The Fibonacci operator approach inspired by Andrews (2004) is explored to investigate $q$-analogs of the generalized Fibonacci and Lucas polynomials introduced by Chu and Vicenti (2003). Their generating functions are compactly expressed in terms of Fibonacci operator fractions. A determinant evaluation on $q$-binomial coefficients is also established which extends a recent result of Sun (2005).

1. INTRODUCTION

The generalized Fibonacci and Lucas polynomials are defined in [11] by

$$F_{n+1}(t) = F_n(t) + tF_{n-1}(t), \quad n \geq 1$$

with the initial conditions $F_0(t) = a$ and $F_1(t) = b$. When $t = 1$, they reduce, for $a = b = 1$ and $a = 2$ and $b = 1$, to Fibonacci and Lucas sequences, respectively, which have been extensively studied for their many beautiful and interesting combinatorial properties.

For the case $a = b = 1$, several slightly different $q$-analogs of $F_n(t)$ have been worked out by Carlitz [4], Cigler [7] and Schur [10]. On the related literature of recurrence relations and generating functions, refer to [1, 8] for the theory of orthogonal polynomials and [1, 2, 8, 9] for the Rogers-Ramanujan identities.

Differently from the works just mentioned, Andrews [3] recently introduced the Fibonacci operator $\eta_x$ by $\eta_x f(x) = f(xq)$ for any given function $f(x)$. Then he obtained an unusual operator expression for the generating function of $q$-Fibonacci polynomials. Inspired by this operator approach, we shall study the full $q$-analogue of $F_n(t)$ for $a$ and $b$ be arbitrary numbers and establish the corresponding generating functions in terms of $\eta$-operator fractions. Then we shall evaluate a determinant related $q$-binomial coefficients. Finally for some particular values of $a$ and $b$, we shall give $q$-analogs of some generating functions established in [6], again in terms of $\eta$-operator fractions. We believe that these results on the $q$-incomplete Fibonacci and Lucas polynomials are new.

For two indeterminate $x$ and $q$, the shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - q^k x) \quad \text{with} \quad n = 1, 2, \ldots.$$
When $|q| < 1$, the infinite product

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n x)$$

is well defined, which leads us to the following expression

$$(x; q)_n = \frac{(x; q)_\infty}{(q^n x; q)_\infty} \text{ for } n \in \mathbb{Z}.$$ 

The Gaussian $q$-binomial coefficient is defined by

$$\left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{(q)_n}{(q; q)_m (q; q)_{n-m}}, \quad 0 \leq m \leq n,$$

otherwise.

## 2. $q$-ANALOGS OF THE GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

The $q$-analogs of generalized Fibonacci and Lucas polynomials are introduced by [3]. Let us define a sequence of polynomial $S_n(t, q)$ by the recurrence relation

$$S_{n+1}(t, q) = S_n(t, q) + t q^{n-2} S_{n-1}(t, q), \quad n \geq 1 \quad (2)$$

where $S_0(t, q) = a, S_1(t, q) = b$. It is obvious that $S_n(t, 1) = F_n(t)$ with $F_n(t)$ being defined by (1).

**Theorem 1**: (The generating function defined by recurrence relation (2)).

$$\sum_{n=0}^{\infty} S_n(t, q) x^n = \frac{1}{1 - x - t x^2 \eta_x} \{a + (b - a)x\}.$$ 

**Proof**: Let $\sigma(x)$ stand for the expression on the left side of the equation in Theorem 1. To prove Theorem 1, we need to check the following equivalent relation:

$$(1 - x - t x^2 \eta_x) \sigma(x) = a + (b - a)x.$$ 

According to the definition of $\sigma(x)$, we have

$$a + bx + \sum_{n \geq 2} S_n(t, q) x^n - \sum_{n \geq 0} S_n(t, q) x^{n+1} - t \sum_{n \geq 0} S_n(t, q) x^{n+2} q^n$$

$$= a + bx - ax + \sum_{n \geq 2} \{S_n(t, q) - S_{n-1}(t, q) - t q^{n-2} S_{n-2}(t, q)\} x^n$$

which reduces to $a + (b - a)x$ in view of recurrence relation (2).
In order to find explicit expression for $S_n(t, q)$, we will need the following lemma.

**Lemma 2**: (The Fibonacci operator composition)

\[
(x + tx^2 \eta_x)^n x = \sum_{j \geq 0} t^j x^j \left[ \begin{array}{c} n \\ j \end{array} \right] q^{j(j+1)}.
\]

**Proof**: We can proceed with induction principle. For $n = 0$, the first equation asserts $x^2 = x^2$. Now suppose the first equation is true for $n$. Then we can verify it for $n + 1$ as follows:

\[
(x + tx^2 \eta_x)^{n+1} x = (x + tx^2 \eta_x)x^{n+2} \sum_{j \geq 0} t^j x^j \left[ \begin{array}{c} n \\ j \end{array} \right] q^{j(j+1)}
\]

\[
= x^{n+3} \sum_{j \geq 0} t^j x^j \left[ \begin{array}{c} n+1 \\ j \end{array} \right] q^{j(j+1)} + x^{n+4} \sum_{j \geq 0} t^{j+1} x^j \left[ \begin{array}{c} n \\ j \end{array} \right] q^{n+2+j(j+2)}
\]

\[
= x^{n+3} \sum_{j \geq 0} t^j x^j \left[ \begin{array}{c} n+1 \\ j \end{array} \right] q^{j(j+1)} + x^{n+3} \sum_{j \geq 0} t^j x^j \left[ \begin{array}{c} n \\ j-1 \end{array} \right] q^{n+1+j^2}
\]

\[
= x^{n+3} \sum_{j \geq 0} t^j x^j q^{j(j+1)} \left\{ \left[ \begin{array}{c} n \\ j \end{array} \right] + q^{n+1-j} \left[ \begin{array}{c} n \\ j-1 \end{array} \right] \right\}
\]

\[
= x^{n+3} \sum_{j \geq 0} t^j x^j q^{j(j+1)} \left[ \begin{array}{c} n+1 \\ j \end{array} \right]
\]

where the last line has been justified by $q$-binomial identity

\[
\left[ \begin{array}{c} n+1 \\ j \end{array} \right] = \left[ \begin{array}{c} n \\ j \end{array} \right] + q^{n+1-j} \left[ \begin{array}{c} n \\ j-1 \end{array} \right].
\]

This proves the first equation. The equation (4) can be established similarly. □

**Corollary 3**: (Explicit expression for $S_n(t, q)$)

\[
S_n(t, q) = a \sum_{j \geq 0} t^{j+1} \left[ \begin{array}{c} n-2-j \\ j \end{array} \right] q^{j(j+1)} + b \sum_{j \geq 0} t^j \left[ \begin{array}{c} n-1-j \\ j \end{array} \right] q^j.
\]
**Proof:** According to the geometric series expansion, we have

\[
\sum_{n \geq 0} S_n(t, q)x^n = \frac{1}{1 - x - tx^2 \eta_x}(a + (b - a)x)
\]

\[
= \sum_{n \geq 0} (x + tx^2 \eta_x)^n(a + (b - a)x)
\]

\[
= \sum_{n \geq 0} (x + tx^2 \eta_x)^na + \sum_{n \geq 0} (x + tx^2 \eta_x)^n(b - a)x
\]

\[
= a \sum_{n \geq 0} (x + tx^2 \eta_x)^{n-1}(x + tx^2) + (b - a) \sum_{n \geq 0} (x + tx^2 \eta_x)^nx
\]

\[
= a \sum_{n \geq 0} x^n \sum_{j \geq 0} x^j t^i \left[ \begin{array}{c} n-1 \\ j \end{array} \right] q^{j^2} + at \sum_{n \geq 0} x^{n+1} \sum_{j \geq 0} x^j t^i \left[ \begin{array}{c} n-1 \\ j \end{array} \right] q^{j(j+1)}
\]

\[
+ (b - a) \sum_{n \geq 0} x^{n+1} \sum_{j \geq 0} x^j t^i \left[ \begin{array}{c} n \\ j \end{array} \right] q^{j^2}.
\]

Extract the coefficient of \(x^n\) and we get Corollary 3. \(\square\)

In view of Corollary 3, we can easily deduce that

\[
S_{n,k} = [t^k]S_n(t, q)
\]

\[
= [t^k] \left\{ a \sum_{j \geq 0} t^{j+1} \left[ \begin{array}{c} n-2-j \\ j \end{array} \right] q^{j(j+1)}
\]

\[
+ b \sum_{j \geq 0} t^{j} \left[ \begin{array}{c} n-1-j \\ j \end{array} \right] q^{j^2} \right\}
\]

\[
= a \left[ \begin{array}{c} n-1-k \\ k-1 \end{array} \right] q^{k(k-1)} + b \left[ \begin{array}{c} n-1-k \\ k \end{array} \right] q^{k^2}.
\]

For \(B_{n,k} = S_{2n+1,n-k}\), it is trivial to see that

\[
B_{n+k+i,k+i} = a \left[ \begin{array}{c} n+2k+2i \\ n-1 \end{array} \right] q^{2\left(\begin{array}{c} 2 \\ n \end{array}\right)} + b \left[ \begin{array}{c} n+2k+2i \\ n \end{array} \right] q^{n^2},
\]

then we have the following determinant evaluation.

**Theorem 4:** (determinant identity on \(q\)-binomial coefficients).

\[
\det_{0 \leq k,n \leq m} [B_{n+k+i,k+i}] = b^{m+1}q^{\frac{m(m+1)}{2}(1+5m+6i)} \prod_{n=0}^{m} \frac{(q^2;q^2)_n}{(q;q)_n}.
\]

When \(q \rightarrow 1\), we recover from this theorem a binomial determinant identity appeared in [11].
Proof: Note that $B_{n+k+i,k+i}$ is a polynomial of degree $n$ in $q^{2k}$ with the leading coefficient
\[ \frac{b(-1)^n}{(q;q)_n} q^{\frac{n}{2}(1+3n+4i)} \]. We can write $B_{n+k+i,k+i}$ formally as
\[ B_{n+k+i,k+i} = \sum_{j=0}^{n} \lambda_j(n) q^{2kj} \quad \text{with} \quad \lambda_n(n) = \frac{b(-1)^n}{(q;q)_n} q^{\frac{n}{2}(1+3n+4i)} \]
where $\{\lambda_j(n)\}_{j=0}^{n}$ are constants independent of $q^k$.

For each $n$ with $0 \leq n \leq m$, defining further
\[ \lambda_j(n) = 0 \quad \text{if} \quad n < j \leq m \]
then we have the following determinant factorization
\[ \det_{0 \leq k,n \leq m} [B_{n+k+i,k+i}] = \det_{0 \leq k,j \leq m} [q^{2kj}] \times \det_{0 \leq j,n \leq m} [\lambda_j(n)]. \]

The former is the Vandermonde determinant whose evaluation reads as
\[
\det_{0 \leq k,j \leq m} [q^{2kj}] = \prod_{0 \leq j < j \leq m} (q^{2j} - q^{2j}) \\
= (-1)^{\left(\frac{1+m}{2}\right)} \prod_{n=0}^{m} q^{2n(m-n)} (q^{2};q^{2})_{n}.
\]

The latter is the determinant of a diagonal matrix, which is evaluated by the product of the diagonal elements:
\[
\det_{0 \leq j,n \leq m} [\lambda_j(n)] = \prod_{n=0}^{m} \lambda_n(n) = b^{1+m} (-1)^{\left(\frac{1+m}{2}\right)} \prod_{n=0}^{m} q^{\frac{n}{2}(1+3n+4i)} (q^{2};q^{2})_{n}.
\]

Multiplying both evaluations just displayed and then simplifying the result, we get the determinant identity stated in the theorem. \( \Box \)

3. \textbf{$q$-ANALOGS OF THE INCOMPLETE FIBONACCI AND LUCAS POLYNOMIALS}

For the initial values $a = b = 1$, (1) reduces to the $q$-Fibonacci polynomial of Calitz [4]. Similarly for $a = 2, b = 1$, (1) reduces to the $q$-analog of the incomplete Lucas polynomial in [6].

For $a = b = 1$ and $a = 2, b = 1$ the generating function of $F_n(t, q)$ and $L_n(t, q)$ are given by Theorem 1 as follows:
\[
\sum_{n=0}^{\infty} F_n(t, q)x^n = \frac{1}{1 - x - tx^2 \eta x} \tag{5}
\]
\[
\sum_{n=0}^{\infty} L_n(t, q)x^n = \frac{1}{1 - x - tx^2 \eta x}(2 - x). \tag{6}
\]
where equation (5) has first been established by Andrews [3].

From them we can derive also the explicit generating functions.

**Theorem 5:** (Generating functions)

\[
\sum_{n \geq 0} F_n(t, q)x^n = \sum_{j \geq 0} \frac{x^{2jt} q^{j(j-1)}}{(x; q)_{j+1}},
\]

\[
\sum_{n \geq 0} L_n(t, q)x^n = \sum_{j \geq 0} \frac{x^{2jt} q^{j(j-1)}}{(x; q)_{j+1}} \left\{ 2 - xq^j \right\}.
\]

**Proof:** By means of geometric series and equations (3)-(4), we can compute

\[
\frac{1}{1 - x - tx^2 \eta x} = \sum_{n \geq 0} (x + tx^2 \eta x)^n 1 = \sum_{n \geq 0} (x + tx^2 \eta x)^{n-1} (x + tx^2)
\]

\[
= \sum_{n \geq 0} x^n \sum_{j \geq 0} x^{2jt} q^{j(j-1)} = \sum_{n \geq 0} x^n \sum_{j \geq 0} x^{2jt} q^{j(j+1)}
\]

\[
= \sum_{n \geq 0} x^n \sum_{j \geq 0} x^{2jt} q^{j(j-1)} = \sum_{n \geq 0} x^{n+1} \sum_{j \geq 0} x^{2jt} q^{j(j+1)}
\]

Recalling (5) and then applying the \(q\)-binomial formula

\[
\sum_{j \geq 0} \left[ \begin{array}{c} n+j \\ j \end{array} \right] x^j = \frac{1}{(x; q)_{n+1}}
\]

we get the generating function (7). Similarly, one can derive the generating function (8).

**Theorem 6:** (Generating functions)

\[
\sum_{k \geq n} F_k(t, q)x^k = x^n \frac{F_n(t, q) + xtq^{n-2} F_{n-1}(t, q)}{1 - x - tx^2 \eta x}
\]

\[
\sum_{k \geq n} L_k(t, q)x^k = x^n \frac{L_n(t, q) + xtq^{n-2} L_{n-1}(t, q)}{1 - x - tx^2 \eta x}.
\]

**Proof:** Let us denote by \(\delta(x)\) the expression on the left side of the equation in (9). We prove equivalently the relation:

\[
(1 - x - tx^2 \eta x)\delta(x) = \{ F_n(t, q) + xtq^{n-2} F_{n-1}(t, q) \} x^n.
\]
This can be accomplished as follows:

\[
(1 - x - tx^2 \eta_x) \sum_{k \geq n} F_k(t, q)x^k
= \sum_{k \geq n} F_k(t, q)x^k - \sum_{k \geq n} F_k(t, q)x^{k+1} - t \sum_{k \geq n} F_k(t, q)x^{k+2} q^k
= F_n(t, q)x^n + F_{n+1}(t, q)x^{n+1} - F_n(t, q)x^{n+1}
+ \sum_{k \geq n} \{F_{k+2}(t, q) - F_{k+1}(t, q) - F_k(t, qtq^{-2})\} x^{k+2}
= \{F_n(t, q) + txq^{n-2}F_{n-1}(t, q)\} x^n.
\]

Therefore (9) is valid. The equation (10) follows in the same way. □

Letting \(a = b = 1\) and \(a = 2, b = 1\) in corollary 3, we have

\[
f_n(t, q) = \sum_{j \geq 0} t^{j+1} \left[\binom{n-2}{j} - j\right]q^{j+1} + \sum_{j \geq 0} t^j \left[\binom{n-1}{j} - j\right]q^j
= \sum_{j \geq 0} t^j \left[\binom{n-1}{j} - j\right]q^j.
\]

\[
l_n(t, q) = 2 \sum_{j \geq 0} t^{j+1} \left[(\binom{n-2}{j} - j)\right]q^{j+1} + \sum_{j \geq 0} t^j \left[(\binom{n-1}{j} - j)\right]q^j
= \sum_{j \geq 0} t^j \left[(\binom{n-1}{j} - j)\right]q^j + \sum_{j \geq 1} t^j \left[(\binom{n-1}{j-1} - 1)\right]q^j.
\]

For two incomplete polynomial sequences defined by

\[
F_{m,n}(t, q) = \sum_{j=0}^{m} t^j \left[\binom{n-j}{j}\right]q^j
\]

and

\[
L_{m,n} = \sum_{j=0}^{m} t^j \left[\binom{n-j}{j}\right]q^j + \sum_{j=1}^{m} t^j \left[\binom{n-1-j}{j-1}\right]q^j.
\]

Their generating functions defined respectively by

\[
\Phi(x, y) := \sum_{m,n=0}^\infty F_{m,n}(t, q)x^my^n \quad \text{where} \quad 0 \leq m \leq \frac{n}{2}
\]

and

\[
\Psi(x, y) := \sum_{m,n=0}^\infty L_{m,n}(t, q)x^my^n \quad \text{where} \quad 0 \leq m \leq \frac{n}{2}
\]
are given by the following:

**Theorem 7**: (Generating function)

\[ \Phi(x, y) = \frac{1}{1 - x} \cdot \frac{1}{1 - y - txy^2 \eta_y} \]  
\[ (11) \]

\[ \Psi(x, y) = \frac{1}{1 - x} \cdot \frac{1}{1 - y - txy^2 \eta_y} (2 - y). \]  
\[ (12) \]

**Proof**: This generating function can be obtained through triple sum

\[ \Phi(x, y) = \sum_{m,n=0}^{\infty} \sum_{j=0}^{m} t^j \left[ \begin{array}{c} n-j \\ j \end{array} \right] q^{j(j-1)} x^m y^n, \]

For the inner sum, changing the summation index by \( n = i + 2j \) and then evaluating it as

\[ y^{2j} \sum_{i=0}^{\infty} y^i \left[ \begin{array}{c} i+j \\ j \end{array} \right] = \frac{y^{2j}}{(y; q)_{j+1}} \]

we can simplify the double sum as follows:

\[ \Phi(x, y) = \sum_{0 \leq j \leq m < \infty} x^m y^{j(j-1)} \frac{y^{2j}}{(y; q)_{j+1}} \]

\[ = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} t^j x^j q^{j(j-1)} \frac{y^{2j}}{(y; q)_{j+1}} \]

\[ = \frac{1}{1 - x} \cdot \frac{1}{1 - y - txy^2 \eta_y} \]

where equation (5) and (7) have been combined for justifying the last step. This proves the identity (11). Similarly we can deduce the identity (12).

**REFERENCES**


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