FIBONACCI DETERMINANTS – A COMBINATORIAL APPROACH

Arthur T. Benjamin
Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
e-mail: benjamin@hmc.edu

Naiomi T. Cameron
Department of Mathematical Sciences, Lewis and Clark College, Portland, OR 97219
e-mail: ncameron@lclark.edu

Jennifer J. Quinn
Interdisciplinary Arts & Sciences, University of Washington, Tacoma, WA 98402
e-mail: jjquinn@u.washington.edu

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ABSTRACT

In this paper, we provide combinatorial interpretations for some determinantal identities involving Fibonacci numbers. We use the method due to Lindström-Gessel-Viennot in which we count nonintersecting n-routes in carefully chosen digraphs in order to gain insight into the nature of some well-known determinantal identities while allowing room to generalize and discover new ones.

1. INTRODUCTION

When a matrix has Fibonacci entries, combinatorial methods can bring deeper understanding to the evaluation of its determinant. While there are several combinatorial interpretations of determinants [6, 15], we choose to exploit the one due to Lindström-Gessel-Viennot [8, 12] that counts nonintersecting paths.

For an $n \times n$ matrix $A = \{a_{ij}\}$, the general idea is to create an acyclic directed graph D with n origin nodes and n destination nodes, so that the number of paths from origin o_i to destination d_j is a_{ij}. For example, if A is the $n \times n$ matrix of binomial coefficients $A = \{\binom{i+j}{i}\}$, where i and j range between 0 and $n-1$, then D can be constructed as follows:

1. the vertices of D are the integer points (i, j) where $0 \leq i, j \leq n-1$;
2. the arcs of D create the grid on the integer lattice and are directed up or to the right;
3. origin o_i is the vertex $(0, n-1-i)$ for $0 \leq i \leq n-1$;
4. destination d_j is the vertex $(j, n-1)$ for $0 \leq j \leq n-1$.

See Figure 1. To get from o_i to d_j takes j horizontal steps and i vertical steps, so there are $\binom{i+j}{i}$ possible paths, as desired.

$$[width = 3.5in]figure01$$

Figure 1: The 4×4 binomial matrix and its associated directed graph.
Origin nodes are enclosed by a circle and destination nodes are enclosed by a square.

An *n-route* in D is a collection of n directed paths, one from each origin, onto the set of destinations. Let X denote the set of indices of our origins and destinations, typically $X = \{0, 1, \ldots, n\}$ or $X = \{1, 2, \ldots, n\}$, and let S_X denote the set of permutations of X. Then every n-route induces a permutation σ in S_X where the ith directed path goes from origin o_i to destination $d_{\sigma(i)}$. For σ in S_X, the number of n-routes associated with σ is the product of the matrix entries corresponding to σ in A, $\prod_{i\in X} a_{i\sigma(i)}$. Figure 2 shows a 4-route associated with the permutation (031)(2) for the matrix of binomial coefficients example. There are $\binom{3}{0}\binom{1}{1}\binom{3}{2}\binom{4}{2} = 18$ 4-routes corresponding to this permutation.

$$[width = 2.0in]figure02$$

Figure 2: One of the eighteen 4-routes associated with the permutation (031)(2).

The permanent of A is the sum of over all permutations σ in S_X of $\prod_{i\in X} a_{i\sigma(i)}$, which counts the n-routes in the associated digraph D. By contrast, the determinant of A, denoted by $\det(A)$ is equal to $\sum_{\sigma\in S_X} \mathrm{sgn}(\sigma)\prod_{i\in X} a_{i\sigma(i)}$, where $\mathrm{sgn}(\sigma)$ is the sign of the permutation. So we interpret the determinant as the number of n-routes induced by even permutations minus the number of n-routes induced by odd permutations. As our next theorem will show, we can simplify the interpretation further by pairing some of the odd and even n-routes.

We call an n-route *intersecting* if there is a vertex of D shared by two of its paths and *nonintersecting* otherwise. For the 4-route given in Figure 2, the vertices of intersection are $(0,1)$, $(0,2)$, $(0,3)$, $(1,2)$, $(1,3)$, and $(2,3)$. For an intersecting n-route, *swapping the tails* of two of its paths at the "first point of intersection" (as described in [1]) introduces a transposition and transforms an even n-route to an odd n-route. Thus, by tailswapping, there is a bijection between even intersecting n-routes and odd intersecting n-routes. This yields the following theorem.

Theorem 1: *Let D be a directed acyclic graph with n designated origin and destination nodes, and let A be the $n\times n$ matrix whose (i,j) entry is the number of paths from the ith origin to the jth destination. Then the determinant of A equals Even(D) $-$ Odd(D), where Even(D) is the number of nonintersecting n-routes corresponding to even permutations, and Odd(D) is the number of* nonintersecting *n-routes corresponding to odd permutations.*

Revisiting the example of $A = \{\binom{i+j}{i}\}$ and its associated digraph D, there is only one nonintersecting n-route and it corresponds to the identity permutation (see Figure 3). Since the identity permutation is even, $\det(A) = 1$.

$$[width = 2.0in]figure03$$

Figure 3: The only nonintersecting 4-route for the digraph given in Figure 1 is associated with the identity permutation. Thus the determinant equals 1.

In fact, we can generalize the binomial determinant by shifting our destination nodes k units to the right. Then $\det\{\binom{i+j+k}{i}\} = 1$ since there remains one nonintersecting n-route on the associated digraph.

In this paper we construct directed graphs that are used to calculate the determinants of matrices involving Fibonacci numbers and generalized Fibonacci numbers. By considering the signed sum of nonintersecting n-routes, we hope to concretely illustrate the beauty and elegance of these determinantal identities.

2. ELEMENTARY FIBONACCI DETERMINANTS

Determinants of matrices with Fibonacci entries have a long history [4, 9, 10]. For $m \geq 1$, Cassini's identity

$$F_{m-1}F_{m+1} - F_m^2 = (-1)^m \tag{1}$$

is classically proved by showing that

$$\begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^m$$

and taking the determinant of both sides. We instead choose to evaluate the determinant on the left directly by counting nonintersecting 2-routes on a related directed graph. We define the *Fibonacci digraph*, $\mathcal{F}_t$ on vertices $v_0, v_1, v_2, \ldots, v_t$ with arcs of the form (v_j, v_{j+1}) (for $0 \le j \le t-1$) called *steps*, and arcs of the form (v_j, v_{j+2}) (for $0 \le j \le t-2$) called *jumps.* The Fibonacci digraph $\mathcal{F}_7$ is illustrated in Figure 4.

$[width = 4.0in]figure04$

Figure 4: The directed graph $\mathcal{F}_7$ associated with Cassini's Fibonacci determinant, $F_{m-1}F_{m+1} - F_m^2 = (-1)^m$, when $m = 7$. The black arcs are required in the nonintersecting 2-route.

It is easy to prove by induction that the number of paths from v_0 to v_j is F_{j+1}. Equivalently, we shall make repeated use of the following: *In the Fibonacci digraph $\mathcal{F}_t$, for all $0 \le i \le j \le t$, the number of paths from v_i to v_j is F_{j-i+1}.* Thus, to combinatorially interpret the matrix

$$A = \begin{bmatrix} F_{m-1} & F_m \\ F_m & F_{m+1} \end{bmatrix},$$

we consider the graph $\mathcal{F}_m$, and designate the origin nodes to be $o_1 = v_1, o_2 = v_0$, and the destinations nodes to be $d_1 = v_{m-1}, d_2 = v_m$. Hence, for $1 \le i, j \le 2$, the number of paths from o_i to d_j is a_{ij}. For example, the number of paths from o_1 to d_1 and d_2 are F_{m-1} and F_m, respectively.

So by Theorem 1, the determinant of A can be calculated by considering the nonintersecting 2-routes in $\mathcal{F}_m$. The only way a 2-route can be nonintersecting is if it consists entirely of jumps. If m is even, the nonintersecting 2-route will be associated with the identity permutation (since o_1 goes to d_1 and o_2 goes to d_2), giving a determinant of 1. If m is odd, the nonintersecting 2-route will be associated with the transposition (12) (since o_1 goes to d_2 and o_2 goes to d_1), giving a determinant of -1. So Cassini's identity given in equation (1) holds.

By allowing the origin and destination nodes to be nonconsecutive, we obtain a generalized form of Cassini's identity. Consider positive integers, m, r, s where $r \le m$. Then the matrix

$$A = \begin{bmatrix} F_{m-r} & F_{m+s-r} \\ F_m & F_{m+s} \end{bmatrix}$$

can be shown to have determinant $(-1)^{m-1-r}F_rF_s$, as follows. In the Fibonacci digraph $\mathcal{F}_{m+s-1}$, we use origin nodes $o_1 = v_r$, $o_2 = v_0$ and destination nodes $d_1 = v_{m-1}$ and $d_2 = v_{m+s-1}$. Hence, for every $1 \le i, j \le 2$, a_{ij} counts the paths from o_i to d_j. Thus by Theorem 1, $\det(A)$ can be calculated by considering the nonintersecting 2-routes in $\mathcal{F}_{m+s-1}$. A nonintersecting 2-route contains some path from v_0 to v_{r-1}, the jumps (v_i, v_{i+2}) for $r-1 \le i \le m-2$, and some path from v_m to v_{m+s-1}. There are F_rF_s such 2-routes, and the parity of $m-r$ determines which permutation is associated with the 2-route; that is, o_1 goes to d_1 in our nonintersecting 2-route if and only if $m-1-r$ is even. So

$$F_{m-r}F_{m+s} - F_mF_{m+s-r} = \det(A) = (-1)^{m-1-r}F_rF_s. \tag{2}$$

$[width = 4.0in]figure05$

Figure 5: The directed graph associated with Cassini's generalized Fibonacci determinant, $F_{m-r}F_{m+s} - F_mF_{m+s-r} = (-1)^{m+1-r}F_rF_s$. The black arcs are required in a nonintersecting 2-route.

Increasing the number of origin and destination nodes under consideration in a Fibonacci digraph changes the size of the associated matrix. For example the determinant,

$$\det \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \\ F_{m+1} & F_{m+2} & F_{m+3} \\ F_{m+2} & F_{m+3} & F_{m+4} \end{bmatrix}, \tag{3}$$

is associated with the nonintersecting 3-routes of $\mathcal{F}_{m+3}$ with origins v_2, v_1, v_0 and destinations $v_{m+1}, v_{m+2}, v_{m+3}$. Algebraically, the determinant is equal to zero because the third row is the sum of the first two. Combinatorially, the determinant is equal to zero because it is impossible to create a nonintersecting 3-route on $\mathcal{F}_{m+3}$.

By spreading out the locations of the origin and destination nodes, we see that a more general 3×3 Fibonacci matrix has determinant equal to zero. Specifically if $0 \le r \le s$ and $0 \le p \le q$, the digraph $\mathcal{F}_{m+q+s-1}$ with origins $o_1 = v_s, o_2 = v_{s-r}, o_3 = v_0$ and destinations $d_1 = v_{m+s-1}, d_2 = v_{m+p+s-1}, d_3 = v_{m+q+s-1}$ has no nonintersecting 3-routes. See Figure 6. So

$$\det \begin{bmatrix} F_m & F_{m+p} & F_{m+q} \\ F_{m+r} & F_{m+p+r} & F_{m+q+r} \\ F_{m+s} & F_{m+p+s} & F_{m+q+s} \end{bmatrix} = 0, \tag{4}$$

which is not as obvious from the algebraic viewpoint. Similarly, Fibonacci matrices of higher order created in the same way must also have determinants equal to zero.

$[width = 4.0in]figure06$

Figure 6: There are no nonintersecting 3-routes on a Fibonacci digraph.

3. GENERALIZING FIBONACCI DETERMINANTS

The Gibonacci Digraph

A shrewd mathematician knows that lurking beneath any Fibonacci identity are related identities involving *Gibonacci* numbers. Next we modify the Fibonacci digraph $\mathcal{F}_t$ to obtain some generalizations.

Recall that a Gibonacci sequence $G_0, G_1, \dots, G_t, \dots$ is defined by initial conditions G_0, G_1 and the recurrence $G_t = G_{t-1} + G_{t-2}$ for $t \ge 2$. (When the initial conditions are $G_0 = 0$ and $G_1 = 1$, these are the Fibonacci numbers.) It is easy to see, by induction on t, that for all $t \ge 1$,

$$G_t = G_1F_t + G_0F_{t-1} \tag{5}$$

and

$$G_{r+t} = G_{r+1}F_t + G_rF_{t-1}. \tag{6}$$

For simplicity, we assume that G_0 and G_1 are positive integers, but we shall remove that assumption later. Now let S be a subset of $\{0, 1, \dots, t-2\}$. The *Gibonacci digraph*, $\mathcal{G}_t(S)$, includes a copy of $\mathcal{F}_t$ on vertices $v_0, v_1, v_2, \dots, v_t$. Additionally, for each element i in S, we include a vertex v_i', along with G_1 copies of the arc (v_i', v_{i+1}) and G_0 copies of the arc (v_i', v_{i+2}). See Figure 7. In all of our applications, S will contain node 0, so that v_0' will be a vertex of our digraph. By considering the first step from v_0', the number of paths from v_0' to v_s is $G_1F_s + G_0F_{s-1} = G_s$ by equation (5). (Notice that when $G_0 = 0$ and $G_1 = 1$, there are no arcs from v_0' to v_2, and so this amounts to the number of paths from v_1 to v_s in the Fibonacci digraph, namely F_s.) In general, for i in S and $t > i$, *the number of paths from* v_i' *to* v_t *is* G_{t-i}.

$$[width = 4.0in]figure07$$

Figure 7: The Gibonacci digraph $\mathcal{G}_7(\{0,1\})$ associated with Cassini's Gibonacci identity.

For the Gibonacci version of Cassini's identity, we calculate the determinant of

$$A = \begin{bmatrix} G_{m-1} & G_m \\ G_m & G_{m+1} \end{bmatrix},$$

by considering nonintersecting 2-routes in $\mathcal{G}_{m+1}(\{0,1\})$ with origins $o_1 = v_1'$, $o_2 = v_0'$ and destinations $d_1 = v_m$, $d_2 = v_{m+1}$. A nonintersecting 2-route contains the arcs (v_i, v_{i+2}) for $2 \le i \le m-1$. See Figure 8.

$$[width = 4.0in]figure08$$

Figure 8: The directed graph associated with Cassini's generalized Gibonacci determinant,
$G_{m-1}G_{m+1} - G_m^2 = (-1)^{m-1}(G_0G_2 - G_1^2)$.
The black arcs are required in a nonintersecting 2-route.

So it remains to determine how origins v_1' and v_0' connect to the rest of the 2-route through the vertices v_2 and v_3 in a nonintersecting way. There are G_0 ways for v_1' to be incident to v_3 and G_2 paths from v_0' to v_2. Hence there are G_0G_2 nonintersecting 2-routes that use an arc (v_1', v_3). The parity of this 2-route is determined by the parity of m, since o_1 goes to d_1 if and only if m is odd; hence all of these 2-routes have parity $(-1)^{m-1}$. Likewise, there are G_1 ways for v_1' to be incident to v_2 and G_1 paths from v_0' to v_3 that can complete a nonintersecting 2-route. Thus there are G_1^2 nonintersecting 2-routes of this type, all of which have parity $(-1)^m$. So the determinant becomes $(-1)^{m-1}G_0G_2 + (-1)^m G_1^2$ giving the identity

$$G_{m-1}G_{m+1} - G_m^2 = \det(A) = (-1)^{m-1}(G_0G_2 - G_1^2). \tag{7}$$

We can again obtain a more general theorem, by spreading out the origins and destinations. For positive integers m, r, s with $r \le m$, we evaluate the determinant of

$$A = \begin{bmatrix} G_{m-r} & G_{m+s-r} \\ G_m & G_{m+s} \end{bmatrix}$$

by letting $o_1 = v_r'$, $o_2 = v_0'$, $d_1 = v_m$, and $d_2 = v_{m+s}$ in $\mathcal{G}_{m+s}(\{0,r\})$. A nonintersecting 2-route in this digraph contains the arcs (v_i, v_{i+2}) for $r+1 \le i \le m-1$ and one of F_s paths between v_{m+1} and v_{m+s}. There are G_0 ways for v_r' to be incident to v_{r+2} and G_{r+1} paths from v_0' to v_{r+1}. The parity of $m-r$ determines which permutation is associated with these 2-routes. There are G_1 ways for v_r' to be incident to v_{r+1} and G_r paths from v_0' to v_r that can be used to complete a nonintersecting 2-route. These 2-routes have opposite sign. So the determinant gives the identity

$$\begin{aligned} G_{m-r}G_{m+s} - G_m G_{m+s-r} = \det(A) &= (-1)^{m-r}(G_0 G_{r+1} - G_r G_1)F_s \\ &= (-1)^{m-r}(G_0 G_{r+s} - G_r G_s) \end{aligned}$$

where the second equality can be shown using equations (5) and (6). In fact, this argument can be generalized one step further. Let H_t be another Gibonacci sequence with initial conditions H_0 and H_1. Modify the digraph $\mathcal{G}_{m+s}(\{0,r\})$ by assigning H_1 arcs from v_0' to v_1 and H_0 arcs from v_0' to v_2. Then virtually the same argument leads to

$$\begin{aligned} \det \begin{bmatrix} G_{m-r} & G_{m+s-r} \\ H_m & H_{m+s} \end{bmatrix} &= (-1)^{m-r}(G_0 H_{r+1} - G_r H_1)F_s \\ &= (-1)^{m-r}(G_0 H_{r+s} - G_r H_s). \end{aligned}$$

As in the case of the general 3×3 Fibonacci determinant given in (4), it is impossible to create nonintersecting n-routes for $n \geq 3$ in a Gibonacci digraph, provided the origin nodes precede the destination nodes. Consequently,

$$\det \begin{bmatrix} G_m & G_{m+p} & G_{m+q} \\ G_{m+r} & G_{m+p+r} & G_{m+q+r} \\ G_{m+s} & G_{m+p+s} & G_{m+q+s} \end{bmatrix} = 0, \tag{8}$$

or more generally, for Gibonacci sequences G_t, H_t, and I_t,

$$\det \begin{bmatrix} G_m & G_{m+p} & G_{m+q} \\ H_{m+r} & H_{m+p+r} & H_{m+q+r} \\ I_{m+s} & I_{m+p+s} & I_{m+q+s} \end{bmatrix} = 0. \tag{9}$$

Similarly, Gibonacci matrices of higher order created in the same way must also have determinants equal to zero.

Combining Binomial and Fibonacci Digraphs

Now that we understand nonintersecting routes in the binomial digraph from Figure 1 and the Fibonacci digraph, $\mathcal{F}_m$, it is reasonable to investigate what happens when we combine them. For example, we can attach the Fibonacci digraph $\mathcal{F}_m$ to the top of a binomial lattice to create $\mathcal{H}_m$ as illustrated in Figure 9.

$[width = 4.0in]figure09$

Figure 9: The digraph $\mathcal{H}_m$ contains the $2 \times m$ binomial digraph plus the arcs $((i,1),(i+2,1))$ for $0 \leq i \leq m-2$.

Assign origins $o_1 = (0,1)$, $o_2 = (0,0)$, and destinations $d_1 = (m-1,1)$, $d_2 = (m,1)$. The number of paths from o_1 to d_1 and d_2 are F_m and F_{m+1} respectively. Any path from o_2 to the destinations must use exactly one vertical arc, which we interpret as the first jump of the path. Since jumps have length 2, the number of paths from o_2 to d_1 equals the number of paths of length $m+1$ that use at least one jump. Thus the number of paths from o_2 to d_1 and d_2 are $F_{m+2}-1$ and $F_{m+3}-1$, respectively. So we are ready to compute the determinant of

$$A = \begin{bmatrix} F_m & F_{m+1} \\ F_{m+2}-1 & F_{m+3}-1 \end{bmatrix}. \tag{10}$$

A nonintersecting 2-route in $\mathcal{H}_m$ uses one arc for the form $((q,0),(q,1))$, $1 \le q \le m$. There are F_q paths from $(0,1)$ to $(q-1,1)$ and the rest of the nonintersecting 2-route is uniquely determined. If m and q have the same parity, the associated permutation is even and the contribution to the determinant is $+F_q$. If m and q have opposite parity, the contribution is $-F_q$. So $Even(\mathcal{H}_m) - Odd(\mathcal{H}_m) = \sum_{q=1}^{m}(-1)^{m-q}F_q = (-1)^{m-1} + F_{m-1}$. The last equality comes from the familiar alternating Fibonacci sum identity

$$\sum_{q=1}^{m}(-1)^q F_q = -1 + (-1)^m F_{m-1}, \tag{11}$$

which can be proved combinatorially (as done in [3]) or by induction on m. Therefore, $\det(A) = (-1)^{m-1} + F_{m-1}$.

It is natural to generalize the graph. Begin by increasing the lattice from $(0,0)$ to $(m,2)$ and define the vertices $o_1 = (0,2)$, $o_2 = (0,0)$, $d_1 = (m-1,2)$, and $d_2 = (m,2)$. This digraph corresponds to the matrix

$$A = \begin{bmatrix} F_m & F_{m+1} \\ F_{m+4}-(m+3) & F_{m+5}-(m+4) \end{bmatrix}$$

since any path from o_2 to the destinations must use at least two jumps. A nonintersecting 2-route uses one arc for the form $((q,1),(q,2))$, $1 \le q \le m$. Given q, there are F_q paths from $(0,2)$ to $(q-1,2)$, there are $q+1$ choices for the first vertical arc, and the rest of the nonintersecting 2-route is uniquely determined. If m and q have the same parity, the associated permutation is even and the contribution to the determinant is $+(q+1)F_q$. If m and q have opposite parity, the contribution is $-(q+1)F_q$. So the determinant of A is $\sum_{q=1}^{m}(-1)^{m-q}(q+1)F_q$. Using the identity

$$\sum_{q=1}^{m} qF_q(-1)^q = (-1)^m(mF_{m-1} + F_{m-3}) - 2 \tag{12}$$

which can also be proved combinatorially or by induction on m, we can derive the determinant of A to be $(m+1)F_{m-1} + F_{m-3} - 3\cdot(-1)^m$.

Now consider the same graph with three origins $o_1 = (0,2)$, $o_2 = (0,1)$, $o_3 = (0,0)$, and three destinations $d_1 = (m-2,2)$, $d_2 = (m-1,2)$, and $d_3 = (m,2)$.
$[width = 4.0in]figure10$

Figure 10: The $3 \times m$ binomial digraph plus the arcs $((i,2),(i+2,2))$ for $0 \le i \le m-2$.

This digraph corresponds to the determinant of

$$A = \begin{bmatrix} F_{m-1} & F_m & F_{m+1} \\ F_{m+1}-1 & F_{m+2}-1 & F_{m+3}-1 \\ F_{m+3}-(m+2) & F_{m+4}-(m+3) & F_{m+5}-(m+4) \end{bmatrix}.$$

In a nonintersecting 3-route, o_3 must map to d_3 using the arc $((m,1),(m,2))$. Then the path from o_2 must use an arc $((q,1),(q,2))$, for $1 \le q \le m-1$. Given q, the path from o_3 must use an arc $((r,0),(r,1))$, for $q+1 \le r \le m$ and there are F_q paths from o_1 to $(q-1,2)$. If m and q have opposite parity, the associated permutation is even and the contribution to the determinant is $+(m-q)F_q$. If m and q have the same parity, the contribution is $-(m-q)F_q$. Counting the nonintersecting 3-routes and using equations (11) and (12), we find that

$$\det(A) = \sum_{q=1}^{m-1} (-1)^{m+1-q}(m-q)F_q = F_{m-3} + (-1)^m(m-2).$$

Clearly there is more work to be done increasing the height of the lattice and the distance between destinations.

Weighting the Arcs

Suppose we take the matrix given in (3) and add a fixed constant k to each entry,

$$\begin{bmatrix} F_m + k & F_{m+1} + k & F_{m+2} + k \\ F_{m+1} + k & F_{m+2} + k & F_{m+3} + k \\ F_{m+2} + k & F_{m+3} + k & F_{m+4} + k \end{bmatrix}.$$

By adding weighted arcs to a directed graph, we will be able to calculate this determinant even when k is not an integer. In the digraph given in Figure 11, all arcs have weight 1, except for arc (o_1, w) of weight k and arc (w, d_3) of weight -1.
$[width = 4.0in]figure11$

Figure 11: Adding vertices and weighted arcs to the Fibonacci digraph changes the determinant.

The *weight* of an n-route is the product of the weights of the arcs used. The 3-route in Figure 11 has weight $-k$. Switching destinations at a point of intersection preserves the weight of an n-route but changes the sign of the associated permutation. So just as in the unweighted situation, the determinant will be the sum of the weights of the even nonintersecting n-routes minus the sum of the weights of the odd nonintersecting n-routes.

Figure 11 is the right picture to verify that

$$\det(A) = \det \begin{bmatrix} F_m + k & F_{m+1} + k & F_{m+2} + k \\ F_{m+1} + k & F_{m+2} + k & F_{m+3} + k \\ F_{m+2} + k & F_{m+3} + k & F_{m+4} + k \end{bmatrix} = (-1)^{m+1}k.$$

To see that the matrix entries count weighted paths from origins to destinations, we verify total weights for the paths starting at the first and second origins. From o_1 to d_1, there are F_m paths of weight 1 and one path of weight k, for a total weight of $F_m + k$. From o_1 to d_2 there are F_{m+1} paths of weight 1 and one path of weight k. From o_1 to d_3 there are F_{m+2} paths of weight 1, *two* paths of weight k, and one path of weight $-k$, for a total weight of $F_{m+2} + k$. Starting from o_2 with an initial horizontal step, the total weight of paths to d_1 is, as before, $F_m + k$, and the number of paths after an initial downward step to v_3 is F_{m-1}; hence the total weight of all paths from o_3 to d_1 is $F_m + k + F_{m-1} = F_{m+1} + k$. The other calculations are derived in a similar way.

The unique nonintersecting 3-route sends o_1 to d_3 via the path of weight $-k$. Origins o_2 and o_3 "jump" to d_1 and d_2 by paths of weight 1. If m is odd, then o_2 goes to d_2 and the resulting permutation, in cycle notation (13), is odd. Otherwise the permutation is (132), which is even. Hence the determinant of A is $(-1)^m(-k) = (-1)^{m+1}k$, as desired.

This identity easily generalizes to Gibonacci entries, even with noninteger initial conditions. The digraph given in Figure 12 can be used to verify

$$\det \begin{bmatrix} G_m + k & G_{m+1} + k & G_{m+2} + k \\ G_{m+1} + k & G_{m+2} + k & G_{m+3} + k \\ G_{m+2} + k & G_{m+3} + k & G_{m+4} + k \end{bmatrix} = (-1)^m k(G_0G_2 - G_1^2). \tag{13}$$

$[width = 4.0in]figure12$

Figure 12: Adding weighted arcs to the Gibonacci digraph. The black arcs are required in a nonintersecting 3-route.

As before, a nonintersecting 3-route sends o_1 to d_3 via the path of weight $-k$. After using the initial weighted arcs, the paths from o_2 and o_3 to destinations d_1 and d_2 are completely determined. If the path from o_2 has weight G_1, then the path from o_3 must also have weight G_1. If the path from o_2 has weight G_0, then the path from o_3 has weight G_0 or G_1 (and thus the weight of all such paths is $G_0 + G_1 = G_2$), but the final destinations will be transposed. The sign of the associated permutation is determined by the parity of m. Hence, the signed sum of the weighted 3-routes is $(-1)^m k(G_0G_2 - G_1^2)$, as promised.

We note that this technique can be applied to our earlier determinant identities with Gibonacci numbers. By replacing the G_0 or G_1 arcs of weight 1 in Figures 7 and 8 with a single arc of weight G_0 or G_1, the earlier identities remain true with essentially the same argument, even when the initial conditions are not positive integers.

The directed graph in Figure 12 is not the most intuitive representation for the determinant in equation (13). However, it has the advantage of a simple analysis of nonintersecting 3-routes. A more intuitive representation is the digraph in Figure 13 which takes $\mathcal{G}_{m+4}(\{0, 1, 2\})$ and adds the vertices x, w, the arcs (v_2', x), (v_1', x), (v_0', x), (w, v_{m+2}) of weight one, the arc (x, w) of weight k, and the arc (w, v_{m+4}) of weight -1. The analysis of nonintersecting 3-routes must now consider which of the three origin vertices gets mapped to d_3 via the path of weight $-k$. The advantage here is that we can generalize equation (13) by introducing different weights at each origin. See Figure 13. Let H_t and I_t be Gibonacci sequences with initial conditions H_0, H_1, and I_0, I_1 respectively. We evaluate the determinant of

$$A = \begin{bmatrix} G_m + k & G_{m+1} + k & G_{m+2} + k \\ H_{m+1} + k & H_{m+2} + k & H_{m+3} + k \\ I_{m+2} + k & I_{m+3} + k & I_{m+4} + k \end{bmatrix} \tag{14}$$

by counting nonintersecting 3-routes in Figure 13.

$[width = 4.0in]figure13$

Figure 13: A more intuitive and more general digraph than Figure 12. The black arcs are required in all nonintersecting 3-routes.

Suppose a nonintersecting 3-route sends o_1 to d_3 via the path of weight $-k$. After using the initial weighted arcs, the paths from o_2 and o_3 to destinations d_1 and d_2 are completely determined. If the path from o_2 has weight H_1, then the path from o_3 has weight I_1. If the path from o_2 has weight H_0, then the path from o_3 has weight I_0 or I_1 (and thus the weight of all such paths is I_2), but the final destinations will be transposed. The sign of the associated permutation is determined by the parity of m. Hence, the contribution towards the signed sum of the weighted 3-routes is $(-1)^m k(H_0I_2 - H_1I_1)$. A similar analysis shows a contribution of $(-1)^m k(G_0I_3 - G_1I_2)$ when o_2 maps to d_3 via the path of weight $-k$ and a contribution of $(-1)^{m+1}k(G_0H_2 - G_1H_1)$ when o_3 maps to d_3 via the path of weight $-k$. Thus

$$\det(A) = (-1)^m k[(H_0I_2 - H_1I_1) + (G_0I_3 - G_1I_2) - (G_0H_2 - G_1H_1)]. \quad (15)$$

Since $G_0G_3 - G_1G_2 = G_0G_2 - G_1^2$, the determinant in (15) specializes to (13) when the Gibonacci sequences coincide. Further, selecting initial conditions $H_0 = G_{r-1}$, $H_1 = G_r$, $I_0 = G_{s-1}$, and $I_1 = G_s$ has the effect of increasing the distance between the origins in the Gibonacci digraph.

More Intriguing Determinants

So far, the matrix entries for our determinants have had increasing indices across rows and columns. This property has facilitated the construction of representative digraphs on which the consideration of nonintersecting n-routes is manageable. Next, we wish to consider a Fibonacci matrix that does not satisfy the increasing index property. The representative digraph increases in complexity and our strategy to pair intersecting n-routes of opposite signs, expands to pair nonintersecting n-routes of opposite signs whenever possible.

The following identity first appeared as a problem in the *Fibonacci Quarterly* [11].

$$\det(A) = \det \begin{bmatrix} F_{m+3} & F_{m+2} & F_{m+1} & F_m \\ F_{m+2} & F_{m+3} & F_m & F_{m+1} \\ F_{m+1} & F_m & F_{m+3} & F_{m+2} \\ F_m & F_{m+1} & F_{m+2} & F_{m+3} \end{bmatrix} = F_{2m}F_{2m+6}. \quad (16)$$

To combinatorially see the determinant, consider the digraph given in Figure 14.

$[width = 4.0in]figure14$

Figure 14: This digraph is used to prove equation (16). For convenience, unlabeled vertices will be referred to by position. So $v_{s,r}$ is the vertex in column s on level r.

A path from o_1 or o_4 to any destination remains at the same level (call it *level one* or *level four*) until its final (vertical) arc. The number of paths from these origins to a particular destination is the Fibonacci number associated with the length of the path. Paths from o_2 to d_1 or d_3 must travel along level one. However a path from o_2 to d_2 may travel along level one or may remain at level two until its final arc. There are F_{m+1} level one paths and F_{m+2} level two paths from o_2 to d_2, for a total of F_{m+3} paths. The number of paths from o_2 to d_4, o_3 to d_1, and o_3 to d_3 can be found similarly.

Counting nonintersecting 4-routes in Figure 14 remains a daunting task. To simplify it, we classify nonintersecting 4-routes according to the number of levels used and then pair off some of those having opposite sign.

Table 1 counts the nonintersecting 4-routes using exactly two levels, namely level one and level four. Notice that the permutations associated with these eight 4-routes depend on the parity of m, but either way, the total signed sum remains the same at -4.

m odd		m even	
Permutation	Signed number of 4-routes	Permutation	Signed number of 4-routes
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_4 & d_3 & d_2 & d_1 \end{pmatrix}$	$+1$	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_4 & d_1 & d_2 \end{pmatrix}$	$+1$
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_4 & d_2 & d_3 & d_1 \end{pmatrix}$	-1	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_2 & d_4 & d_1 & d_3 \end{pmatrix}$	-1
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_4 & d_1 & d_2 & d_3 \end{pmatrix}$	-2	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_4 & d_3 & d_2 \end{pmatrix}$	-2
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_2 & d_3 & d_4 & d_1 \end{pmatrix}$	-2	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_2 & d_1 & d_4 \end{pmatrix}$	-2
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_2 & d_1 & d_4 & d_3 \end{pmatrix}$	$+1$	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$	$+1$
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_3 & d_2 & d_4 \end{pmatrix}$	-1	$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_1 & d_4 & d_2 \end{pmatrix}$	-1
Total	-4	Total	-4

Table 1: Counting nonintersecting 4-routes restricted to levels one and four in Figure 14 .

Next we consider the 4-routes using three or more levels. Let $P = (P_1, P_2, P_3, P_4)$ be a 4-route where σ is the associated permutation and P_i is the path from o_i to $d_{\sigma(i)}$ $(i = 1, \dots, 4)$. We show that the signed sum of 4-routes having $|\sigma(1)-\sigma(2)| = 2$ is zero. If $|\sigma(1)-\sigma(2)| = 2$ and P_2 is a level two path, moving P_1 to level two and P_2 to level one exchanges the destinations of o_1 and o_2, producing a 4-route of opposite sign. So $P' = (P_2, P_1, P_3, P_4)$ is a 4-route associated to $\sigma' = \sigma(12)$. If $|\sigma(1)-\sigma(2)| = 2$ and P_2 is not a level two path, then P_3 must be a level three path. Moving P_3 to level four and P_4 to level three exchanges the destinations of o_3 and o_4, producing a 4-route of opposite sign. On the set of 4-routes using three or more levels with the restriction that $|\sigma(1)-\sigma(2)| = 2$, this exchange produces a bijection between the even and odd members. Thus it remains to consider only those 4-routes on three or more levels where $|\sigma(1) - \sigma(2)| = 1$ or 3.

Table 2 counts the nonintersecting 4-routes using all four levels where $|\sigma(1)-\sigma(2)|$ is odd. Notice that the signed sum is independent of the parity of m.

Permutation	Signed number of 4-routes	Configurations
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$	$+F_{m+2}^2 F_{m+3}^2$	[width=1.5in]table2image1
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_1 & d_4 & d_3 & d_2 \end{pmatrix}$	$-F_m F_{m+1} F_{m+2} F_{m+3}$	[width=1.5in]table2image2
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_2 & d_1 & d_4 \end{pmatrix}$	$-F_m F_{m+1} F_{m+2} F_{m+3}$	[width=1.5in]table2image3
$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ d_3 & d_4 & d_1 & d_2 \end{pmatrix}$	$+F_m^2 F_{m+1}^2$	[width=1.5in]table2image4
Total	$(F_{m+2}F_{m+3} - F_m F_{m+1})^2$	

Table 2: Counting nonintersecting 4-routes using all four levels in Figure 14 where the destination indices of o_1 and o_2 differ by an odd number.

The next step is to pair off the remaining 4-routes, which use exactly three levels and satisfy $|\sigma(1)-\sigma(2)| = 1$ or 3. We present the details for 4-routes using level two—the symmetry of the digraph guarantees that the argument using level three is similar. Suppose $\mathcal{P}$ is the set of nonintersecting 4-routes using levels one, two, and four with $|\sigma(1) - \sigma(2)|$ odd. If we partition $\mathcal{P}$ into subsets $\mathcal{P}_{ij}$ where $\sigma(1) = i$ and $\sigma(2) = j$, we find that $\mathcal{P} = \mathcal{P}_{12} \cup \mathcal{P}_{14} \cup \mathcal{P}_{32} \cup \mathcal{P}_{34}$. The three correspondences below completely match the even and odd elements of $\mathcal{P}$.

1. *$\mathcal{P}_{34}$ is in one-to-one correspondence with members of $\mathcal{P}_{14}$ where the level one path ends with two steps.*

   Given a 4-route in $\mathcal{P}_{34}$, exchange the roles of d_1 and d_3 to create a 4-route in $\mathcal{P}_{14}$. Add two steps $(v_{m,1}, v_{m+1,1})$ and $(v_{m+1,1}, v_{m+2,1})$. Extend the path that previously terminated at d_1 by the jump $(v_{m+2,4}, v_{m,4})$. Move the terminal vertical arcs appropriately. The associated permutations differ by the transposition (13) and so they are of opposite sign. See Figure 15.

   $$[width = 4.0in]figure15$$

   Figure 15: Illustrating the bijections between $\mathcal{P}_{34}$ and members of $\mathcal{P}_{14}$ where the level one path ends with two steps.

2. *$\mathcal{P}_{32}$ is in one-to-one correspondence with members of $\mathcal{P}_{12}$ where the level one path uses the vertex $v_{m,1}$.*

   Given a 4-route in $\mathcal{P}_{32}$, exchange the roles of d_1 and d_3 to create a 4-route in $\mathcal{P}_{12}$. On level four, take the arcs in the length two path from $v_{m+1,4}$ to $v_{m-1,4}$ and use these arcs to extend the path from $v_{m,1}$ to $v_{m+2,1}$. Add the jumps $(v_{m+2,4}, v_{m,4})$ and $(v_{m+1,4}, v_{m-1,4})$. Move the terminal vertical arcs appropriately. See Figure 16.

   $$[width = 4.0in]figure16$$

Figure 16: Illustrating the bijections between $\mathcal{P}_{32}$ and members of $\mathcal{P}_{12}$ where the level one path uses the vertex $v_{m,1}$.

3. *The members of* $\mathcal{P}_{14}$ *where the level one path does not end with two steps are in one-to-one correspondence with the members of* $\mathcal{P}_{12}$ *where the level one path does not pass through vertex* $v_{m,1}$.

Given a 4-route in $\mathcal{P}_{14}$ that does not end in two steps, exchange the roles of d_2 and d_4 to create a 4-route in $\mathcal{P}_{12}$. The path on level one has the form $Q(v_{m,1}, v_{m+2,1})$ or $Q'(v_{m-1,1}, v_{m+1,1})(v_{m+1,1}, v_{m+2,1})$ where Q is a path of length m and Q' is a path of length $m-1$. Level two contains a path of length $m-1$, call it R. Move R to level one and insert the jump $(v_{m-1,1}, v_{m+1,1})$ and the step $(v_{m+1,1}, v_{m+2,1})$. The path on level two becomes either $Q(v_{m+1,2}, v_{m+2,2})$ or $Q'(v_{m,2}, v_{m+2,2})$ depending on the form of the original path on level one. Finally, add a jump $(v_{m+1,4}, v_{m-1,4})$. Move the terminal vertical arcs appropriately. See Figure 17.

$$[width = 4.0in]figure17$$

Figure 17: Illustrating the bijections between members of $\mathcal{P}_{14}$ where the level one path does not end and members of $\mathcal{P}_{12}$ where the level one path does not pass through vertex $v_{m,1}$.

Each correspondence is reversible and exchanges two destinations, so the permutations associated to the paired 4-routes have opposite sign and their signed sum will be zero. The only 4-routes to contribute to the determinant are those counted in Tables 1 and 2. Thus $\det(A) = (F_{m+2}F_{m+3} - F_mF_{m+1})^2 - 4$. Identities 15 and 88 from [2] can be used to simplify the result to $(F_{m+2}F_{m+3} - F_mF_{m+1})^2 - 4 = F_{2m+3}^2 - F_3^2 = F_{2m}F_{2m+6}$.

We can extend the determinant in (16) to have Gibonacci entries. The modified digraph weights the arcs from each origin vertex as shown in Figure 18.

$$[width = 4.5in]figure18$$

Figure 18: This digraph is used to prove equation (17). The arcs exiting each origin have weight G_0 or G_1 as indicated above.

Using an argument similar to the proof of equation (7), each nonintersecting 4-route given in Table 1 contributes $(G_0G_2 - G_1^2)^2$ towards the signed sum of the weighted 4-routes of the modified graph. Using three or more levels, the same bijections match 4-routes of equal weight but opposite sign. Using exactly three levels when $|\sigma(1) - \sigma(2)|$ is odd changes the entries in Table 2 from Fibonacci numbers to Gibonacci numbers. We conclude that

$$\det \begin{bmatrix} G_{m+3} & G_{m+2} & G_{m+1} & G_m \\ G_{m+2} & G_{m+3} & G_m & G_{m+1} \\ G_{m+1} & G_m & G_{m+3} & G_{m+2} \\ G_m & G_{m+1} & G_{m+2} & G_{m+3} \end{bmatrix} = (G_{m+2}G_{m+3} - G_mG_{m+1})^2 - 4(G_0G_2 - G_1^2)^2. \quad (17)$$

While every determinant can be represented by some weighted digraph, the focus of this paper was coming up with particular presentations that made the computations more intuitive and generalizable. Further work remains designing useful digraphs to discover and understand other identities.

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