DOUBLY INTERSPERSED SEQUENCES, DOUBLE INTERSPERSIONS, AND FRACTAL SEQUENCES

CLARK KIMBERLING

Abstract. It is proved that distinct positive Fibonacci sequences eventually intersperse or else doubly intersperse. Necessary and sufficient conditions for the latter are given. A rectangular array of infinitely many rows is called a double interspersion if every pair of its row sequences doubly intersperse. The union of the odd-numbered columns of a double interspersion yield, as an order array, an interspersion, as does the union of even-numbered columns. A special example of a double interspersion whose rows are Fibonacci sequences is examined; the columns of the associated pair of interspersions all satisfy a certain 3rd-order linear recurrence.

1. Introduction

We begin with four definitions. Sequences $s = (s_1, s_2, \ldots)$ and $x = (x_1, x_2, \ldots)$ *eventually intersperse* if there exist i and j such that

$$0 < s_i < x_j < s_{i+1} < x_{j+1} < s_{i+2} < x_{j+2} < \cdots ; \tag{1.1}$$

that is, if $s_{i+h} < x_{j+h}$ for all $h \geq 0$. Sequences s and x *eventually doubly intersperse* if there exist i and j such that

$$0 < s_i < s_{i+1} < x_j < x_{j+1} < s_{i+2} < s_{i+3} < x_{j+2} < x_{j+3} < \cdots . \tag{1.2}$$

If i is least possible in (1.2), then x *doubly intersperses* s from s_i and x_j, and we write $D(s, x) = (i, j)$.

A *Fibonacci sequence* is a sequence $s = (s_1, s_2, \ldots)$ of positive real numbers (not necessarily integers) satisfying

$$s_m = s_{m-1} + s_{m-2} \tag{1.3}$$

for $m \geq 3$. The classical Fibonacci sequence (F_m) is given by (1.3) using initial terms $F_1 = s_1 = 1$ and $F_2 = s_2 = 1$.

The main result we shall prove is that each pair of distinct Fibonacci sequences eventually intersperse or eventually doubly intersperse. We shall also examine arrays consisting of infinitely many Fibonacci row sequences, every pair of which eventually doubly intersperse. Double interspersions yield pairs of ordinary interspersions as defined in [1], and we consider their fractal sequences.

2. Sequences: Interspersed, Doubly Interspersed

Lemma 1. *Suppose that s and x are Fibonacci sequences satisfying*

$$s_i < x_j < s_{i+1} < x_{j+1} < s_{i+2} \tag{2.1}$$

for some i and j. Then

$$s_{i+h} < x_{j+h} < s_{i+h+1} \tag{2.2}$$

for all $h \geq 0$ (so that s and x eventually intersperse).

Proof. For $h = 0$ and $h = 1$, the inequalities (2.2) hold by hypothesis. Assume for arbitrary $h \geq 2$ that

$$s_{i+h-2} < x_{j+h-2} < s_{i+h-1} \quad \text{and} \quad s_{i+h-1} < x_{j+h-1} < s_{i+h}.$$

Then

$$s_{i+h-2} + s_{i+h-1} < x_{i+h-2} + x_{j+h-1} < s_{i+h-1} + s_{i+h},$$

so that (2.2) holds. □

Lemma 2. *Suppose that s and x are Fibonacci sequences satisfying $x_k = s_k$ and $x_{k+h} = s_{k+h}$ for some $k \geq 1$ and $h \geq 1$. Then $x = s$.*

Proof. Given that $x_k = s_k$, if $x_{k+1} = s_{k+1}$, then clearly $x = s$. On the other hand, if $x_{k+1} < s_{k+1}$, then

$$x_{k+2} = x_k + x_{k+1} < s_k + s_{k+1} = s_{k+2},$$

and inductively, $x_{k+h} < s_{k+h}$ for all $h \geq 1$, contrary to the hypothesis. If $s_{k+1} < x_{k+1}$ then inductively $s_{k+h} = x_{k+h}$ for all $h \geq 1$, a contradiction. Therefore, $x = s$. □

Lemma 3. *If s is a Fibonacci sequence, then*

$$s_n = s_1 F_{n-3} + s_2 F_{n-2}$$

for all $n \geq 3$, where $F_0 = 0$.

Proof. By induction. □

Theorem 1. *Distinct positive Fibonacci sequences s and x eventually doubly intersperse if and only if there exist $a \geq 1$ and $b \geq 1$ such that*

$$x_b - s_a = \tau(s_{a+1} - x_{b+1}), \tag{2.3}$$

where τ denotes the golden ratio, $(1 + \sqrt{5})/2$. Otherwise, s and x eventually intersperse.

Proof. Suppose that s and x are distinct positive Fibonacci sequences and i and j satisfy

$$s_{i-1} \leq s_i < x_j. \tag{2.4}$$

The following cases are exhaustive:

Case 1. $s_i < x_j < s_{i+1} < x_{j+1} < s_{i+2}$;

Case 2. $s_i < x_j < x_{j+1} \leq s_{i+1}$;

Case 3. $s_i < x_j < s_{i+1} < s_{i+2} \leq x_{j+1}$.

In Case 1, the sequences s and x eventually intersperse by Lemma 1. In Case 3, there are two subcases:

$$x_{j-1} \leq s_i < x_j < s_{i+1} < s_{i+2} \leq x_{j+1}; \tag{2.5}$$

$$s_i < x_{j-1} < x_j < s_{i+1} < s_{i+2} \leq x_{j+1}. \tag{2.6}$$

However, for (2.5), the inequalities $x_{j-1} \leq s_i$ and $x_j < s_{i+1}$ imply $x_{j+1} < s_{i+2}$, a contradiction. For (2.6), clearly $s_{i+1} < x_{j+1} < s_{i+3}$. Now if $s_{i+3} < x_{j+2}$, then s and x eventually intersperse by Lemma 1. On the other hand, if

$$s_{i+1} < x_{j+1} < x_{j+2} \leq s_{i+3},$$

then this possibility is covered by Case 1 (with a shift of indices). Therefore all that remains is Case 2, which breaks into two cases.

Case 2.1. $x_{j+2} < s_{i+2}$;

Case 2.2. $x_{j+2} \geq s_{i+2}$.

In Case 2.1, we have $s_{i-1} \le s_i < x_j$ and $s_i < x_{j+1}$, so that

$$s_{i+1} = s_{i-1} + s_i < x_j + x_{j+1} = x_{j+2}.$$

Consequently,

$$s_i < x_j < x_{j+1} \le s_{i+1} < x_{j+2} < s_{i+2},$$

which implies

$$s_{i+1} < x_{j+2} < s_{i+2} < x_{j+3} < s_{i+3},$$

so that s and x eventually intersperse by Lemma 1.

In Case 2.2, if there exists $h \ge 1$ for which

$$s_{i+h} < x_{j+h} < s_{i+h+1} < x_{j+h+1} < s_{i+h+2},$$

then s and x eventually intersperse by Lemma 1. In the remaining case that no such h exists, we have

$$s_i < x_j < x_{j+1} \le s_{i+1} < s_{i+2} \le x_{j+2} < x_{j+3} \le s_{i+3} < s_{i+4} \le \cdots . \tag{2.7}$$

Although (2.7) shows $x_{j+h} \le s_{i+h}$ for infinitely many h, at most one of these is $x_{j+h} = s_{i+h}$ by Lemma 2. Likewise, at most one of the inequalities $s_{i+k} \le x_{j+k}$ in (2.7) is actually $s_{i+k} = x_{j+k}$ Therefore, in (2.7) we can (and do) assume i large enough that

$$s_i < x_j < x_{j+1} < s_{i+1} < s_{i+2} < x_{j+2} < x_{j+3} < s_{i+3} < s_{i+4} < \cdots , \tag{2.8}$$

which is to say that s and x eventually doubly intersperse. From (2.8) we extract two families of inequalities:

$$\begin{aligned} s_i &< x_j \\ s_i + s_{i+1} = s_{i+2} &< x_{j+2} = x_j + x_{j+1} \\ 2s_i + 3s_{i+1} = s_{i+4} &< x_{j+4} = 2x_j + 3x_{j+1} \\ &\vdots \\ F_n s_i + F_{n+1} s_{i+1} &< F_n x_j + F_{n+1} x_{j+1}, \text{ for } n = 1, 3, 5, \cdots \end{aligned}$$

and

$$\begin{aligned} x_{j+1} &< s_{i+1} \\ x_j + 2x_{j+1} = x_{j+3} &< s_{i+3} = s_i + 2s_{i+1} \\ 3x_j + 5x_{j+1} = x_{j+5} &< s_{i+5} = 3s_i + 5s_{i+1} \\ &\vdots \\ F_n x_j + F_{n+1} x_{j+1} &< F_n s_i + F_{n+1} s_{i+1}, \text{ for } n = 2, 4, 6, \cdots . \end{aligned}$$

These two families are equivalent to the following pair:

$$\frac{F_{n+1}}{F_n} < \frac{x_j - s_i}{s_{i+1} - x_{j+1}} \text{ for } n = 1, 3, 5, \cdots ;$$
$$\frac{F_{n+1}}{F_n} > \frac{x_j - s_i}{s_{i+1} - x_{j+1}} \text{ for } n = 2, 4, 6, \cdots .$$

Taking limits as $n \to \infty$ results in (2.3). To summarize, if s and x eventually doubly intersperse, then (2.3) holds. The converse follows from the fact that the steps leading to (2.3) are reversible. □

According to Theorem 1, any s and x that do *not* eventually *doubly* intersperse must eventually intersperse, and one should surely then ask "where the interspersion commences." A practical way to answer this question for given specific sequences is to determine the least i for which there is a j satisfying (2.1).

To determine a and b as in (2.3), one can list the first few terms of the sequences $(x_n - s_n)$, $(x_{n+1} - s_n)$, $(x_{n+2} - s_n), \ldots$, writing all the terms in the form $h - k\tau$. Then look for a term $h - k\tau$ whose successor is $h + k - h\tau$. The reason this works is that if $x_b - s_a = h - k\tau$, then $x_b - s_a = (-\tau)(h + k - h\tau)$ so that $h + k - h\tau$ is the desired number $x_{b+1} - s_{a+1}$.

Example 1. *Define* $s_1 = 1$, $s_2 = 4$, $x_1 = \tau + 1$, *and* $x_2 = 3$, *so that*

$$s = (1, 4, 5, 9, 14, 23, \ldots),$$
$$x = (\tau + 1, 3, \tau + 4, \tau + 7, 2\tau + 11, \ldots),$$

and x *doubly intersperses* s *from* $s_2 = 4$ *and* $x_3 = \tau + 4$. *The necessary and sufficient condition (2.3) holds for* $a = b = 1$.

Corollary 1. *Suppose that* s *and* x *eventually doubly intersperse and that* a, b *are indices for which (2.3) holds. Then*

$$s_{a+n} - x_{b+n} = (F_{n+1} - \tau F_n)(s_a - x_b)$$

for $n = 0, 1, 2, \ldots$.

Proof. The hypothesis (2.3) is equivalent to

$$s_{a+1} - x_{b+1} = (1 - \tau)(s_a - x_b).$$

Then

$$\begin{aligned} s_{a+2} - x_{b+2} &= (s_{a+1} - x_{b+1}) + (s_a - x_b) \\ &= (F_3 - \tau F_2)(s_a - x_b), \end{aligned}$$

and the proof follows by induction. □

Corollary 2. *Suppose that* s *and* x *eventually doubly intersperse and that* a, b *are indices for which (2.3) holds. Then*

$$x_{b+n} = (F_{n+1} - \tau F_n)x_b + F_n(s_{a+1} + (\tau - 1)s_a) \tag{2.9}$$

for $n = 0, 1, 2, \ldots$.

Proof. By (2.3),

$$\begin{aligned} x_{b+1} &= s_{a+1} - (1/\tau)(x_b - s_a) \\ &= (1 - \tau)x_b + s_{a+1} + (\tau - 1)\, s_a \\ &= (F_2 - \tau F_1)x_b + F_1(s_{a+1} + (\tau - 1)\, s_a), \end{aligned}$$

and (2.9) follows by induction. □

Example 2. *It is easy to construct sequences which eventually doubly intersperse the Fibonacci numbers. Here we choose to insert* x_1 *and* x_2 *between* 5 *and* 8. *Choosing* $x_1 = 6$, *we determine the other terms of* x *by (2.9) using* $a = b = 2$:

$$s = (5, 8, 13, 21, 34, \ldots),$$
$$x = (6, 9 - \tau, 15 - \tau, 24 - 2\tau, 39 - 3\tau, \ldots, F_{n+4} + F_n - \tau F_{n-1}, \ldots).$$

Having succeeded with $x_1 = 6$, it is easy to check that if $x_1 = 7$, there is no choice of x_2 for which (2.8) holds using $i = j = 1$. In order to find an upper bound for x_1 in cases such as this, we turn to another corollary.

Corollary 3. *Suppose that $s = (s_1, s_2, \ldots)$ and $x = (x_1, x_2 \ldots)$ are positive Fibonacci sequences satisfying*

$$s_1 < x_1 < x_2 < s_2 < s_3 < x_3 < x_4 < s_4 < s_5 < \cdots . \tag{2.10}$$

Then

$$x_1 < (2 - \tau)s_1 + (\tau - 1)s_2.$$

Proof. Taking $n = 1$, $a = 1$, and $b = 1$ in (2.9) gives

$$(1 - \tau)x_1 + s_2 + (\tau - 1)s_1 = x_2 > x_1,$$

so that

$$x_1 < \frac{s_2 + (\tau - 1)s_1}{\tau} = (2 - \tau)s_1 + (\tau - 1)s_2.$$

□

Example 3. *Extending Example 2, for $s = (5, 8, 13, \ldots)$, we can make a sequence x satisfying (2.10) (so that x doubly intersperses s from s_2 and x_3) by choosing any x_1 in the interval $(5, 3\tau + 2)$ and then applying (2.9). One such sequence is*

$$x = (4\tau, 5\tau - 1, 9\tau - 1, 14\tau - 2, 23\tau - 3, \ldots, (F_{n+3} + F_{n+2})\tau - F_{n-1}, \ldots).$$

3. Double Interspersions

A Stolarsky interspersion [3] is an array of positive integers, each occurring exactly once, such that the rows of the array are Fibonacci sequences, each pair of which eventually intersperse, and such that the first column of the array is increasing. This last condition ensures that the first row is the sequence $(F_2, F_3, \ldots) = (1, 2, 3, 5, 8, \ldots)$ of distinct Fibonacci numbers. Among Stolarsky interspersions are the original Stolarsky array [8], the Wythoff array [5, 7], the golden array [2], and the Wythoff dual array [2].

An array of real numbers will be called a *double interspersion* if every pair of its row sequences eventually doubly intersperse. We shall present a double interspersion $R = \{R(n, k)\}$ in which every row is a positive Fibonacci sequence. By (2.3), R cannot consist solely of integers; however, we shall construct from R an array which is a double interspersion consisting solely of positive integers, although the rows are not Fibonacci sequences. The array R, defined by

$$R(n, k) = \begin{cases} F_{k+1} & \text{if } n = 1 \\ F_{n+k-1} + F_{k+2} - \tau F_{k+1} & \text{if } n \geq 2, \end{cases} \tag{3.1}$$

begins thus:

$$\begin{array}{ccccc} 1 & 2 & 3 & 5 & 8 \\ 3 - \tau & 5 - 2\tau & 8 - 3\tau & 13 - 5\tau & 21 - 8\tau \\ 4 - \tau & 6 - 2\tau & 10 - 3\tau & 16 - 5\tau & 26 - 8\tau \\ 5 - \tau & 8 - 2\tau & 13 - 3\tau & 21 - 5\tau & 34 - 8\tau \\ 7 - \tau & 11 - 2\tau & 18 - 3\tau & 29 - 5\tau & 47 - 8\tau \\ 10 - \tau & 16 - 2\tau & 26 - 3\tau & 42 - 5\tau & 68 - 8\tau \end{array} .$$

If the numbers in R are jointly ranked in increasing order and then each is replaced by its rank, the resulting array (A161179 in the Online Encyclopedia of Integer Sequences [6]) begins thus:

1	4	7	12	17	24	31	40	49
2	3	8	11	18	23	32	39	50
5	6	13	16	25	30	41	48	61
9	10	19	22	33	38	51	58	73
14	15	26	29	42	47	62	69	86
20	21	34	37	52	57	74	81	100 .

Denoting this *order array* by $\widehat{R}$, we find inductively that

$$\widehat{R}(n,k) = \begin{cases} (k^2+2k-1)/2 & \text{if } k \text{ is odd and } n=1 \\ (k^2+2k)/2 & \text{if } k \text{ is even and } n=1 \\ [n^2+(2k-1)n+k^2-2k-1]/2 & \text{if } k \text{ is odd and } n \geq 2 \\ [n^2+(2k-3)n+k^2-2k]/2 & \text{if } k \text{ is even and } n \geq 2. \end{cases} \tag{3.2}$$

The arrays $R(n,k)$ and $\widehat{R}(n,k)$ are clearly double interspersions which share the same commencement function D, defined just after (1.2). This function, one can easily show, is given by

$$D(n,k) = \begin{cases} (2,3) & \text{if } (n,k)=(1,1) \\ (k-n,1) & \text{otherwise.} \end{cases}$$

4. From Double to Two Singles

An arbitrary double interspersion R necessarily splits into two interspersions $\widehat{R}_1$ and $\widehat{R}_2$ in the following way: $\widehat{R}_1$ is the order array of the union of the odd-numbered columns of R, and $\widehat{R}_2$ is the order array of the union of the even-numbered columns of R *after the first two rows have been swapped.* Taking R to be the array (3.1), or equivalently, taking R to the array $\widehat{R}$ in (3.2), we find inductively that

$$\widehat{R}_1(n,k) = \begin{cases} k^2 & \text{if } n=1 \\ k^2+(n-2)k+(n^2-2n+4)/4 & \text{if } n \text{ is even} \\ k^2+(n-2)k+(n^2-2n+1)/4 & \text{if } n \text{ is odd and } n \geq 3; \end{cases}$$

this array is indexed [6] as A163253. Shown here is a corner of $\widehat{R}_1$:

1	4	9	16	25	36	49	64	81
2	5	10	17	26	37	50	65	82
3	7	13	21	31	43	57	73	91
6	11	18	27	38	51	66	83	102
8	14	22	32	44	58	74	92	112
12	19	28	39	52	67	84	103	124 .

The columns of $\widehat{R}_1$, beginning with row 5, satisfy a 3rd-order recurrence:

$$a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1. \tag{4.1}$$

It is noteworthy that the set of non-squares (A000037 in [6]) can be partitioned into a set of sequences satisfying (4.1). To do so, merely remove the first row from $\widehat{R}_1$, obtaining A163254, of which the order array is an interspersion, A163255.

The array $\widehat{R}_2$ is given by

$$\widehat{R}_2(n,k) = \begin{cases} k^2+k-1 & \text{if } n=1 \\ k^2+k & \text{if } n=2 \\ k^2+(n-1)k+(n-1)(n-3)/4 & \text{if } n \text{ is odd and } n \geq 3 \\ k^2+(n-1)k+n(n-4)/4 & \text{if } n \text{ is even and } n \geq 4. \end{cases}$$

Beginning with row 6, the columns of $\widehat{R}_2$ satisfy (4.1). For each n, row n of $\widehat{R}_2$ has exactly one term that is also in row n of $\widehat{R}_1$.

It is proved elsewhere [4] that every interspersion $I(n,k)$ begets a fractal sequence a by the rule

$$a(m) = \text{the number } n \text{ such that } m = I(n,k).$$

The fractal sequence A163256 of $\widehat{R}_1$ is especially simple if broken into segments beginning with 1:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 \\ 1 & 2 & 4 & 3 & 5 \\ 1 & 2 & 4 & 6 & 3 & 5 & 7 \\ 1 & 2 & 4 & 6 & 8 & 3 & 5 & 7 & 9 \\ 1 & 2 & 4 & 6 & 8 & 10 & 3 & 5 & 7 & 9 & 11 \end{array} \quad .$$

If $n > 1$, then row n comes from row $n-1$ by inserting $2n$ just before 3 and $2n+1$ at the end. The resulting sequence

$$(1,2,3,1,2,4,3,5,1,2,4,6,3,5,7,\ldots)$$

is "fractal" in the sense that it contains infinitely many proper subsequences that are identical to the sequence itself. (For the full definition of fractal sequence, see [4]. The array $\widehat{R}_2$ and its fractal sequence appear in [6] as A163257 and A163258, respectively.)

It is easy to construct other double interspersions and associated interspersions and fractal sequences. Another particularly simple one, in some sense a dual to R, is given by

$$T(n,k) = \begin{cases} F_{k+1} & \text{if } n=1; \\ \tau(F_{n+k-2}+1)F_{k+2} - F_{k+1} & \text{if } n \geq 2, \end{cases}$$

it begins thus:

$$\begin{array}{ccccc} 1 & 2 & 3 & 5 & 8 \\ 2\tau-2 & 3\tau-3 & 5\tau-5 & 8\tau-8 & 13\tau-13 \\ 3\tau-1 & 4\tau-2 & 7\tau-3 & 11\tau-5 & 18\tau-8 \\ 4\tau-1 & 6\tau-2 & 10\tau-3 & 16\tau-5 & 26\tau-8 \\ 6\tau-1 & 9\tau-2 & 15\tau-3 & 24\tau-5 & 39\tau-8 \\ 9\tau-1 & 14\tau-2 & 23\tau-3 & 37\tau-5 & 60\tau-8 \end{array} \quad .$$

References

[1] C. Kimberling, *Interspersions and dispersions*, Proceedings of the American Mathematical Society, **117** (1993), 313–321.

[2] C. Kimberling, *Ordering the set of all positive Fibonacci sequences*, Applications of Fibonacci Numbers, Proceedings of the Fifth International Conference on Fibonacci Numbers and Their Applications, Vol. 5, G. E. Bergum, A. N. Philippou, A. F. Horadam, editors, Kluwer Academic Publishers, Dordrecht, 1993, 405–416.

[3] C. Kimberling, *Stolarsky interspersions*, Ars Combinatoria, **35** (1995), 129–138.
[4] C. Kimberling, *Fractal sequences and interspersions*, Ars Combinatoria, **45** (1997), 157–168.
[5] D. R. Morrison, *A Stolarsky array of Wythoff pairs*, A Collection of Manuscripts Related to the Fibonacci Sequence, Fibonacci Association, Santa Clara, 1980, 134–136.
[6] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/.
[7] N. J. A. Sloane, *Classic Sequences*, http://www.research.att.com/~njas/sequences/classic.html.
[8] K. B. Stolarsky, *A set of generalized Fibonacci sequences such that each natural number belongs to exactly one*, The Fibonacci Quarterly, **15** (1977), 224.

MSC2000: 11B39

Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, IN 47722
E-mail address: ck6@evansville.edu