PERIODIC RECURRENCE RELATIONS AND CONTINUED FRACTIONS

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ABSTRACT

The Fibonacci series represents the simplest series whose successive members obey a periodic 3-term relation, wherein the coefficients and the period are all equal to 1. Here the most general case where these parameters are all arbitrary is treated. For a series of quantities or elements, related by a periodic 3-term recurrence relation between adjacent elements, it is shown that there is also a 3-term invariant recurrence relation between corresponding elements within adjacent periods. Application to the numerators and denominators of the convergents of a periodic continued fraction follows naturally.

1. THEOREM

Given a periodic 3-term recurrence relation between the elements E_i of the form:

$$c_i E_i = b_i E_{i-1} + a_i E_{i-2} \tag{1}$$

or, equivalently:

$$a_{i+2} E_i + b_{i+2} E_{i+1} - c_{i+2} E_{i+2} = 0, \tag{2}$$

where the a_i, b_i, c_i are constants which repeat with period n, such that $a_{i+n} = a_i, b_{i+n} = b_i, c_{i+n} = c_i$ then

$$CE_{(r+2)n+s} = BE_{(r+1)n+s} + AE_{rn+s}, \tag{3}$$

where A, B, C are constants for all integer values of r, s.

(Illustrative examples are provided in a later section of this paper).

2. PROOF

Let $M(a_i, b_i, -c_i)$ be the tridiagonal array composed of the quantities $a_i, b_i, -c_i$, and let $D_{l,m}$ be the associated tridiagonal determinant whose first row is $(b_l, -c_l)$ and last row (a_m, b_m). From the $m-l+1$ equations (1) for $i = l, m$ in $m-l+3$ unknowns by backward elimination successively of the $m-l$ unknowns $E_{m-1}, \dots, E_l$ one obtains

$$\Pi_l^m(c_i) E_m = D_{l,m} E_{l-1} + a_l D_{l+1,m} E_{l-2} \tag{4}$$

or, putting $l = 1, m = n$

$$CE_n = D_{1,n} E_0 + a_1 D_{2,n} E_{-1}, \tag{5}$$

where $C = \Pi_1^n(c_i)$. Similarly from the $m-l$ equations (1) with $i = l+1, m$ in $m-l+2$ unknowns by successive forward elimination of the $m-l-1$ unknowns $E_l, \dots, E_{m-2}$ one obtains

$$\Pi_{l+1}^m(-a_i) E_{l-1} = D_{l+1,m} E_{m-1} - c_m D_{l+1,m-1} E_m \tag{6}$$

or, putting $l = 1, m = n$

$$A^{'}E_0 = D_{2,n}E_{n-1} - c_n D_{2,n-1}E_n, \tag{7}$$

where $A^{'} = \Pi_2^n(-a_i) = (-1)^{n-1}\Pi_2^n(a_i)$. From the periodicity of the a_i, b_i, c_i it follows immediately, by full n-cycle shifts of the indices i, from (5), by a shift of $r+1$ cycles, that

$$CE_{(r+2)n} = D_{1,n}E_{(r+1)n} + a_1 D_{2,n}E_{(r+1)n-1} \tag{8}$$

and from (6), by a shift of r cycles, that

$$A^{'}E_{rn} = D_{2,n}E_{(r+1)n-1} - c_n D_{2,n-1}E_{(r+1)n}. \tag{9}$$

Hence, by elimination of $E_{(r+1)n-1}$ between (8) and (9)

$$CE_{(r+2)n} = BE_{(r+1)n} + AE_{rn}, \tag{10}$$

where $B = D_{1,n} + a_1 c_n D_{2,n-1}$ and $A = a_1 A' = (-1)^{n-1}\Pi_1^n(a_i)$. Clearly A and C are invariant under any re-ordering of the a_i and c_i. It remains to be shown that B is invariant under any cyclic shift of the a_i, b_i, c_i. Subjecting B to a single cyclic shift of i to $i+1$ and n to $n+1$, using $c_{n+1} = c_1$

$$B_{;1} = D_{2,n+1} + a_2 c_{n+1} D_{3,n}, \tag{11}$$

where $;s$ signifies the result of increasing all the indices on the a_i, b_i, c_i by an amount s. Now expansion of $D_{2,n+1}$ gives

$$D_{2,n+1} = b_1 D_{2,n} + a_1 c_n D_{2,n-1} \tag{12}$$

while similarly,

$$D_{1,n} = b_1 D_{2,n} + a_2 c_1 D_{3,n} \tag{13}$$

and thus it follows that

$$B_{;1} = D_{1,n} + a_1 c_n D_{2,n-1} = B. \tag{14}$$

A result similar to (13) holds for an arbitrary cyclic shift s giving

$$B_{;s} = B_{;s-1} = \cdots = B_{;1} = B \tag{15}$$

and (3) is therefore valid for all $r, s = 0, 1, 2\ldots,$ thus proving the theorem. For consistency in application of the above results $D_{i,i} = b_i, D_{i,i-1} = 1, D_{i,i-2} = 0$. Since the indexing of the a_i, b_i, c_i and E_i is arbitrary, then (3) is valid also for negative values of r and s, provided the recurrence relations (1) are satisfied. Thus it is evident from (15) that a triplet obeying (3) may be considered to begin anywhere within a period.

3. COROLLARY

A direct consequence of the above theorem is that, any sequence satisfying a recurrence relation of period unity, such as for example the Fibonacci and Lucas sequences, F_n and L_n, also satisfies a recurrence relation of arbitrary period n. For each such value of n, the relation (3) holds with constants A_n, B_n, C_n, which are independent of the initial values E_0, E_1

and therefore are the same for all sequences with the same a_i, b_i, c_i. For the specific case $a_i = b_i = c_i = 1$ (all i), then $A_n = (-1)^{n-1}$ and $C_n = 1$. The expression (14) for B_n can be most easily evaluated using the representations $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $L_n = (\alpha^n + \beta^n)$ where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$ with $\alpha\beta = -1$. Using the results

$$(\alpha^{2n+s} - \beta^{2n+s}) = (\alpha^n + \beta^n)(\alpha^{n+s} - \beta^{n+s}) - (\alpha\beta)^n(\alpha^s - \beta^s)$$

$$(\alpha^{2n+s} + \beta^{2n+s}) = (\alpha^n + \beta^n)(\alpha^{n+s} + \beta^{n+s}) - (\alpha\beta)^n(\alpha^s + \beta^s)$$

it follows that $B_n = L_n$. In the particular case of the Fibonacci sequence for which $F_0 = 0$, putting $s = 0$ recovers the well-known result $F_{2n} = L_n F_n$ as a special case of the more general theorem. However, for the Lucas sequence with $L_0 = 2$ the corresponding result is $L_{2n} = (L_n)^2 + 2(-1)^{n-1}$.

4. APPLICATIONS

An immediate application of the result (10) is to the accelerated evaluation of high series members of the sequence E_i starting from lower members. Such a situation applies to the higher approximants of a periodic continued fraction. A typical continued fraction of the form (employing an obvious notation)

$$F = b_0 + a_1/b_1 + a_2/b_2 + \cdots \tag{16}$$

has successive approximants which may be written $F_i = A_i/B_i$ whose numerators A_i and denominators B_i separately satisfy a 3-term recurrence relation of the form (1) with $c_i = 1$

$$A_i = b_i A_{i-1} + a_i A_{i-2} \tag{17}$$

$$B_i = b_i B_{i-1} + a_i B_{i-2} \tag{18}$$

with, for consistency,

$$A_0 = b_0, A_{-1} = 1, B_0 = 1, B_{-1} = 0. \tag{19}$$

Setting $l = 1$ in (4)

$$A_m = b_0 D_{1,m} + a_1 D_{2,m} = D_{0,m} \tag{20}$$

$$B_m = D_{1,m} = D_{0,m-1;1} = A_{m-1;1} \tag{21}$$

thus recovering results given by Perron[3]. A general periodic continued fraction may consist of a non-periodic part followed by a periodic part, as in

$$F = b_0 + a_1/b_1 + \cdots + a_k/R_k, \tag{22}$$

where

$$R_k = b_k + a_{k+1}/b_{k+1} + \cdots + a_{k+n}/R_k \tag{23}$$

repeats with period n, so that $a_{k+n} = a_k$ and $b_{k+n} = b_k$. When $R_k = F$ and there is no non-periodic part the continued fraction is said to be purely periodic, otherwise it is mixed periodic. Periodic continued fractions occur in the development of quadratic surds such as $S = P + Q\sqrt{R}$ where P and Q are rational with Q non-zero and R is a positive integer, not a perfect square. These give rise to simple continued fractions where $a_i = 1$, $c_i = 1$ and the b_i are positive integers. Then in the recurrence relation (3) $C = 1$ and $A = (-1)^{n-1}$ with B a positive integer. The evaluation of high order periodic approximants is of interest in the solution of Diophantine equations of the Pell type. A different (2-term) recurrence relation, connecting only two adjacent periods, and which applies to the numerators and denominators of approximants to only purely periodic continued fractions, has been given also by Perron [4]. Putting $l = n$ in (4), with all $c_i = 1$ gives

$$E_m = D_{n,m}E_{n-1} + a_n D_{n+1,m}E_{n-2} \tag{24}$$

which, using (20) and (21) becomes

$$E_m = A_{m-n;n}E_{n-1} + a_n B_{m-n;n}E_{n-2}. \tag{25}$$

Letting $m = rn + s, r = 1, 2, 3, \dots ; s = 0, 1, 2, \dots$ and using $A_{m;n} = A_m, B_{m;n} = B_m$

$$E_{rn+s} = A_{(r-1)n+s}E_{n-1} + a_n B_{(r-1)n+s}E_{n-2} \tag{26}$$

which, when E_i is replaced by A_i or B_i expresses each in terms of linear combination of both A_{i-n} and B_{i-n} with the fixed coefficients A_{n-1}, A_{n-2} and B_{n-1}, B_{n-2}, respectively. Thus,

$$A_{rn+s} = A_{n-1}A_{(r-1)n+s} + a_n A_{n-2}B_{(r-1)n+s} \tag{27}$$
$$B_{rn+s} = B_{n-1}A_{(r-1)n+s} + a_n B_{n-2}B_{(r-1)n+s}. \tag{28}$$

For the special case of r=1, s=0, (26), (27) and (28) all reduce to the form

$$E_n = A_0 E_{n-1} + a_n B_0 E_{n-2} \tag{29}$$

or, using (19) together with $a_0 = a_n, b_0 = b_n$

$$E_n = b_n E_{n-1} + a_n E_{n-2} \tag{30}$$

in accord with the defining relation (1) for $c_i = 1$ with $i = n$.

5. RELATED WORK

H. R. P. Ferguson [2] considers periodic recurrence systems generated by a one-parameter class of linear recurrences of the form

$$f_n(t) = a_n f_{n-1} + t b_{n-1} f_{n-2} \tag{31}$$

where the sequences a_i and b_i are periodic. The Fibonacci pseudogroup is invoked to facilitate the description of such systems, and to evaluate the characteristic polynomial in the parameter t. Cooper [1] discusses periodic recurrence systems of the type (1), for the special case where all $c_i = 1$, and has evaluated by direct calculation in each case what are essentially our constants A, B, C for periodicities k =1,2,3,4,5,6,7 and indicated how results can be obtained in the general case. His 'tree' operations replace basically the matrix arithmetic involved in (14), but relate only to the case $s = 0$. I am extremely grateful to Professor Cooper for bringing these works to my attention.

6. ILLUSTRATIVE EXAMPLES

1. Let (1,2,3); (2,3,1); (3,1,2) be the (arbitrary) values of a_i, b_i, c_i defining a recurrence sequence E_i of period $n = 3$. Setting (again arbitrarily) $E_0 = 0, E_1 = 1$ the subsequent members E_2 to E_{10} are 2/3, 4, 3; 10/3, 16, 13; 14, 68, 55 and the values of A, B, C are found to be 6, 24, 6 or 1, 4, 1 when reduced to their simplest terms. Then clearly the triplets E_i with indices $i =$ (0,3,6); (1,4,7); (2,5,8); (3,6,9); (4,7,10) all obey the relation (3).

2. It is readily seen that $\sqrt{7}$ can be developed as the periodic continued fraction (using the notation of (16))

$$\sqrt{7} = 2 + (1/1 + 1/1 + 1/1 + 1/4) \tag{32}$$

where the brackets (...) enclose the recurrent partial fractions with period $n = 4$, giving $A = -1, B = 16, C = 1$. When expanded the successive approximants A_0/B_0 to A_{12}/B_{12} are given by

$$A_i = 2; 3, 5, 8, 37; 45, 82, 127, 590; 717, 1307, 2024, 9403 \tag{33}$$

$$B_i = 1; 1, 2, 3, 14; 17, 31, 48, 223; 271, 494, 765, 3554 \tag{34}$$

from which it is readily found that the triplets of A_i and B_i with the indices $i =$ (0,4,8); (1,5,9); (2,6,10); (3,7,11); (4,8,12) also all obey the relation (3).

REFERENCES

[1] C. Cooper. "An Identity for Period k Second Order Linear Recurrence Systems." *Proceedings of the 12th International Conference on Fibonacci Numbers and Their Applications*, (to appear) or <http://www.math-cs.ucmo.edu/∼curtisc/articles.html>.

[2] H.R.P. Ferguson. "The Fibonacci Pseudogroup, Characteristic Polynomials and Eigenvalues of Tridiagonal Matrices, Periodic Linear Recurrence Systems and Application to Quantum Mechanics." *The Fibonacci Quarterly* **16** (1978): 435-447.

[3] O. Perron, *Die Lehre von den Kettenbruchen,* Band I, S. 8-9, Teubner, 1977.

[4] O. Perron, *Die Lehre von den Kettenbruchen,* Band II, S. 83, Teubner, 1977.

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