

# FIBONACCI NUMBERS AND TRIGONOMETRIC IDENTITIES

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ABSTRACT. In 1969 Webb and Parberry proved a startling trigonometric identity involving Fibonacci numbers. This identity has remained isolated up to now, despite the amount of work on related polynomials. We provide a wide generalization of this identity together with what we believe (and hope!) to be its proper understanding.

## 1. INTRODUCTION

Fibonacci numbers verify a wealth of identities, see e.g. [5], [6], [7], [12]. By specifying  $x$  and  $y$  to 1 in Corollary 10 of [4], we get an intriguing one which states that for  $n \geq 1$ :

$$F_n = \prod_{k=1}^{[(n-1)/2]} \left( 1 + 4 \cos^2 \frac{k\pi}{n} \right) = \prod_{k=1}^{[(n-1)/2]} \left( 3 + 2 \cos \frac{2k\pi}{n} \right). \quad (1)$$

Webb and Parberry's paper [15] contains all the necessary material to write this identity, but they do not state it explicitly. This formula is indeed intriguing: the left hand side satisfies a second order recursion formula while no such recursion arises from the right hand side expression. Indeed, how could we connect  $\cos \frac{2k\pi}{n}$  and  $\cos \frac{2k\pi}{n+1}$ ? Taking a number theoretic point of view leads to more dismay. Fibonacci numbers are linked with the arithmetic of  $\mathbb{Q}(\sqrt{5})$  and not with that of  $\mathbb{Q}(\exp(2i\pi/n))$ .

The mystery gets somewhat lifted by the proof of the above identity. We introduce the second order Chebyshev polynomials  $U_n$  by

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad (2)$$

so that they verify

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}. \quad (3)$$

This last expression leads to the recursion above as well as to the formula

$$U_n(x) = 2^n \prod_{k=1}^n \left( x - \cos \frac{k\pi}{n+1} \right). \quad (4)$$

Since it is not difficult to discover that  $F_n = i^{n-1}U_{n-1}(-i/2)$  we get the result. This factorization has been noted and studied by [1], [4], [15], and [16]. The main arguments of this proof are the recurrence relation (2) and the rule of additions of sine. However this sheds no light on how the arithmetic of  $\mathbb{Q}(\sqrt{5})$  and  $\mathbb{Q}(\exp(2i\pi/n))$  get entangled.

Elaborating on this argument, Webb & Parberry in [15] and Hoggatt & Long in [4] defined a sequence of polynomials by

$$u_0(x, y) = 0, \quad u_1(x, y) = 1, \quad u_{n+2}(x, y) = xu_{n+1}(x, y) + yu_n(x, y). \quad (5)$$

The case  $y = 1$ , the only one considered in [15], leads to what is sometimes called Fibonacci polynomials. A further generalization led to Morgan-Voyce polynomials in [8], [13] and [14], and more recently to Brahmagupta's polynomials in [10] and [11]. In particular, it is proved that

$$u_n(x, y) = \prod_{k=1}^n \left( x - 2i\sqrt{y} \cos \frac{k\pi}{n} \right) \tag{6}$$

and this leads to amazing identities like the following.

**Corollary 1.** *We have for  $n \geq 3$*

$$\prod_{1 \leq \ell \leq n} \left( 1 + 4 \sin^2 \frac{2\pi\ell}{n} \right) = (1 + F_n - 2F_{n+1} + (-1)^n)^2$$

where  $F_n$  is the  $n$ th Fibonacci number, with  $F_0 = 0$  and  $F_1 = 1$ .

We also have  $1 + F_n - 2F_{n+1} + (-1)^n = 1 - L_n + (-1)^n$  where  $L_n$  is the  $n$ th Lucas number ( $L_n = F_{n-1} + F_{n+1}$ ).

This corollary does not appear as such in [4], but can be derived from the material presented therein. We provide a simple proof later. Again the LHS satisfies a linear recursion, which we now compute explicitly. First express  $F_n$  in terms of  $\phi$  and  $-1/\phi$  where  $\phi$ , the golden ratio, is the larger root of  $x^2 - x - 1$  and expands the square. There comes a linear combination of terms of the form  $p^n$ , where all the  $p$ 's appearing are roots of  $(x^2 - 3x + 1)(x^2 - x - 1)(x + 1)(x - 1) = x^6 - 4x^5 + 2x^4 + 6x^3 - 4x^2 - 2x + 1$  since  $\phi^2$  and  $1/\phi^2$  are the roots of  $x^2 - 3x + 1$ . If we call  $g(n)$  the LHS of the quantity computed in the above corollary, we have

$$g(n + 6) = 2g(n + 5) + 4g(n + 4) - 6g(n + 3) - 2g(n + 2) + 4g(n + 1) - g(n)$$

since each sequence  $(p^n)$  satisfies it.

The next question is whether one can obtain such identities with three homogeneous variables  $x, y$  and  $z$ . One path to such a generalization has been to study the dynamics of the zeros of Fibonacci polynomials in [3], getting interesting by-products but no trigonometric identity.

## 2. RESULTS AND PROOFS

We prove here such a trigonometric identity. Let us start with a simple case.

**Theorem 1.** *We have for  $n \geq 3$  and  $\xi = \exp(2i\pi/n)$*

$$\prod_{1 \leq \ell \leq n} (1 - x\xi^\ell - y\xi^{2\ell}) = 1 - G_n + (-y)^n$$

where  $G_n$  is defined by the recursion  $G_n = xG_{n-1} + yG_{n-2}$  for  $n \geq 2$  with  $G_0 = 2, G_1 = x$  and  $G_2 = x^2 + 2y$ .

The polynomials  $G_n$  are a special case of Brahmagupta polynomials (take  $t = 1$ ) introduced and studied in [10], [11]. These are linked with Morgan-Voyce polynomials, see [8] and are the Fibonacci polynomials when we further specialize  $y$  to be equal to 1.

At this level, our proof was a mystery to us, we did in fact stumble on it while studying a completely different problem. It was from then onwards tempting to prove such trigonometric relations with  $r$  terms instead of 3. This led us to a very simple and direct proof of a

general relation, via considerations of circulant matrices. Most earlier connections between determinants and Fibonacci numbers, or Chebyshev polynomials involved continuants. Determinant of a family of increasing size tridiagonal matrices whose first elements are properly chosen while the later ones are fixed (see [1], [2], and exercise 24 page 85 of [5]).

And while studying this proof, we discovered that the heart of such identities was a hundred years old one liner (see [9], paragraph 136, formula (4)).

**Fundamental Lemma.** *Let  $(x_s)_{0 \leq s \leq r}$  be complex numbers and set  $\xi = \exp(2i\pi/n)$ . We have*

$$\prod_{0 \leq \ell \leq n-1} \sum_{0 \leq s \leq r} x_s \xi^{s\ell} = (-1)^{rn} x_r^n \prod_{\rho/P(\rho)=0} (\rho^n - 1)$$

where  $P(Y) = \sum_{s=0}^r x_s Y^s$ .

Simply because the quantity to be computed is up to the sign of the resultant of  $P(Y)$  and  $Q_n(Y) = Y^n - 1$  for which we have

$$\text{Res}(P, Q_n) = (-1)^{rn} \prod_{u/Q_n(u)=0} P(u) = x_r^n \prod_{\rho/P(\rho)=0} Q_n(\rho). \tag{7}$$

So, on one side, we have the roots of one polynomial, while on the other one we have the roots of another polynomial. This identity being so fundamental, we recall its one line proof. Consider two polynomials  $A$  and  $B$  with respective leading terms  $a_m$  and  $b_n$ , degrees  $m$  and  $n$  and roots  $(\alpha_i)_{1 \leq i \leq m}$  and  $(\beta_j)_{1 \leq j \leq n}$  repeated with multiplicity. Then, and it can be taken as a definition,

$$\text{Res}(A, B) = a_m^n b_n^m \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (\alpha_i - \beta_j), \tag{8}$$

from which (7) follows trivially. There is an expression of this resultant as a (Sylvester) determinant whose entries are the coefficients of  $A$  and  $B$ , as well as some 0's, but we will not invoke such an expression.

Let us use our fundamental lemma on an example. Taking  $P = Y^2 - Y - 1$ , with roots the golden ratio  $\phi = (1 + \sqrt{5})/2$  and  $-1/\phi$  we get

$$\begin{aligned} \prod_{0 \leq \ell \leq n-1} \left( 1 + 2i \sin \frac{2\pi\ell}{n} \right) &= 1 - \phi^n - (-\phi)^{-n} + (-1)^n \\ &= 1 + F_n - 2F_{n+1} + (-1)^n \end{aligned}$$

since  $\phi^n = \phi^{-1}F_n + F_{n+1}$  and  $(-\phi)^{-n} = -\phi F_n + F_{n+1}$ , yielding an illuminating proof of Corollary 1!

Proving (1) along these lines requires some tuning: we start from the polynomial  $P = 1 + iY + Y^2$  and use only even indices  $2n$ . This leads after some manipulations to the square of the right-hand side of (1). We simplify the squareroot of the left-hand side by noting that

$$2F_{2n+1} - F_{2n} - 2(-1)^n = 5F_n^2. \tag{9}$$

We saw in these examples how linear recurrent sequences can be introduced and we now address the general case treated in the Theorem. The adaptation is straightforward. To avoid denominators, we set  $P(Y) = Y^2 - xY - y$  with roots  $\rho_1$  and  $\rho_2$  in  $\mathbb{C}(x, y)$ . By our

fundamental lemma, we have

$$\begin{aligned} \prod_{1 \leq \ell \leq n} (\xi^{2\ell} - x\xi^\ell - y) &= (\rho_1\rho_2)^n - \rho_1^n - \rho_2^n + 1 \\ &= (-y)^n - \rho_1^n - \rho_2^n + 1. \end{aligned}$$

We set  $G_n(x, y) = \rho_1^n + \rho_2^n$ , which yields  $G_0 = 2$ ,  $G_1 = x$  and  $G_{n+2} = xG_{n+1} + yG_n$ . We need only to factor  $\xi^{2\ell}$  out and exchange  $\ell$  by  $-\ell$  to get our Theorem!

Let us now treat the general case. Let  $x_0, \dots, x_r$  be  $r$  indeterminates. Let  $\rho_1, \dots, \rho_r$  be the  $r$  roots of  $P(Y) = \sum_{0 \leq s \leq r} x_s Y^s$  in an algebraic closure of  $K(x_0, \dots, x_r)$ . We set

$$H_n = (-1)^{rn} x_r^n \prod_{1 \leq s \leq r} (\rho_s^n - 1) \tag{10}$$

which in fact belongs to  $K[x_0, \dots, x_r]$  and

$$\mathcal{H}(X) = \sum_{n \geq 0} H_n X^n = \sum_{n \geq 0} \sum_{S \subset \{1, \dots, r\}} (-1)^{r-|S|} \prod_{s \in S} \rho_s^n ((-1)^r x_r X)^n \tag{11}$$

$$= \sum_{S \subset \{1, \dots, r\}} \frac{(-1)^{r-|S|}}{1 - (-1)^r \prod_{s \in S} \rho_s x_r X}. \tag{12}$$

This shows that  $\mathcal{H}(X)$  is a rational fraction with denominator of degree at most  $2^r$ . As a consequence  $H_n$  verifies a linear recursion of degree at most  $2^r$  which we easily establish by writing  $\mathcal{H}(X) = \mathcal{A}(X)/\mathcal{B}(X)$  with  $\mathcal{A}(X)$  and  $\mathcal{B}(X)$  polynomials. When equating coefficients in the equation

$$\mathcal{B}(X) \sum_{n \geq 0} H_n X^n = \mathcal{A}(X)$$

we recover an explicit form of the required recursion. Summarizing, we get

**Theorem 2.** *Let  $(x_s)_{0 \leq s \leq r}$  be complex numbers and set  $\xi = \exp(2i\pi/n)$ . We have*

$$\prod_{0 \leq \ell \leq n-1} \sum_{0 \leq s \leq r} x_s \xi^{s\ell} = H_n$$

where  $P(Y) = \sum_{s=0}^r x_s Y^s$  and  $H_n$  defined by (10) satisfies a linear recursion of degree at most  $2^r$ .

Since  $\mathcal{B}(X)$  is fairly universal and determines the coefficients of the recursion satisfied by  $H_n$ , it would be satisfactory to have a complete description of it solely in terms of the  $x_s$ 's. We have not been able to derive such a description. We can of course group together the products  $1 - (-1)^r \prod_{s \in S} \rho_s x_r X$  over  $S$  with a fixed cardinality. As symmetric expressions of the roots, each can be expressed as a polynomial of  $\mathbb{C}[x_0, x_1, \dots, x_r]$ . For  $|S| = 0$ , we get  $1 - (-1)^r x_r X$ . For  $|S| = 1$ , we get  $(-x_r X)^r P((-1)^r/(x_r X))$ .

We end this section with yet another identity.

**Corollary 2.** *We have for  $n \geq 5$*

$$\prod_{k=1}^{[(n-4)/4]} \left( 5 + 4 \sin^2 \frac{2k\pi}{n} \right) = \begin{cases} F_n & \text{when } n \text{ is even,} \\ \sqrt{F_n^2 - 4} & \text{when } n \text{ is odd.} \end{cases}$$

We achieve this by considering the polynomial  $P = Y^2 - 3Y + 1$  with roots  $\phi^2$  and  $\phi^{-2}$ . We then follow the path described above together with some symmetry and identity (9). Here are the details to the proof.

$$\begin{aligned} \prod_{k=0}^{n-1} (1 - 3\xi^k + \xi^{2k}) &= \prod_{k=0}^{n-1} \xi^k \prod_{k=0}^{n-1} \left(-3 + 2 \cos \frac{2\pi k}{n}\right) \\ &= (-1)^{n-1} \prod_{k=0}^{n-1} \left(-3 + 2 \cos \frac{2\pi k}{n}\right). \\ &= (-1)^n \prod_{k=1}^{(n-1)/2} \left(-3 + 2 \cos \frac{2\pi k}{n}\right) \times \begin{cases} -5 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

### 3. EXTENSIONS AND LIMITATIONS

The above approach would work if we were to replace  $Y^n - 1$  by  $Q(Y^n)$  for a fixed polynomial  $Q$ . It would also extend to the case when the coefficients of  $Q$  are polynomials in  $n$ . The same remark holds for the coefficients of  $P$ . In these cases, the roots do not depend on roots of unity, which means that the rule of addition of sine does not in fact intervene. We can even handle the more difficult case  $Q_n = Y^{2n} + Y^{n+1} + 1$ .

However, we are not able to treat the case  $Y^{n^2} - 1$  or any other non subsequence of  $Y^n - 1$  where the exponents would not be taken in an arithmetic progression. No linear recursion in that case exists, for it would mean a linear recursion for the values  $H_{n^2}$ , and their growth is too steep to allow such a fact (once we specialize the  $x_s$ 's).

Let us end this paper with a related problem. Restricting the product to indices  $\ell$  prime to  $n$  in our theorems, one gets the norm of  $1 - x\xi - x\xi^2$  in  $\mathbb{Q}(\xi)$ :

**Corollary 3.** *We have*

$$\prod_{\substack{1 \leq \ell \leq n \\ (\ell, n) = 1}} (1 - x\xi^\ell - y\xi^{2\ell}) = \prod_{d|n} (1 - G_d + (-y)^d)^{\mu(n/d)}$$

where  $\mu$  denotes the Möbius function.

The question arose to decide whether this norm verifies a linear recursion as above or not. Our approach via a resultant supports a negative answer. We have

$$\prod_{\substack{1 \leq \ell \leq n \\ (\ell, n) = 1}} (1 - x\xi^\ell - y\xi^{2\ell}) = (-1)^{(r-1)n} y^n \prod_{\rho/P(\rho)=0} \Phi_n(\rho)$$

where  $\Phi_n$  is the  $n$ th cyclotomic polynomial. In general, this polynomial does not have a finite number of monomes (for instance when  $n$  is prime), which ruins the approach we used, but also make us believe no recursion does indeed exist.

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