# ON A SUM OF MELHAM AND ITS VARIANTS

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Abstract. We prove a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions.

## 1. Introduction

Wiemann and Cooper [2] report about some conjectures of Melham [1] related to the sum

$$\sum_{k=1}^{n} F_{2k}^{2m+1}.$$

We don't know what these conjectures are, but an expansion for $m = 2$ is cited:

$$\sum_{k=1}^{n} F_{2k}^{5} = \frac{1}{L_1 L_3 L_5}\Big[4F_{2n+1}^{5} - 15F_{2n+1}^{3} + 25F_{2n+1} - 14\Big].$$

In this paper, we will prove the general formula

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \sum_{l=0}^{m} \lambda_{m,l} F_{2n+1}^{2l+1} + C_m,$$

with

$$\lambda_{m,l} = 5^{l-m}(-1)^{m-l}\frac{1}{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{L_{2m-2j+1}}$$

and

$$C_m = \frac{1}{5^m}\sum_{j=0}^{m}(-1)^{j-1}\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}.$$

(We use Fibonacci and Lucas numbers, as usual.)

In September 2008, after I obtained these results, I learned from C. Cooper that somebody anticipated them by 2 months! So I decided to go ahead and find more results. These are the evaluations of

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon},$$

for $\delta, \varepsilon \in \{0, 1\}$, as well as the evaluations of the corresponding sums for Lucas numbers. All in all there are 8 formulas of the Melham type. At the end, we will apply them to compute *iterates* of the original Melham sum.

We decided to present the original proof of the Melham sum, since the remaining instances are done in a similar style, hence we present then only some key steps.

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## 2. Proof

We will make extensive use of the Binet formulas

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2};$$

note that $\beta = -1/\alpha$.

We need a Lemma.

**Lemma 2.1.**

$$x^{2k+1} + x^{-2k-1} = \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} \left(x + \tfrac{1}{x}\right)^{2l+1}. \tag{2.1}$$

*Proof.* The proof will be by induction, the instance $k = 0$ being clear. We compute

$$\begin{aligned}
x^{2k+3} + x^{-2k-3} &= (x^{2k+1} + x^{-2k-1})(x^2 + x^{-2}) - (x^{2k-1} + x^{-2k+1}) \\
&= (x^2 + x^{-2}) \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} (x + \tfrac{1}{x})^{2l+1} \\
&\qquad - \sum_{l=0}^{k-1} (-1)^{k-1-l} \frac{2k-1}{2l+1} \binom{k-1+l}{k-1-l} (x + \tfrac{1}{x})^{2l+1} \\
&= \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} (x + \tfrac{1}{x})^{2l+3} \\
&\qquad - 2 \sum_{l=0}^{k} (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} (x + \tfrac{1}{x})^{2l+1} \\
&\qquad - \sum_{l=0}^{k-1} (-1)^{k-1-l} \frac{2k-1}{2l+1} \binom{k-1+l}{k-1-l} (x + \tfrac{1}{x})^{2l+1}.
\end{aligned}$$

We compute the coefficient of $(x + \frac{1}{x})^{2l+1}$:

$$\begin{aligned}
(-1)^{k-l-1} \frac{2k+1}{2l-1} \binom{k+l-1}{k-l+1} &- 2(-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} \\
&- (-1)^{k-1-l} \frac{2k-1}{2l+1} \binom{k-1+l}{k-1-l} \\
&= (-1)^{k+1-l} \frac{2k+3}{2l+1} \binom{k+1+l}{k+1-l},
\end{aligned}$$

as desired. □

And now comes the big computation:

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m \sqrt{5}} \sum_{k=1}^{n} (\alpha^{2k} - \beta^{2k})^{2m+1}$$

$$= \frac{1}{5^m\sqrt{5}} \sum_{j=0}^{2m+1} (-1)^{j-1} \binom{2m+1}{j} \sum_{k=1}^{n} \alpha^{2k(2j-2m-1)}$$
$$= \frac{1}{5^m\sqrt{5}} \sum_{j=0}^{2m+1} (-1)^{j-1} \binom{2m+1}{j} \frac{\alpha^{2(n+1)(2j-2m-1)} - \alpha^{2(2j-2m-1)}}{\alpha^{2(2j-2m-1)} - 1}$$
$$= \frac{1}{5^m\sqrt{5}} \sum_{j=0}^{m} (-1)^{j-1} \binom{2m+1}{j} \Bigg[ \frac{\alpha^{2(n+1)(2j-2m-1)} - \alpha^{2(2j-2m-1)}}{\alpha^{2(2j-2m-1)} - 1} - \frac{\alpha^{2(n+1)(2m-2j+1)} - \alpha^{2(2m-2j+1)}}{\alpha^{2(2m-2j+1)} - 1} \Bigg]$$
$$= \frac{1}{5^m\sqrt{5}} \sum_{j=0}^{m} (-1)^{j-1} \binom{2m+1}{j} \frac{1 - \alpha^{2n(2j-2m-1)} - \alpha^{2(n+1)(2m-2j+1)} + \alpha^{2(2m-2j+1)}}{\alpha^{2(2m-2j+1)} - 1}$$
$$= \frac{1}{5^m\sqrt{5}} \sum_{j=0}^{m} (-1)^{j-1} \binom{2m+1}{j} \frac{-\beta^{(2m-2j+1)} + \beta^{(2n+1)(2m-2j+1)} - \alpha^{(2n+1)(2m-2j+1)} + \alpha^{(2m-2j+1)}}{\alpha^{(2m-2j+1)} + \beta^{(2m-2j+1)}}$$
$$= \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{F_{(2n+1)(2m-2j+1)} - F_{2m-2j+1}}{L_{2m-2j+1}}$$
$$= \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{F_{(2n+1)(2m-2j+1)}}{L_{2m-2j+1}} - \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}}.$$

This displays already the constant term $C_m$.

Now we use the lemma with $x = \alpha^{2n+1}$, and get the formula

$$F_{(2k+1)(2n+1)} = \sum_{l=0}^{k} 5^l (-1)^{k-l} \frac{2k+1}{2l+1} \binom{k+l}{k-l} F_{2n+1}^{2l+1}.$$

So we can rewrite our formula:

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{F_{(2n+1)(2m-2j+1)}}{L_{2m-2j+1}} - \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}}$$
$$= \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{1}{L_{2m-2j+1}} \times \sum_{l=0}^{m-j} 5^l (-1)^{m-j-l} \frac{2m-2j+1}{2l+1} \binom{m-j+l}{m-j-l} F_{2n+1}^{2l+1} - \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{j} \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}}$$

$$= \sum_{l=0}^{m} F_{2n+1}^{2l+1} 5^{l-m} (-1)^{m-l} \frac{1}{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{L_{2m-2j+1}}$$
$$- \frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}},$$

as announced.

For the reader's convenience, here is a little list:

$$\sum_{k=1}^{n} F_{2k}^{1} = F_{2n+1}^{1} - 1,$$
$$\sum_{k=1}^{n} F_{2k}^{3} = \tfrac{1}{4} F_{2n+1}^{3} - \tfrac{3}{4} F_{2n+1}^{1} + \tfrac{1}{2}$$
$$\sum_{k=1}^{n} F_{2k}^{5} = \tfrac{1}{11} F_{2n+1}^{5} - \tfrac{15}{44} F_{2n+1}^{3} + \tfrac{25}{44} F_{2n+1}^{1} - \tfrac{7}{22},$$
$$\sum_{k=1}^{n} F_{2k}^{7} = \tfrac{1}{29} F_{2n+1}^{7} - \tfrac{56}{319} F_{2n+1}^{5} + \tfrac{455}{1276} F_{2n+1}^{3} - \tfrac{553}{1276} F_{2n+1}^{1} + \tfrac{139}{638},$$
$$\sum_{k=1}^{n} F_{2k}^{9} = \tfrac{1}{76} F_{2n+1}^{9} - \tfrac{189}{2204} F_{2n+1}^{7} + \tfrac{5625}{24244} F_{2n+1}^{5} - \tfrac{4083}{12122} F_{2n+1}^{3} + \tfrac{8055}{24244} F_{2n+1}^{1} - \tfrac{1877}{12122}.$$

## 3. Three more summation formulas of the Melham type

A similar approach as before produces

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} \binom{2m+1}{j} \frac{F_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}}.$$

Now we use the identity

$$x^{2k+1} - x^{-2k-1} = \sum_{l=0}^{k} \binom{k+l}{k-l} \frac{2k+1}{2l+1} \left(x - \tfrac{1}{x}\right)^{2l+1} \tag{3.1}$$

and its corollary ($x = \alpha^{2(n+1)}$)

$$F_{2(n+1)(2m+1-2j)} = \sum_{l=0}^{m-j} 5^l \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{2l+1} F_{2(n+1)}^{2l+1}$$

to continue:

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \frac{1}{5^m} \sum_{j=0}^{m} \binom{2m+1}{j} \frac{F_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}}$$
$$= \sum_{j=0}^{m} \binom{2m+1}{j} \frac{1}{L_{2m+1-2j}} \sum_{l=0}^{m-j} 5^{l-m} \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{2l+1} F_{2(n+1)}^{2l+1}$$

$$= \sum_{l=0}^{m} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \frac{5^{l-m}}{L_{2m+1-2j}} \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{2l+1}.$$

Now we turn to the next instance:

$$\sum_{k=0}^{n} F_{2k}^{2m} = \frac{1}{5^m} \sum_{j=0}^{m-1} \binom{2m}{j} (-1)^j \frac{F_{(2n+1)(2m-2j)}}{F_{2m-2j}} + \frac{1}{5^m} \binom{2m}{m} (-1)^m (n + \tfrac{1}{2}).$$

Here, we need a formula of a slightly different type.

$$x^{2N} - x^{-2N} = \left(x + \tfrac{1}{x}\right) \sum_{l=0}^{N-1} \binom{N+l}{N-l-1} \left(x - \tfrac{1}{x}\right)^{2l+1}. \tag{3.2}$$

The factor $x + \frac{1}{x}$ is necessary here; without it, it would be like expanding an even periodic function as a Fourier series, but using only sine functions!

The corollary is ($x = \alpha^{2n+1}$)

$$F_{(2n+1)(2m-2j)} = F_{2n+1} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1}.$$

Thus,

$$\begin{aligned}\sum_{k=0}^{n} F_{2k}^{2m} &= \frac{1}{5^m} \sum_{j=0}^{m-1} \binom{2m}{j} (-1)^j \frac{F_{2n+1}}{F_{2m-2j}} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1} \\ &\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m (n + \tfrac{1}{2}) \\ &= \frac{F_{2n+1}}{5^m} \sum_{l=0}^{m-1} L_{2n+1}^{2l+1} \sum_{j=0}^{m-l-1} \binom{2m}{j} (-1)^j \frac{1}{F_{2m-2j}} \binom{m-j+l}{m-j-l-1} \\ &\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m (n + \tfrac{1}{2}).\end{aligned}$$

And now we turn to the last formula:

$$\sum_{k=0}^{n} F_{2k+1}^{2m} = \frac{1}{5^m} \sum_{j=0}^{m-1} \binom{2m}{j} \frac{F_{2(n+1)(2m-2j)}}{F_{2m-2j}} + \frac{n+1}{5^m} \binom{2m}{m}.$$

We use (3.2) again, with $x = \alpha^{2(n+1)}$:

$$F_{2(n+1)(2m-2j)} = L_{2(n+1)} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} 5^l F_{2(n+1)}^{2l+1},$$

therefore,

$$\begin{aligned}\sum_{k=0}^{n} F_{2k+1}^{2m} &= \frac{1}{5^m} \sum_{j=0}^{m-1} \binom{2m}{j} \frac{1}{F_{2m-2j}} L_{2(n+1)} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} 5^l F_{2(n+1)}^{2l+1} \\ &\quad + \frac{n+1}{5^m} \binom{2m}{m}\end{aligned}$$

$$= L_{2(n+1)} \sum_{l=0}^{m-1} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l-1} \binom{2m}{j} \frac{1}{F_{2m-2j}} \binom{m-j+l}{m-j-l-1} 5^{l-m} + \frac{n+1}{5^m} \binom{2m}{m}.$$

## 4. Summary of results

For the reader's convenience, we collect the 4 formulas of the Melham Fibonacci type here; they hold for all $n, m \in \mathbb{N}_0$:

$$\sum_{k=0}^{n} F_{2k}^{2m+1} = \sum_{l=0}^{m} F_{2n+1}^{2l+1} \sum_{j=0}^{m-l} (-1)^{m-l} \frac{5^{l-m}}{L_{2m-2j+1}} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{2l+1} - \frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{F_{2m-2j+1}}{L_{2m-2j+1}},$$

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \sum_{l=0}^{m} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l} \frac{5^{l-m}}{L_{2m-2j+1}} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m-2j+1}{2l+1},$$

$$\sum_{k=0}^{n} F_{2k}^{2m} = \frac{F_{2n+1}}{5^m} \sum_{l=0}^{m-1} L_{2n+1}^{2l+1} \sum_{j=0}^{m-l-1} \binom{2m}{j} \binom{m-j+l}{m-j-l-1} (-1)^j \frac{1}{F_{2m-2j}} + \frac{1}{5^m} \binom{2m}{m} (-1)^m (n + \tfrac{1}{2}),$$

$$\sum_{k=0}^{n} F_{2k+1}^{2m} = L_{2(n+1)} \sum_{l=0}^{m-1} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l-1} \binom{2m}{j} \binom{m-j+l}{m-j-l-1} \frac{1}{F_{2m-2j}} 5^{l-m} + \frac{n+1}{5^m} \binom{2m}{m}.$$

## 5. Lucas types of Melham's sum

We start with

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} \frac{L_{(2n+1)(2m+1-2j)}}{L_{2m+1-2j}} + 4^m.$$

We use (3.1) with $x = \alpha^{2n+1}$:

$$L_{(2n+1)(2m+1-2j)} = \sum_{l=0}^{m-j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} L_{2n+1}^{2l+1}.$$

Therefore,

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} \frac{1}{L_{2m+1-2j}} \sum_{l=0}^{m-j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} L_{2n+1}^{2l+1} + 4^m$$

$$= \sum_{l=0}^{m} L_{2n+1}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \frac{1}{L_{2m+1-2j}} + 4^m.$$

The next one:

$$\sum_{k=0}^{n} L_{2k+1}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{L_{2(n+1)(2m+1-2j)}}{L_{2m+1-2j}} - \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}}.$$

We use (2.1) with $x = \alpha^{2(n+1)}$ to get

$$L_{2(n+1)(2m+1-2j)} = \sum_{l=0}^{m-j} (-1)^{m-j-l} \frac{2m+1-2j}{2l+1} \binom{m-j+l}{m-j-l} L_{2(n+1)}^{2l+1}.$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{n} L_{2k+1}^{2m+1} &= \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{1}{L_{2m+1-2j}} \\
&\qquad \times \sum_{l=0}^{m-j} (-1)^{m-j-l} \frac{2m+1-2j}{2l+1} \binom{m-j+l}{m-j-l} L_{2(n+1)}^{2l+1} \\
&\qquad - \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}} \\
&= \sum_{l=0}^{m} L_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \frac{(-1)^{m-l}}{L_{2m+1-2j}} \\
&\qquad - \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}}.
\end{aligned}$$

Furthermore,

$$\sum_{k=0}^{n} L_{2k}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} \frac{F_{(2n+1)(2m-2j)}}{F_{2m-2j}} + 2^{2m-1} + \binom{2m}{m} \left(n + \tfrac{1}{2}\right).$$

Now we need the formula (3.2) and its corollary for $x = \alpha^{2n+1}$:

$$F_{(2n+1)(2m-2j)} = F_{2n+1} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1}.$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{n} L_{2k}^{2m} &= \sum_{j=0}^{m-1} \binom{2m}{j} \frac{F_{2n+1}}{F_{2m-2j}} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} L_{2n+1}^{2l+1} + 2^{2m-1} + \binom{2m}{m} \left(n + \tfrac{1}{2}\right) \\
&= F_{2n+1} \sum_{l=0}^{m-1} L_{2n+1}^{2l+1} \sum_{j=0}^{m-1-l} \binom{2m}{j} \binom{m-j+l}{m-j-l-1} \frac{1}{F_{2m-2j}} \\
&\qquad + 2^{2m-1} + \binom{2m}{m} \left(n + \tfrac{1}{2}\right).
\end{aligned}$$

Finally,

$$\sum_{k=0}^{n} L_{2k+1}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} (-1)^j \frac{F_{2(n+1)(2m-2j)}}{F_{2m-2j}} + \binom{2m}{m} (-1)^m (n+1).$$

We use (3.1) again to get

$$F_{2(n+1)(2m-2j)} = L_{2(n+1)} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} 5^l F_{2(n+1)}^{2l+1}.$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{n} L_{2k+1}^{2m} &= \sum_{j=0}^{m-1} \binom{2m}{j} (-1)^j \frac{1}{F_{2m-2j}} L_{2(n+1)} \sum_{l=0}^{m-j-1} \binom{m-j+l}{m-j-l-1} 5^l F_{2(n+1)}^{2l+1} \\ &\quad + \binom{2m}{m} (-1)^m (n+1) \\ &= L_{2(n+1)} \sum_{l=0}^{m-1} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-1-l} (-1)^j \binom{2m}{j} \binom{m-j+l}{m-j-l-1} \frac{5^l}{F_{2m-2j}} \\ &\quad + \binom{2m}{m} (-1)^m (n+1). \end{aligned}$$

## 6. Collection of the Lucas Melham formulas

Here is this list of formulas, valid again for $n, m \in \mathbb{N}_0$:

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{l=0}^{m} L_{2n+1}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \frac{1}{L_{2m+1-2j}} + 4^m,$$

$$\begin{aligned} \sum_{k=0}^{n} L_{2k+1}^{2m+1} &= \sum_{l=0}^{m} L_{2(n+1)}^{2l+1} \sum_{j=0}^{m-l} \binom{2m+1}{j} \binom{m-j+l}{m-j-l} \frac{2m+1-2j}{2l+1} \frac{(-1)^{m-l}}{L_{2m+1-2j}} \\ &\quad - \sum_{j=0}^{m} \binom{2m+1}{j} (-1)^j \frac{2}{L_{2m+1-2j}} \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{n} L_{2k}^{2m} &= F_{2n+1} \sum_{l=0}^{m-1} L_{2n+1}^{2l+1} \sum_{j=0}^{m-1-l} \binom{2m}{j} \binom{m-j+l}{m-j-l-1} \frac{1}{F_{2m-2j}} \\ &\quad + 2^{2m-1} + \binom{2m}{m} (n + \tfrac{1}{2}), \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{n} L_{2k+1}^{2m} &= L_{2(n+1)} \sum_{l=0}^{m-1} F_{2(n+1)}^{2l+1} \sum_{j=0}^{m-1-l} (-1)^j \binom{2m}{j} \binom{m-j+l}{m-j-l-1} \frac{5^l}{F_{2m-2j}} \\ &\quad + \binom{2m}{m} (-1)^m (n+1). \end{aligned}$$

## 7. Applications

With these results we can iterate the summations. For instance we can *sum* the Melham sum, viz.

$$\begin{aligned}\sum_{n=0}^{N}\sum_{k=0}^{n}F_{2k}^{2m+1} &= \sum_{n=0}^{N}\sum_{l=0}^{m}F_{2n+1}^{2l+1}5^{l-m}(-1)^{m-l}\frac{1}{2l+1}\\ &\qquad\times\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{L_{2m-2j+1}}\\ &\qquad-\sum_{n=0}^{N}\frac{1}{5^m}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}\\ &= \sum_{l=0}^{m}5^{l-m}(-1)^{m-l}\frac{1}{2l+1}\sum_{j=0}^{m-l}\binom{2m+1}{j}\binom{m-j+l}{m-j-l}\frac{2m-2j+1}{L_{2m-2j+1}}\\ &\qquad\times\sum_{a=0}^{l}F_{2(N+1)}^{2a+1}\sum_{b=0}^{l-a}\frac{5^{a-l}}{L_{2l-2b+1}}\binom{2l+1}{b}\binom{l-b+a}{l-b-a}\frac{2l-2b+1}{2a+1}\\ &\qquad-(N+1)\frac{1}{5^m}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\frac{F_{2m-2j+1}}{L_{2m-2j+1}}.\end{aligned}$$

The coefficients of $F_{2(N+1)}^{2a+1}$ are now triple sums. In principle, further iterations can be performed, but the results are not too attractive.

## 8. Acknowledgement

A referee has given a pointer to the paper [3]. In this one, expressions for $F_{(2n+1)m}$, etc. are derived (without summing them, as it is done here). They are rederived here as intermediate steps. We decided to stick to our presentation, to keep it self-contained.

## References

[1] R. Melham. Private communication to C. Cooper.
[2] M. Wiemann and C. Cooper, *Divisibility of an F-L Type Convolution*, Applications of Fibonacci Numbers, Vol. 9, Kluwer Acad. Publ., Dordrecht, 2004, 267–287.
[3] M. N. S. Swamy, *On Certain Identities Involving Fibonacci and Lucas Numbers*, The Fibonacci Quarterly, **35.3** (1997), 230–232.

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