the degenerate Stirling numbers as the pair $S(n,k;a,1,0)$ and $S(n,k;1,a,0)$ (or, in his notation, $S(n,k|a)$ and $(-1)^{n-k}S_1(n,k|a)$). These are essentially equivalent to the C-numbers of Charalambides [6], which are given by

$$C(n,k,a) = a^k S(n,k;1,a,0).$$

In order to keep the discussion from becoming too complex, we consider only the particular case where $a=-1$. The numbers $(-1)^k S(n,k;1,-1,0)$ are known as Lah numbers (see [11] and [12, pages 43-44]). Equation (8) shows that $S(n,k;-1,1,0)$ and $S(n,k;1,-1,0)$ differ only by a sign factor:

$$S(n,k;1,-1,0) = (-1)^{n-k}S(n,k;-1,1,0). \tag{13}$$

We first refer to a combinatorial definition of these numbers. Define R_{nl} to be the set of all partitions of the set $\{1,2,\ldots,n\}$ into l ordered subsets. Specializing the discussion of section 6.2 of reference [2] to the case $a=-1$ and using (13), we find that

$$|R_{nl}| = (-1)^{n-l}S(n,l;1,-1,0) = S(n,l;-1,1,0). \tag{14}$$

Since

$$|R_{nn}| = 1 = S(n,n;1,-1,0) = S(n,n;-1,1,0), \tag{15}$$

it follows that the only non-trivial cases of (9) are those for which n is greater than l. We now address these cases, using the same form of argument as in section 2. We say that a partition belonging to the set R_{nl} possesses property C_u, $(u=1,2,\ldots,n)$, if u is not the first element of the subset to which it belongs. For example, the partition $\{\{5,1,3,6\}\{2,4\}\}$, belonging to $R_{6,2}$, possesses properties C_1, C_3, C_4 and C_6. If n is greater than l the principle of inclusion and exclusion implies that

$$|R_{nl}| - \sum_u N(C_u) + \sum_{u<v} N(C_uC_v) - \sum_{u<v<w} N(C_uC_vC_w) + \cdots + (-1)^n N(C_1C_2\ldots C_n) = 0, \tag{16}$$

where $N(C_uC_v\ldots)$ is the number of elements of R_{nl} that possess property C_u and property C_v and property $\ldots$.

To assemble partitions possessing $C_{u_1}, C_{u_2}, \ldots, C_{u_j}$, where $1 \le u_1 < u_2 < \cdots < u_j \le n$, we construct $n-j$ building blocks which are ordered subsets whose non-leading elements are $u_1, u_2, \ldots, u_j$. These building blocks are then assembled in all possible orders into l larger sets forming elements of R_{nl} that possess properties $C_{u_1}, C_{u_2}, \ldots, C_{u_j}$. The number of ways this can be done is $|R_{n-j,l}|$. For example, each of the sets of building blocks $\{\{1\},\{3,2\},\{4\},\{6,5\}\}$ and $\{\{1\},\{4,5,2\},\{3\},\{6\}\}$ can be assembled in $|R_{4,2}|$ ways into elements of $R_{6,2}$, such as $\{\{3,2,1\},\{4,6,5\}\}$ or $\{\{6,1,4,5,2\},\{3\}\}$ respectively, each of which possess properties C_2 and C_5. Conversely, any element of R_{nl} possessing properties $C_{u_1}, C_{u_2}, \dots, C_{u_j}$ can be uniquely decomposed into its building blocks. Taking account of all values of $u_1, u_2, \dots, u_j$, the number of ways we can construct the $n-j$ building blocks is simply the number of ways we can split the set $\{1,2,\dots,n\}$ into $n-j$ ordered subsets, that is to say, $|R_{n,n-j}|$. The $(j+1)$th term in (16) is therefore $(-1)^j|R_{n,n-j}||R_{n-j,l}|$. It is impossible for more than $n-l$ elements simultaneously to be non-leading elements of the subsets to which they belong, so equation (16) becomes

$$|R_{nl}| + \sum_{j=1}^{n-l}(-1)^j|R_{n,n-j}||R_{n-j,l}| = 0.$$

An application of (14) and (15) gives the required orthogonality relations:

$$\sum_{k=l}^{n} S(n,k;1,-1,0)S(k,l;-1,1,0) = 0 = \sum_{k=l}^{n} S(n,k;-1,1,0)S(k,l;1,-1,0), \qquad n > l.$$

Weighted (non-central) Stirling numbers and r-Stirling numbers: A number of authors have discussed weighted Stirling pairs in the case where $a = 0$. Carlitz's weighted Stirling numbers [5], $R(n,k,c)$ and $R_1(n,k,c)$, are equal to $S(n,k;0,1,c)$ and $(-1)^{n-k}S(n,k;1,0,-c)$ respectively. (The first of these had in fact been used much earlier by Riordan [13]). Koutras's non-central Stirling numbers [10] are essentially the same (apart from the $(-1)^{n-k}$ factor) whereas Broder [3] defined r-Stirling numbers that are equal to $S(n-c,k-c;0,1,c)$ and $(-1)^{n-k}S(n-c,k-c;1,0,-c)$.

Suppose that c is a positive integer. Let T^c_{nk} be the set of all partitions of the set $\{1,2,\dots,n\}$ into c distinguishable 'boxes' (some or all of which may be empty), $[\dots]_1, [\dots]_2, \dots, [\dots]_c$, and k non-empty subsets; in the boxes the elements are ordered in all possible ways, and in the subsets the elements are ordered with the smallest element appearing first. For example, a member of $T^2_{6,2}$ is $\{[6,3]_1, [\quad]_2, \{1\}, \{2,5,4\}\}$. Let U^c_{nk} be defined similarly, except that now the elements in each box and each subset must appear in increasing order of magnitude. (Using the terminology of section 2, we refer to these as ascending subsets and ascending boxes.) We can deduce from [5] and [2, section 6.3] the following combinatorial interpretation of the weighted Stirling numbers:

$$|T^c_{nk}| = (-1)^{n-k}S(n,k;1,0,-c), \tag{17}$$

$$|U^c_{nk}| = S(n,k;0,1,c). \tag{18}$$

As usual, we note that

$$|T^c_{nn}| = |U^c_{nn}| = 1, \qquad n \geq 0. \tag{19}$$

Our argument now is similar to that of section 2, with due regard being paid to the existence of the c boxes. We say that a partition belonging to T^c_{nl} possesses property D_u, $(u = 1,2,\dots,n)$, if, whenever u is in a box, the element immediately to its left (if any) is less than u; whenever u is not in a box, u has a smaller element immediately to its left. The usual arguments, with the usual notation, show that, for n greater than l,

$$|T^c_{nl}| - \sum_u N(D_u) + \sum_{u<v} N(D_u D_v) - \sum_{u<v<w} N(D_u D_v D_w) + \cdots + (-1)^n N(D_1 D_2 \ldots D_n) = 0. \tag{20}$$

The $n-j$ building blocks (and, possibly, a number of 'building boxes') that we use to assemble partitions possessing properties $D_{u_1}, D_{u_2}, \ldots, D_{u_j}$ (where $u_1 < u_2 < \ldots < u_j$) are constructed by first taking all elements except $u_1, u_2, \ldots, u_j$, and putting them as the first elements of $n-j$ subsets. We then assign $u_1, u_2, \ldots, u_j$ in turn either to these $n-j$ subsets, or to one or more of the boxes, so as to form ascending subsets and ascending boxes. We take the $n-j$ building blocks, place some or none of them in boxes, in all possible orders to the right of any elements already in the boxes, and from the remainder of the building blocks we construct l non-empty subsets in each of which the smallest element is at the left. This operation can be carried out in $|T^c_{n-j,l}|$ ways and results in an element of T^c_{nl} having properties $D_{u_1}, D_{u_2}, \ldots, D_{u_j}$. Taking account of all values of $u_1, u_2, \ldots, u_j$, the building blocks can be chosen in $|U^c_{n,n-j}|$ ways. Once more, we note that any element of T^c_{nl} having properties $D_{u_1}, D_{u_2}, \ldots, D_{u_j}$ can be uniquely unravelled into its building blocks and building boxes by putting a pair of braces in front of all elements except $u_1, u_2, \ldots, u_j$. We conclude that

$$\sum_{u_1 < \ldots < u_j} N(D_{u_1} D_{u_2} \ldots D_{u_j}) = |U^c_{n,n-j}||T^c_{n-j,l}|.$$

Therefore (20) becomes

$$|T^c_{nl}| + \sum_{j=1}^{n-l} (-1)^j |U^c_{n,n-j}||T^c_{n-j,l}| = 0.$$

Equations (17), (18) and (19) translate this into the orthogonality relation

$$\sum_{k=l}^{n} S(n,k;0,1,c) S(k,l;1,0,-c) = 0, \qquad n > l.$$

In order to derive the orthogonality relation with the Stirling numbers in the reverse order, we declare that a partition belonging to T^c_{nl} possesses property E_u, $(u = 1, 2, \ldots, n)$, if, whenever u is in a box, all the elements to its right (if any) are greater than u; whenever u is not in a box, u has an element immediately to its left, and, again, all the elements to its right (if any) are greater than u. Then, for n greater than l,

$$|T^c_{nl}| - \sum_u N(E_u) + \sum_{u<v} N(E_u E_v) - \sum_{u<v<w} N(E_u E_v E_w) + \cdots + (-1)^n N(E_1 E_2 \ldots E_n) = 0. \tag{21}$$

The building boxes and the $l+j$ building blocks that we use to assemble partitions possessing properties $E_{u_1}, E_{u_2}, \ldots, E_{u_j}$ are constructed by first putting all elements $u_1, u_2, \ldots, u_j$ together with l of the remaining elements as the first elements of $l+j$ subsets. Each remaining element is then assigned either to one of these $l + j$ subsets whose leftmost element is less than it, or to one or more of the boxes, arranged in all possible orders. Some or none of the building blocks headed by $u_1, u_2, \ldots, u_j$ are placed in boxes, to the right of any elements already there, in such a way that the leftmost elements of the building blocks within any box increase from left to right; the remainder of the building blocks headed by $u_1, u_2, \ldots, u_j$ are, in turn, adjoined to the right of one of the other l building blocks in such a way that, again, the leftmost elements of the building blocks within any subset increase from left to right. This assembly of an element of T^c_{nl} having properties $E_{u_1}, E_{u_2}, \ldots, E_{u_j}$ can be uniquely inverted by putting a pair of braces in front of $u_1, u_2, \ldots, u_j$. Summing over all values of $u_1, u_2, \ldots, u_j$ we see that the building blocks can be constructed in $|T^c_{n,l+j}|$ ways, and the building blocks can then be joined together in $|U^c_{l+j,l}|$ ways. Hence (21) becomes

$$|T^c_{nl}| + \sum_{j=1}^{n-l} (-1)^j |T^c_{n,l+j}||U^c_{l+j,l}| = 0.$$

Use of equations (17), (18) and (19) leads to the required result

$$\sum_{k=l}^{n} S(n, k; 1, 0, -c) S(k, l; 0, 1, c) = 0, \qquad n > l.$$

Degenerate weighted Stirling numbers: Howard [8] combined degenerate and weighted Stirling numbers to give degenerate weighted Stirling numbers, which he denoted by $S(n, k; c|a)$ and $S_1(n, k; a + c|a)$. These are equal to $S(n, k; a, 1, c)$ and $(-1)^{n-k} S(n, k; 1, a, -c)$. As in the first part of this section, we will restrict our discussion to the case $a = -1$. These numbers are related by (8):

$$S(n, k; 1, -1, -c) = (-1)^{n-k} S(n, k; -1, 1, c).$$

The combinatorial interpretation is obtained by modifying the discussion for degenerate Stirling numbers by the addition of c boxes, as in the previous subsection. Let V^c_{nk} be the set of all partitions of the set $\{1, 2, \ldots, n\}$ into c distinguishable boxes (some or all of which may be empty), and k non-empty subsets; in the boxes and in the subsets the elements are ordered in all possible ways. Then, (see section 6.4 of [2])

$$|V^c_{nk}| = (-1)^{n-k} S(n, k; 1, -1, -c) = S(n, k; -1, 1, c). \tag{22}$$

A partition is said to possess property F_u if u is in a box or if it is not the first element in a subset. The $n - j$ building blocks (and, possibly, some building boxes) that we use to assemble partitions possessing properties $F_{u_1}, F_{u_2}, \ldots, F_{u_j}$ are in this case constructed by first taking all elements except $u_1, u_2, \ldots, u_j$, and putting them as the first elements of $n - j$ subsets. We then assign $u_1, u_2, \ldots, u_j$ in turn either to these $n - j$ subsets, or to one or more of the boxes, in all possible orders. We take the $n - j$ building blocks, place some or none of them in boxes, in all possible orders to the right of any elements already in the boxes, and from the remainder of the building blocks we construct l non-empty subsets by joining the building blocks in all possible orders. This (invertible) operation can be carried out in $|V^c_{n-j,l}|$ ways and it leads to an element of V^c_{nl} possessing properties $F_{u_1}, F_{u_2}, \ldots, F_{u_j}$. Taking account of all values of

$u_1, u_2, \ldots, u_j$, the building blocks can be constructed in $|V^c_{n,n-j}|$ ways. Arguing as in previous subsections, we conclude that

$$\sum_{u_1<\ldots<u_j} N(F_{u_1}F_{u_2}\ldots F_{u_j}) = |V^c_{n,n-j}||V^c_{n-j,l}|,$$

so that

$$|V^c_{nl}| + \sum_{j=1}^{n-l}(-1)^j|V^c_{n,n-j}||V^c_{n-j,l}| = 0, \qquad n > l.$$

Using the fact that $|V^c_{nn}| = 1$ together with (22) we obtain

$$\sum_{k=l}^{n} S(n,k;1,-1,-c)S(k,l;-1,1,c) = \delta_{nl} = \sum_{k=l}^{n} S(n,k;-1,1,c)S(k,l;1,-1,-c).$$

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REFERENCES

[1] A.T. Benjamin & J.J. Quinn. *Proofs that Really Count: the Art of Combinatorial Proof*, Washington DC: Mathematical Association of America, 2003.

[2] D. Branson. "Stirling Numbers and Bell Numbers: Their Role in Combinatorics and Probability." *Math. Scientist* **25** (2000): 1-31.

[3] A.Z. Broder. "The r-Stirling Numbers." *Discrete Math.* **49** (1984): 241-259.

[4] L. Carlitz. "Degenerate Stirling, Bernoulli and Eulerian Numbers." *Utilitas Mathematica* **15** (1979): 51-88.

[5] L. Carlitz. "Weighted Stirling Numbers of the First and Second Kind - I." *Fibonacci Quart.* **18** (1980): 147-162.

[6] C.A. Charalambides. "A New Kind of Number Appearing in the n-fold Convolution of Truncated Binomial and Negative Binomial Distributions." *SIAM J. Appl. Math.* **33** (1977): 279-288.

[7] R.B. Corcino. "Some Theorems on Generalized Stirling Numbers." *Ars Combinatoria* **60** (2001): 273-286.

[8] F.T. Howard. "Degenerate Weighted Stirling Numbers." *Discrete Math.* **57** (1985): 45-58.

[9] L.C. Hsu & P.J.-S. Shiue. "A Unified Approach to Generalized Stirling Numbers." *Adv. Appl. Math.* **20** (1998): 366-384.

[10] M. Koutras. "Non-central Stirling Numbers and Some Applications." *Discrete Math.* **42** (1982): 73-89.

[11] I. Lah. "Eine Neue Art von Zahren, ihre Eigenschaften und Anwendung in der Mathematischen Statistik." *Mitteilungsbl. Math. Statist.* **7** (1955): 203-212.

[12] J. Riordan. *An Introduction to Combinatorial Analysis*, New York: Wiley, 1958.

[13] J. Riordan. "Moment Recurrence Relations for Binomial, Poisson and Hypergeometric Frequency Distributions." *Ann. Math. Statist.* **8** (1937): 103-111.

[14] H.Q. Yu. "A Generalization of Stirling Numbers." *Fibonacci Quart.* **36** (1998): 252-258.

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