SELF MATCHING IN $\lfloor n\alpha \rfloor$

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ABSTRACT

For an arbitrary real number α with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \lfloor (n+q_i)\alpha \rfloor - \lfloor n\alpha \rfloor$ is equal to p_i, and so is independent of n, except at a small specified number of values of n. For fixed n, this relation holds for all or for all except a finite number of values of i.

1. INTRODUCTION

Bunder and Tognetti noted in [3] that any section of the graph of $\lfloor n\tau \rfloor$ where $\tau = \frac{1}{2}(\sqrt{5} - 1))$ is "matched" for larger values of n. More precisely they proved:

$$\lfloor (n + F_i)\tau \rfloor - \lfloor n\tau \rfloor = F_{i-1}$$

except at $n = kF_{i+1} + \lfloor k\tau \rfloor F_i$ where

$$\lfloor (n + F_i)\tau \rfloor - \lfloor n\tau \rfloor = F_{i-1} - (-1)^i.$$

In this paper we will generalize this result to:

$$\lfloor (n + q_i)\alpha \rfloor - \lfloor n\alpha \rfloor = p_i \quad (\text{possibly} - (-1)^i)$$

where α is an arbitrary positive irrational number and $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ are the convergents of α - just as $\frac{F_0}{F_1}, \frac{F_1}{F_2}, \frac{F_2}{F_3}, \dots$ are the convergents of τ.

2. CONTINUED FRACTIONS AND CONVERGENTS

Definition 1: If α is any real number and $\alpha = [a_0, a_1, a_2, \dots]$ in continued fraction form, then the i**th convergent** of α is given by:

$$\frac{p_i}{q_i} = [a_0, a_1, a_2, \dots, a_i].$$

We quote the following properties of convergents from Khintchine [5]:

Lemma 1: If $\alpha = [a_0, a_1, a_2, \dots]$ then
(i) $p_{-1} = q_{-2} = 1$ and $p_{-2} = q_{-1} = 0$.
(ii) For $i \geq 0$:
  (a) $p_i = a_i p_{i-1} + p_{i-2}$
  (b) $q_i = a_i q_{i-1} + q_{i-2}$.

3. THE MAIN RESULT

We will be using work of Fraenkel, Levitt and Shimshoni [4] (their result assumes $1 < \alpha < 2$, but holds for $\alpha \geq 0$). In particular they use a generalization of the Zeckendorf expansion of an integer (used in [3]). This generalization, as pointed out in Allouche and Shallit [1], is in fact due to Ostrowski 1922 (see [6]).

Theorem 1: Given a positive irrational α with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$, every positive integer n can be represented uniquely by the Ostrowski α-numeration:

$$n = \sum_{i=h}^{m} k_i q_i$$

where $k_h \neq 0$, $m \geq h \geq 0$ and the k_i satisfy the following conditions:
(i) For each i, $0 \leq k_i \leq a_{i+1}$.
(ii) If $i > 0$ and $k_i = a_{i+1}$, $k_{i-1} = 0$.
(iii) If $h = 0, k_h < a_1$.

Note: In the remainder of the paper, every such representation of an integer will be assumed to be an Ostrowski α-numeration.

Theorem 2: Given a positive irrational α with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$, if

$$n = \sum_{i=h}^{m} k_i q_i$$

then

$$\lfloor n\alpha \rfloor = \sum_{i=h}^{m} k_i p_i + (-1 \text{ if } h \text{ is odd}).$$

Proof: Let $\alpha = \alpha' + r$ where $1 < \alpha' < 2$ and r is an integer ≥ -1. The convergents for α' are $\frac{p'_0}{q_0}, \frac{p'_1}{q_1}, \frac{p'_2}{q_2}, \ldots$ where $p'_i + rq_i = p_i$.

Theorem 1 gives us the unique numeration for n, which is the same for α and for α', and by Fraenkel, Levitt and Shimshoni [4]:

$$\lfloor n\alpha' \rfloor = \sum_{i=h}^{m} k_i p'_i + (-1 \text{ if } h \text{ is odd}).$$

So,

$$\begin{aligned}
\lfloor n\alpha \rfloor &= nr + \lfloor n\alpha' \rfloor \\
&= \sum_{i=h}^{m} rk_i q_i + \sum_{i=h}^{m} k_i(p_i - rq_i) + (-1 \text{ if } h \text{ is odd}) \\
&= \sum_{i=h}^{m} k_i p_i + (-1 \text{ if } h \text{ is odd}).
\end{aligned}$$

We can now prove our main result.

Theorem 3: If α is a positive irrational with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ and $n = \sum_{i=h}^{m} k_i q_i$, then

$$\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = \begin{cases} p_j & \text{if } h \leq j \text{ or } j+h \text{ is even,} \\ p_j + (-1)^j & \text{if } j < h \text{ and } j+h \text{ is odd.} \end{cases}$$

Proof: Let $n = \sum_{i=h}^{m} k_i q_i$. In each of the cases below we find the corresponding Ostrowski α-enumeration of $n + q_j$ and then use Theorem 2.

Case 1: If $j < h - 1$ or $j = h - 1$ and $k_h < a_{h+1}$ then

$$n + q_j = q_j + \sum_{i=h}^{m} k_i q_i.$$

Then $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j + (-1 \text{ if } j \text{ is odd}) + (1 \text{ if } h \text{ is odd})$, which gives the result.

Case 2: If $j = h - 1$ and $k_h = a_{h+1}$,

$$n = \sum_{k=0}^{r} a_{h+2k+1} q_{h+2k} + \sum_{i=h+2r+1}^{m} k_i q_i$$

where $r \geq 0, k_{h+2r+1} < a_{h+2r+2}$, and $k_{h+2r+2} < a_{h+2r+3}$ or $m = h + 2r$. Then

$$n + q_j = (k_{h+2r+1} + 1) q_{h+2r+1} + \sum_{i=h+2r+2}^{m} k_i q_i.$$

Then $j + h$ is odd and:

$$\begin{aligned} \lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor &= p_{h+2r+1} + (-1 \text{ if } h \text{ is even}) - \sum_{k=0}^{r} a_{h+2k+1} p_{h+2k} + (1 \text{ if } h \text{ is odd}) \\ &= p_j + (-1)^j. \end{aligned}$$

Case 3: If $m \geq j \geq h$ and either $0 < k_j < a_{j+1} - 1$, $0 < k_j = a_{j+1} - 1$ and $k_{j-1} = 0$, or $k_j = 0$ and $k_{j+1} < a_{j+2}$, then,

$$n + q_j = \sum_{i=h}^{j-1} k_i q_i + (k_j + 1) q_j + \sum_{i=j+1}^{m} k_i q_i$$

and

$$\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j.$$

Case 4: If $m \geq j \geq h$, $k_j = a_{j+1} - 1$, $k_{j+1} < a_{j+2}$ and $k_{j-1} > 0$ then,

$$n + q_j = \sum_{i=h}^{j-2} k_i q_i + (k_{j-1} - 1) q_{j-1} + (k_{j+1} + 1) q_{j+1} + \sum_{i=j+2}^{m} k_i q_i$$

and

$$\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = -p_{j-1} + p_{j+1} - k_j p_j = p_j.$$

Case 5: If $m \geq j = h = 0$ and $k_0 = a_1 - 1$,

$$n = (a_1 - 1)q_0 + \sum_{k=1}^{r} a_{2k+1}q_{2k} + \sum_{i=2r+1}^{m} k_i q_i$$

where $k_{2r+1} < a_{2r+2}$ and $r \geq 0$ (as $h = 0, k_0 > 1$). Then,

$$n + q_j = (k_{2r+1} + 1)q_{2r+1} + \sum_{i=2r+2}^{m} k_i q_i$$

and

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_{2r+1} - 1 - (a_1 - 1)p_0 - \sum_{k=1}^{r} a_{2k+1}p_{2k} = p_1 - 1 - a_1 p_0 + p_0 = p_0.$$

Case 6: If $m \geq j \geq h$, $k_j = a_{j+1} - 1$ and $k_{j+1} = a_{j+2}$ then $a_{j+1} = 1$ and

$$n = \sum_{i=h}^{j-1} k_i q_i + \sum_{k=0}^{r} a_{j+2k+2}q_{j+2k+1} + \sum_{i=j+2r+2}^{m} k_i q_i$$

where $r \geq 0$ and $k_{j+2r+2} < a_{j+2r+3}$ or $j + 2r + 1 = m$, then,

$$n + q_j = \sum_{i=h}^{j-1} k_i q_i + (k_{j+2r+2} + 1)q_{j+2r+2} + \sum_{i=j+2r+3}^{m} k_i q_i,$$

and

$$\lfloor (n + q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_{j+2r+2} - \sum_{k=0}^{r} a_{j+2k+2}p_{j+2k+1} = p_j.$$

Case 7: If $m \geq j \geq h$ and $k_j = a_{j+1}$ then $j > 0$ and $k_{j-1} = 0$ or $j = h$ and

$$n = \sum_{i=h}^{j-2r-2} k_i q_i + \sum_{k=0}^{r+s} a_{j+2k-2r+1}q_{j+2k-2r} + \sum_{i=j+2s+1}^{m} k_i q_i$$

where $r, s \geq 0$, $h \leq j - 2r - 2$ or $h = j - 2r$, and $j + 2s \leq m$, there are several cases. If $m = j + 2s$, the last summation is zero. If $m > j + 2s, k_{j+2s+1} < a_{j+2s+2}$. If $j - 2r - 2 < h$ the first summation is zero. If $j - 2r - 2 \geq h$, $k_{j-2r-2} < a_{j+2r-1}$. If $j - 2r = 0$, as $q_{-1} = 0$, the second summation sums to q_{j+2s+1}, i.e. $h = j + 2s + 1$, which is impossible. So $j - 2r > 0$. If $j - 2r > 1$, $k_{j-2r-2} + 1 < a_{j-2r-1}$ or $k_{j-2r-2} + 1 = a_{j-2r-1}$, $j - 2r - 2 > 0$ and either $k_{j-2r-3} = 0$ or $h = j - 2r - 2$,

$$n + q_j = \sum_{i=h}^{j-2r-3} k_i q_i + (k_{j-2r-2} + 1)q_{j-2r-2} + (a_{j-2r} - 1)q_{j-2r-1} +$$

$$\sum_{k=1}^{r} a_{j-2r+2k}q_{j-2r+2k-1} + (k_{j+2s+1} + 1)q_{j+2s+1} + \sum_{i=j+2s+2}^{m} k_i q_i.$$

If $h = j - 2r > 1$, we have the same but with zero for the first summation and k_{j-2r-2}.

Then in these cases:

$$\begin{aligned}\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor &= p_{j-2r-2} + (a_{j-2r}-1)p_{j-2r-1} \\ &+ \sum_{k=1}^{r} a_{j-2r+2k}p_{j-2r+2k-1} + p_{j+2s+1} - \sum_{k=0}^{r+s} a_{j+2k-2r+1}p_{j+2k-2r} = p_j.\end{aligned}$$

If $j-2r>1$, $k_{j-2r-2}+1 = a_{j-2r-1}$, $h \le j-2r-3$ and $k_{j-2r-3}>0$,

$$\begin{aligned}n+q_j &= \sum_{i=h}^{j-2r-4} k_i q_i + (k_{j-2r-3}-1)q_{j-2r-3} \\ &+ \sum_{k=0}^{r} a_{j-2r+2k}q_{j-2r+2k-1} + (k_{j+2s+1}+1)q_{j+2s+1} + \sum_{i=j+2s+2}^{m} k_i q_i.\end{aligned}$$

If $h = j-2r-2 = 0$ and $k_0+1 = a_1$, the expansion of $n+q_j$ is the same but with zero for the first sum and for the q_{j-2r-3} term.

In these cases:

$$\begin{aligned}\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor &= -p_{j-2r-3} + \sum_{k=0}^{r} a_{j-2r+2k}p_{j-2r+2k-1} \\ &+ p_{j+2s+1} - \sum_{k=0}^{r+s} a_{j+2k-2r+1}p_{j+2k-2r} - k_{j-2r-2}p_{j-2r-2} = p_j.\end{aligned}$$

If $j-2r=1$, h is 1,

$$n+q_j = (a_1-1)q_0 + \sum_{k=1}^{r} a_{2k+1}q_{2k} + (k_{2r+2s+2}+1)q_{2r+2s+2} + \sum_{i=2r+2s+3}^{m} k_i q_i$$

and

$$\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = (a_1-1)p_0 + \sum_{k=1}^{r} a_{2k+1}p_{2k} + p_{2r+2s+2} - \sum_{k=0}^{r+s} a_{2k+2}p_{2k+1} + 1 = p_j.$$

Case 8: If $j>m+1$ or $j=m+1$ then $n+q_j = \sum_{i=h}^{m} k_i q_i + q_j$ and $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j$.

Case 9: If $j=m+1$ and $a_{m+2}=1$ then $n+q_j = \sum_{i=h}^{m-1} k_i q_i + (k_m-1)q_m + q_{m+2}$ and $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = -p_m + p_{m+2} = p_j$.

4. THE MISMATCH POINTS

It follows that the values of n (called j-mismatch points in [3]) where $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor \neq p_j$ are those with $n = \sum_{i=j+2r+1}^{m} k_i q_i$, where $r \ge 0$.

If $n = \sum_{i=h}^{m} k_i q_i$ is fixed, $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor \neq p_j$ only when $j = h-1$, $h-3$, …, $h - 2\lfloor \frac{h-1}{2} \rfloor - 1$.

If $h=0$, $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j$ for all j.

5. A SPECIAL CASE

In the special case where a_i is a constant i.e., $\alpha = [a, a, a, \ldots] = \frac{1}{2}\ (a + (a^2+4)^{1/2})$, it is easy to show from Lemma 1 that $p_i = q_{i+1}$.

We now show that the numbers n where

$$\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j + (-1)^j$$

(the j-mismatch points) are exactly the numbers of the form

$$n = kq_{j+1} + \lfloor k\alpha \rfloor q_j.$$

First we need a lemma:

Lemma 2: If $\alpha = [a, a, a, \ldots]$ and $t, i \geq 0$ then $q_i q_t + q_{i+1} q_{t+1} = q_{i+t+2}$.

Proof:

$$\begin{aligned} q_i q_t + q_{i+1} q_{t+1} &= q_i q_t + (a q_i + q_{i-1}) q_{t+1} \\ &= q_{i-1} q_{t+1} + q_i (a q_{t+1} + q_t) \\ &= q_{i-1} q_{t+1} + q_i q_{t+2} \\ &= q_{i-2} q_{t+2} + q_{i-1} q_{t+3} \\ &= \ldots \\ &= q_{-1} q_{t+i+1} + q_0 q_{t+i+2} \\ &= q_{i+t+2} \text{ as } q_{-1} = 0 \text{ and } q_0 = a q_{-1} + q_{-2} = 1. \end{aligned}$$

Theorem 4: Given $\alpha = [a, a, a, a \ldots] = \frac{1}{2}\ (a + (a^2+4)^{1/2})$,
(a) If n is not of the form $kq_{j-1} + \lfloor k\alpha \rfloor q_j$, then $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j$.
(b) If n is of the form $kq_{j-1} + \lfloor k\alpha \rfloor q_j$, then $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j + (-1)^j$.

Proof: (a) If $n = \sum_{i=h}^{m} k_i q_i$ and $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor \neq p_j$, then $j < h$ and $j + h$ (and so $h - j$) is odd by Theorem 3.

Let $k = \sum_{i=h}^{m} k_i q_{i-j-1}$, then by Theorem 3, using $p_{i-j-1} = q_{i-j}$ and Lemma 2,

$$kq_{j-1} + \lfloor k\alpha \rfloor q_j = \sum_{i=h}^{m} k_i (q_{j-1} q_{i-j-1} + q_j q_{i-j})$$

$$= \sum_{i=h}^{m} k_i q_i = n.$$

Hence if n is not of the form $kq_{j-1} + \lfloor k\alpha \rfloor q_j$ then $\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j$.
(b) Let $k = \sum_{i=h_1}^{m} k_i q_i$, then, as above, if $n = kq_{j-1} + \lfloor k\alpha \rfloor q_j$

$$n = \sum_{i=h_1}^{m} k_i q_{i+j+1} + (-q_j \text{ if } h_1 \text{ is odd}).$$

If h_1 is even we have $j < h_1 + j + 1 = h(\text{for } n)$ and $j + h$ is odd.

If h_1 is odd $q_{h_1+j+1} - q_j = a \sum_{r=j+1}^{h_1+j} q_r$ so $h(\text{for } n) = j + 1 > j$ and $j + h$ is odd.

So in either case, by Theorem 3:

$$\lfloor (n+q_j)\alpha \rfloor - \lfloor n\alpha \rfloor = p_j + (-1)^j.$$

The j-mismatch points for $\alpha = [b, a, a, a, ...]$ can be shown to be $kq_{j-1} + \lfloor k(\alpha + a - b) \rfloor q_j$, but the result does not generalize, in an obvious way, to αs representable as other repeated continued fractions.

6. AN ALTERNATIVE TO THEOREM 2

The $0 < \alpha < 1$, and so $p_0 = 0$, case of the following alternative to Theorem 2 appears in Brown [2] and in Allouche and Shallit [1]. It follows easily from our Theorems 2 and 3.

Theorem 5: If α is a positive irrational number with convergents $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ and n has Ostrowski α-numeration $\sum_{i=h}^{m} k_i q_i$, then $\lfloor (n+1)\alpha \rfloor = \sum_{i=h}^{m} k_i p_i + p_0$.

Proof: By Theorems 3 and 2, as $q_0 = 1$:

$$\begin{aligned}\lfloor (n+1)\alpha \rfloor &= \lfloor n\alpha \rfloor + \begin{cases} p_0 & \text{if } h \text{ is even,} \\ p_0 + 1 & \text{if } h \text{ is odd.} \end{cases} \\ &= \sum_{i=h}^{m} k_i p_i + p_0 \end{aligned}$$

The results in Theorems 2 and 5 look quite different, however we could have used (the $0 < \alpha < 1$ case of) the latter, instead of Theorem 2, to prove Theorem 3 in a similar way to the above.

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