ON THE NUMBER OF PRIMITIVE PYTHAGOREAN TRIANGLES WITH A GIVEN INRADIUS

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ABSTRACT

The inradius of a triangle is the radius of the inscribed circle. In particular, the inradius of a primitive Pythagorean triangle is always an integer. We show how to find the number of primitive Pythagorean triangles with a given inradius.

1. INTRODUCTION

Let a right triangle have sides whose lengths are x, y, z, where z is the length of the hypotenuse. Such a triple is called *primitive* if $GCD(x, y, z) = 1$. It is well-known that x, y, z is a primitive Pythagorean triple (we will abbreviate PPT) if and only if

$$x = 2ab, \quad y = a^2 - b^2, \quad z = a^2 + b^2 \tag{1}$$

where a, b are integers such that

$$a > b > 0, \quad (a, b) = 1, \quad a \not\equiv b \pmod 2 \tag{2}$$

(see [1], Theorem 4.17). The *inradius* of a triangle is the radius of the inscribed circle. Let r be the inradius of PPT x, y, z. The following theorem, which uses a simple geometric argument, allows us to express r in terms of x, y, and z.

Theorem 1: If r, x, y, z are defined as above, then

$$r = \frac{x + y - z}{2} \quad . \tag{3}$$

Proof: Let the perpendicular sides of the triangle meet at point C. Draw radii from the center of the circle to the three points of tangency. Consider the quadrilateral consisting of the radii to the sides of length x and y, and the tangent lines drawn to those sides from point C. Since angle C is a right angle by hypothesis, and since a tangent to a circle is perpendicular to a radius drawn to the point of contact, the quadrilateral has 3 right angles. Therefore the fourth angle must also be a right angle, so the quadrilateral is a rectangle. Since two adjacent sides (namely the radii) have equal length, it follows that the quadrilateral is a square whose side has length r. Thus the points of tangency split the sides of the triangle as follows: (1) the side of length x is split into segments of length r and $x - r$; (2) the side of length y is split into segments of length r and $y - r$; (3) the hypotenuse is split into segments of length $x - r$ and $y - r$. (The last statement follows from the fact that tangents drawn from an external point to a circle have equal length.) Thus we have $(x-r)+(y-r)=z$, from which the conclusion follows.

Corollary 1: If r is the inradius of PPT x, y, z, then r is a positive integer.

Proof: This follows from Theorem 1, the fact z and exactly one of x, y are odd, and the triangle ineqality.

Let $T(r)$ denote the number of PPT's with inradius r. Theorem 1 below enables us to compute $T(r)$.

Definition 1: If the natural number $r \geq 2$, let $\omega(r)$ denote the number of distinct prime factors of r.

Definition 2: If the natural number $r \geq 2$, let $\omega^*(r)$ denote the number of distinct odd prime divisors of r, so that

$$\omega^*(r) = \begin{cases} \omega(r) \text{ if } \ r \text{ is } \ \text{odd} \\ \omega(r)-1 \text{ if r is even} \end{cases}. \tag{4}$$

Theorem 2: If r is a natural number, then $T(r) = 2^{\omega^*(r)}$.

Proof: If we combine (1) and (3), we obtain $r = b(a-b)$.

Now $(a,b)=1 \rightarrow (b,a-b)=1$ and $a \not\equiv b \pmod 2 \rightarrow a-b$ is odd.

If r is odd , $r>1$, and $b|r$, let

$$r = \prod_{i=1}^{n} p_i^{e_i}, \quad b = \prod_{i=1}^{n} p_i^{f_i}, \quad a-b = \prod_{i=1}^{n} p_i^{e_i-f_i}$$

where the p_i are distinct odd primes and $0 \leq f_i \leq e_i$ for all i. Now

$$(b,a-b)=1 \rightarrow Min\{f_i, e_i-f_i\} = 0 \ for \ all \ i \rightarrow f_i = 0 \ or \ e_i \ for \ all \ i \ .$$

Thus we have $2^n = 2^{\omega(r)} = 2^{\omega^*(r)}$ possible values of b.

If $r = 2^k m$ where $k \geq 1$ and m is odd, then we have $b = 2^j d,\ a-b = 2^{k-j}\delta$ where $0 \leq j \leq k$ and $d\delta = m$. As before, $(b,a-b)=1 \rightarrow j=0 \ or \ j=k$. But if $j=0$, then $2^k|(a-b)$, so $a-b$ is even, an impossibility. Therefore $b = 2^k d,\ a-b=\delta$. As in the proof for the case when r is odd, the number of choices of d is $2^{\omega(m)} = 2^{\omega^*(r)}$. $\square$

Corollary 2: $T(r) = 1$ if and only if $r = 2^k$ for some $k \geq 0$.

Proof: If $k \geq 1$, then the corollary follows directly from Theorem 1. If $k=0$, then $1 = r = b(a-b) \rightarrow b=1$, $a=2 \rightarrow x=4$, $y=3$, $z=5$.

REFERENCES

[1] N. Robbins. *Beginning Number Theory.* (2nd ed.) (2006). Jones and Bartlett Publishers, Sudbury, MA.

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