

A GENERALIZED FILBERT MATRIX

EMRAH KILIC AND HELMUT PRODINGER

ABSTRACT. A generalized Filbert matrix is introduced, sharing properties of the Hilbert matrix and Fibonacci numbers. Explicit formulae are derived for the LU-decomposition, their inverses, and the Cholesky factorization. The approach is to use q -analysis and to leave the justification of the necessary identities to the q -version of Zeilberger's celebrated algorithm.

1. INTRODUCTION

The Filbert matrix $H_n = (h_{ij})_{i,j=1}^n$ is defined by $h_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the n th Fibonacci number. It has been defined and studied by Richardson [3].

In this paper we will study the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. The size of the matrix does not really matter, and we can think about an infinite matrix \mathcal{F} and restrict it whenever necessary to the first n rows resp. columns and write \mathcal{F}_n .

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$. All the identities we are going to derive hold for general q , and results about Fibonacci numbers come out as corollaries for the special choice of q .

Throughout this paper we will use the following notations: $(x; q)_n = (1-x)(1-xq) \cdots (1-xq^{n-1})$ and the Gaussian q -binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

Furthermore, we will use *Fibonomial coefficients*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 \cdots F_k}.$$

The link between the two notations is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{with } q = -\alpha^{-2}.$$

We will obtain the LU-decomposition $\mathcal{F} = L \cdot U$.

Theorem 1.1. *For $1 \leq d \leq n$ we have*

$$L_{n,d} = q^{\frac{n-d}{2}} \mathbf{i}^{n+d} (-1)^n \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \begin{bmatrix} 2d+r \\ d \end{bmatrix} \begin{bmatrix} n+d+r \\ d \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$L_{n,d} = \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix} \begin{Bmatrix} 2d+r \\ d \end{Bmatrix} \begin{Bmatrix} n+d+r \\ d \end{Bmatrix}^{-1}.$$

Theorem 1.2. For $1 \leq d \leq n$ we have

$$U_{d,n} = q^{\frac{n-d-r-1}{2}+d^2+rd} \mathbf{i}^{n+d+r+1} (-1)^{n+d+r} \begin{bmatrix} 2d+r-1 \\ d-1 \end{bmatrix}^{-1} \begin{bmatrix} n+d+r \\ d \end{bmatrix}^{-1} \begin{bmatrix} n \\ d \end{bmatrix} \frac{1-q}{1-q^n}$$

and its Fibonacci corollary

$$U_{d,n} = (-1)^{r(d+1)} \begin{Bmatrix} 2d+r-1 \\ d-1 \end{Bmatrix}^{-1} \begin{Bmatrix} n+d+r \\ d \end{Bmatrix}^{-1} \begin{Bmatrix} n \\ d \end{Bmatrix} \frac{1}{F_n}.$$

We could also determine the inverses of the matrices L and U .

Theorem 1.3. For $1 \leq d \leq n$ we have

$$L_{n,d}^{-1} = q^{\frac{(n-d)^2}{2}} \mathbf{i}^{n+d} (-1)^d \begin{bmatrix} n+r \\ d+r \end{bmatrix} \begin{bmatrix} n+d-1+r \\ d-1 \end{bmatrix} \begin{bmatrix} 2n-1+r \\ n-1 \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$L_{n,d}^{-1} = (-1)^{(n+1)d+\frac{n(n+1)}{2}+\frac{d(d+1)}{2}} \begin{Bmatrix} n+r \\ d+r \end{Bmatrix} \begin{Bmatrix} n+d-1+r \\ d-1 \end{Bmatrix} \begin{Bmatrix} 2n-1+r \\ n-1 \end{Bmatrix}^{-1}.$$

Theorem 1.4. For $1 \leq d \leq n$ we have

$$U_{d,n}^{-1} = q^{-\frac{n^2}{2}+\frac{d^2}{2}+\frac{r+1}{2}-(d+r)n} \mathbf{i}^{n+d-1+r} (-1)^{n-d} \begin{bmatrix} 2n+r \\ n \end{bmatrix} \begin{bmatrix} n+d+r-1 \\ d+r \end{bmatrix} \begin{bmatrix} n-1 \\ d-1 \end{bmatrix} \frac{1-q^n}{1-q}$$

and its Fibonacci corollary

$$U_{d,n}^{-1} = (-1)^{\frac{n(n+1)}{2}+\frac{d(d-1)}{2}-dn-rn+r} \begin{Bmatrix} 2n+r \\ n \end{Bmatrix} \begin{Bmatrix} n+d+r-1 \\ d+r \end{Bmatrix} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix} F_n.$$

As a consequence we can compute the determinant of \mathcal{F}_n , since it is simply evaluated as $U_{1,1} \cdots U_{n,n}$ (we only state the Fibonacci version).

Theorem 1.5.

$$\det \mathcal{F}_n = (-1)^{\frac{rn(n-1)}{2}} \prod_{d=1}^n \begin{Bmatrix} 2d+r-1 \\ d-1 \end{Bmatrix}^{-1} \begin{Bmatrix} 2d+r \\ d \end{Bmatrix}^{-1} \frac{1}{F_d}.$$

Now we determine the inverse of the matrix \mathcal{F} . This time it depends on the dimension, so we compute $(\mathcal{F}_n)^{-1}$.

Theorem 1.6. For $1 \leq i, j \leq n$:

$$\begin{aligned} (\mathcal{F}_n)_{i,j}^{-1} &= q^{\frac{i^2+j^2+r+1}{2}-(i+j+r)n} \mathbf{i}^{i+j+r+1} (-1)^{i+j+1} \\ &\quad \times \begin{bmatrix} n+r+i \\ n \end{bmatrix} \begin{bmatrix} n+r+j \\ n \end{bmatrix} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} \frac{(1-q^n)^2}{(1-q^{r+i+j})(1-q)} \end{aligned}$$

and its Fibonacci corollary

$$(\mathcal{F}_n)_{i,j}^{-1} = (-1)^{\frac{i(i-1)}{2} + \frac{j(j-1)}{2} + n(i+j+r)+r} \\ \times \begin{Bmatrix} n+r+i \\ n \end{Bmatrix} \begin{Bmatrix} n+r+j \\ n \end{Bmatrix} \begin{Bmatrix} n-1 \\ i-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix} \frac{F_n^2}{F_{r+i+j}}.$$

The results about the inverse matrix are not new, compare [1, 4], but have been included for completeness.

We can also find the Cholesky decomposition $\mathcal{F} = \mathcal{C} \cdot \mathcal{C}^T$ with a lower triangular matrix \mathcal{C} .

Theorem 1.7. For $n \geq d$:

$$\mathcal{C}_{n,d} = (-1)^n \mathbf{i}^{n+r+\frac{r+1}{2}} q^{\frac{n}{2} - \frac{r+1}{4} + \frac{d(d-1)}{2} + \frac{rd}{2}} \frac{\sqrt{(1-q^{2d+r})(1-q)}}{1-q^{2n+r}} \begin{bmatrix} 2n+r \\ n-d \end{bmatrix} \begin{bmatrix} 2n+r-1 \\ n-1 \end{bmatrix}^{-1}$$

and its Fibonacci corollary

$$\mathcal{C}_{n,d} = (-1)^{\frac{d(d-1)}{2} + \frac{r(d+1)}{2}} \frac{\sqrt{F_{2d+r}}}{F_{2n+r}} \begin{Bmatrix} 2n+r \\ n-d \end{Bmatrix} \begin{Bmatrix} 2n+r-1 \\ n-1 \end{Bmatrix}^{-1}.$$

Notice that for odd r , even the Fibonacci version may contain complex numbers.

2. PROOFS

In order to show that indeed $\mathcal{F} = L \cdot U$, we need to show that for any m, n :

$$\sum_d L_{m,d} U_{d,n} = \mathcal{F}_{m,n} = \alpha^{-m-n-r+1} \frac{1-q}{1-q^{m+n+r}}.$$

In rewritten form the formula to be proved reads

$$\sum_d (q^{d^2+(r-1)d-r} - q^{d^2+(r+1)d}) \begin{bmatrix} 2m+r \\ m-d \end{bmatrix} \begin{bmatrix} 2n+r \\ n-d \end{bmatrix} \\ = \frac{(1-q^{2n+r})(1-q^{2m+r})}{1-q^{m+n+r}} \begin{bmatrix} 2m+r-1 \\ m-1 \end{bmatrix} \begin{bmatrix} 2n+r-1 \\ n-1 \end{bmatrix}.$$

Nowadays, such identities are a routine verification using the q -Zeilberger algorithm, as described in the book [2]. We used Zeilberger's own version [6], which is a Maple program. Mathematica users would get the same results using a package called `qZeil` [5].

For interest, we also state (as a corollary) the corresponding Fibonacci identity:

$$\sum_d (-1)^{r(d-1)} F_{2d+r} \begin{Bmatrix} 2m+r \\ m-d \end{Bmatrix} \begin{Bmatrix} 2n+r \\ n-d \end{Bmatrix} = \frac{F_{2n+r} F_{2m+r}}{F_{m+n+r}} \begin{Bmatrix} 2m+r-1 \\ m-1 \end{Bmatrix} \begin{Bmatrix} 2n+r-1 \\ n-1 \end{Bmatrix}.$$

Now we move to the inverse matrices. Since L and L^{-1} are lower triangular matrices, we only need to look at the entries indexed by (m, n) with $m \geq n$:

$$\sum_{n \leq d \leq m} L_{m,d} L_{d,n}^{-1} \\ = \sum_{n \leq d \leq m} q^{\frac{m-d}{2}} \mathbf{i}^{m+d} (-1)^m \begin{bmatrix} m-1 \\ d-1 \end{bmatrix} \begin{bmatrix} 2d+r \\ d \end{bmatrix} \begin{bmatrix} m+d+r \\ d \end{bmatrix}^{-1} \\ \times q^{\frac{(n-d)^2}{2}} \mathbf{i}^{n+d} (-1)^n \begin{bmatrix} d+r \\ n+r \end{bmatrix} \begin{bmatrix} n+d-1+r \\ n-1 \end{bmatrix} \begin{bmatrix} 2d-1+r \\ d-1 \end{bmatrix}^{-1}$$

$$= \frac{1}{1 - q^{2m+r}} \begin{bmatrix} 2m+r-1 \\ m-1 \end{bmatrix}^{-1} \mathbf{i}^{m+n} (-1)^{m+n} \\ \times \sum_{n \leq d \leq m} q^{\frac{m-d}{2} + \frac{(n-d)^2}{2}} (1 - q^{2d+r}) (-1)^d \begin{bmatrix} 2m+r \\ m-d \end{bmatrix} \begin{bmatrix} n+d-1+r \\ n+r \end{bmatrix} \begin{bmatrix} d-1 \\ n-1 \end{bmatrix}.$$

The q -Zeilberger algorithm can evaluate the sum, and it is indeed $[m = n]$, as predicted.

The argument for $U \cdot U^{-1}$ is similar:

$$\sum_{m \leq d \leq n} U_{m,d} U_{d,n}^{-1} \\ = (-1)^{m+n} \mathbf{i}^{m+n} q^{-\frac{m}{2} + m^2 + rm - \frac{n^2}{2} - rn} \begin{bmatrix} 2m+r-1 \\ m-1 \end{bmatrix}^{-1} \begin{bmatrix} 2n+r-1 \\ n-1 \end{bmatrix} \frac{1 - q^{2n+r}}{1 - q^{n+m+r}} \\ \times \sum_{m \leq d \leq n} (-1)^d q^{\frac{d(d+1)}{2} - dn} \begin{bmatrix} n+d+r-1 \\ d-m \end{bmatrix} \begin{bmatrix} n+m+r \\ n-d \end{bmatrix}.$$

Again, the q -Zeilberger algorithm evaluates this to $[m = n]$.

Now we turn to the inverse matrix:

$$((\mathcal{F}_n)^{-1} \mathcal{F}_n)_{i,k} = \mathbf{i}^{i-k} (-1)^i q^{\frac{i^2+k}{2} - (i+r)n+r} (1 - q^n)^2 \begin{bmatrix} n+r+i \\ n \end{bmatrix} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \\ \times \sum_{j=1}^n q^{\frac{j(j+1)}{2} - jn} (-1)^j \begin{bmatrix} n+r+j \\ n \end{bmatrix} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} \frac{1}{(1 - q^{j+k+r})(1 - q^{r+i+j})}.$$

And the q -Zeilberger algorithm evaluates this again to $[i = k]$.

The Cholesky verification goes like this:

$$\sum_{d=1}^{\min\{m,n\}} \mathcal{C}_{m,d} \mathcal{C}_{n,d} \\ = (-1)^{m+n+r} \mathbf{i}^{m+n+r+1} q^{\frac{m+n-r-1}{2}} \frac{1 - q}{(1 - q^{2m+r})(1 - q^{2n+r})} \\ \times \begin{bmatrix} 2n+r-1 \\ n-1 \end{bmatrix}^{-1} \begin{bmatrix} 2m+r-1 \\ m-1 \end{bmatrix}^{-1} \sum_d q^{d(d-1)+rd} (1 - q^{2d+r}) \begin{bmatrix} 2m+r \\ m-d \end{bmatrix} \begin{bmatrix} 2n+r \\ n-d \end{bmatrix}.$$

And again the q -Zeilberger algorithm evaluates this to be

$$\frac{1 - q}{1 - q^{m+n+r}} \mathbf{i}^{m+n+r-1} q^{\frac{m+n+r-1}{2}},$$

as it should.

REFERENCES

- [1] J. E. Anderson and C. Berg, *Quantum Hilbert matrices and orthogonal polynomials*, J. Comput. Appl. Math. **223.3** (2009), 723–729.
- [2] M. Petkovsek, H. Wilf, and D. Zeilberger, *A = B*, A K Peters, Wellesley, MA, 1996.
- [3] T. Richardson, *The Filbert matrix*, The Fibonacci Quarterly, **39.3**(2001), 268–275.
- [4] T. M. Richardson, *Generalized Cauchy determinants*, Integers, (8) (2008), A44, 11 pages.
- [5] A. Riese, <http://www.risc.uni-linz.ac.at/research/combinat/software/qZeil/index.php>.
- [6] D. Zeilberger, <http://www.math.rutgers.edu/~zeilberg/tokhniot/qEKHAD>.

MSC2000: 11B39

TOBB UNIVERSITY OF ECONOMICS AND TECHNOLOGY, MATHEMATICS DEPARTMENT, 06560 ANKARA
TURKEY

E-mail address: `ekilic@etu.edu.tr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH, 7602 STELLENBOSCH, SOUTH AFRICA

E-mail address: `hproding@sun.ac.za`