USE OF DETERMINANTS TO PRESENT IDENTITIES INVOLVING FIBONACCI AND RELATED NUMBERS

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Abstract. Let $\mathcal{S}_1$ denote a sequence of variables y_n, $n \in \mathbb{Z}$, subject to some difference equation. Let $\mathcal{S}_2$ denote a sequence of $n \times n$ determinants T_n, with elements defined in terms of the members of some sequence of type $\mathcal{S}_1$, in such a way that the T_n also obey a difference equation, proved as Proposition 1. This is used to produce determinantal identities. From a wide range of examples studied, a selection of these identities is presented, some quite striking, in which the Fibonacci, and sometimes Lucas or Jacobsthal numbers appear in either the y_n or the T_n role, or in some cases both roles.

1. Introduction

Let $\mathcal{S}_1$ denote a sequence $\{y_0, y_1, y_2, \dots\}$ of real quantities y_n governed by a difference equation

$$y_n = ay_{n-1} + by_{n-2}, \tag{1.1}$$

where a, b, y_0, y_1 are given real numbers in terms of which all y_n, $n \in \mathbb{Z}$ are determined.

Let $\mathcal{S}_2$ denote a sequence of $n \times n$ determinants

$$T_n(x, y_0, y_1, a, b) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_{n-2} & y_{n-1} & y_n \\ -x & y_1 & y_2 & \cdots & y_{n-3} & y_{n-2} & y_{n-1} \\ 0 & -x & y_1 & \cdots & y_{n-4} & y_{n-3} & y_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & y_1 & y_2 & y_3 \\ 0 & 0 & 0 & \cdots & -x & y_1 & y_2 \\ 0 & 0 & 0 & \dots & 0 & -x & y_1 \end{vmatrix}, \tag{1.2}$$

where x is an indeterminate. Here x is normally a suitably chosen real number, but T_n may also be viewed as a polynomial of degree $(n-1)$ in x. The sequence $\mathcal{S}_2$ may be viewed as a transform of the sequence $\mathcal{S}_1$ in the sense of the definition [14, 20] of the Hankel transform.

In Section 2 below, we present and prove Proposition 1 for the T_n. The purpose of this paper is to exploit Proposition 1 to obtain identities in which the Fibonacci numbers F_n appear in a starring role, and the Lucas numbers L_n, and the Jacobsthal numbers J_n, occur occasionally.

The subsections of Section 3 contain various examples. Examples 1 and 2 in Sections 3.1 and 3.2 use simple sequences $\mathcal{S}_1$ to generate determinantal formulas for the F_n. Examples 3 and 4 use the F_n to define $\mathcal{S}_1$. Example 5 is an example chosen so that the F_n appear as elements of both $\mathcal{S}_1$ and $\mathcal{S}_2$. Section 3.6 shows that Example 5 is a special case of a class of examples wherein both $\mathcal{S}_1$ and $\mathcal{S}_2$ involve the same sequences. Section 3.7 uses the Jacobsthal numbers to define $\mathcal{S}_1$, and produces determinantal formulas for the F_n in terms of the J_n. The origin of this example, which provided the starting point of the present study, is indicated in Section 3.7.

Section 4 takes a brief look at what can be learned when the y_n of $\mathcal{S}_1$ obey a third order difference equation.

A key feature of the present studies is the systematic use of Proposition 1 in the production of determinantal identities. It is to be noted that all of these therefore feature the determinants of $n \times n$ matrices of Hessenberg type which are not tridiagonal. Most of the ideas involved here are echoes of themes familiar from the existing literature. There is of course a long tradition of work in the general area of present interest involving many types of determinants of $n \times n$ matrices, from the early items [3, see B-24, proposed by Br. U. Alfred, B-12 proposed by P. F. Byrd, and B-13 proposed by S. L. Basin, with solutions to B-12 and B-13 by M. Bicknell], [15] to [4]; see also [19]. Such studies like those on the determinants of tridiagonal matrices [8, 7, 6] however seldom, feature results that appear below or can be recast in a form seen here. Papers in which the fact that the elements of the determinants involved obey a recurrence relation is utilized, as here, include [12, 2, 11, 1, 13]. Contact with the results given here should most naturally be expected in papers which explicitly focus on Hessenberg matrices like [18, 8, 10, 5, 24]. Many results, whose value is acknowledged, resemble results seen below but even special cases thereof which can then be found below are few in number and of a simple nature. However the result $\mathcal{E}_{n,t=0} = F_n$ arising from Proposition 2 of [10] for $t = 0$ is equivalent to the result of Example 1 below; the same applies to the result $|\mathrm{A}_{n,0}| = F_n$ from Proposition 2.1 of [5] for $t = 0$. Of course the papers from which these two cases have been picked out contain a good range of results beyond the simple facts mentioned but developed along lines that are different than those of the present work.

1.1. **Fibonacci, Lucas, and Jacobsthal Numbers.** These well-known sequences of numbers occur regularly throughout the formalism of this paper. Information about each of them can be found in [22]: go to pages 629, 1111 and 951 for the F_n, L_n, and J_n. For a compendium of identities including several used below for the Fibonacci and Lucas numbers, see [9, 21].

The Fibonacci and Lucas numbers are governed by the same difference equation

$$y_{n+2} = y_{n+1} + y_n, \qquad y_n = F_n \quad \text{or} \quad y_n = L_n,$$

but different initial conditions

$$F_0 = 0, \quad F_1 = 1, \qquad L_0 = 2, \quad L_1 = 1.$$

For the usual Jabobsthal numbers J_n, and their relatives j_n, called Jacobsthal-Lucas numbers for an obvious reason, we have

$$z_{n+2} = z_{n+1} + 2z_n, \qquad z_n = J_n \quad \text{or} \quad z_n = j_n,$$

and

$$J_0 = 0, \quad J_1 = 1, \qquad j_0 = 2 \quad j_1 = 1.$$

$n =$	-1	0	1	2	3	4	5	6	7	8
$F_n =$	1	0	1	1	2	3	5	8	13	21
$L_n =$	-1	2	1	3	4	7	11	18	29	47
$J_n =$	1/2	0	1	1	3	5	11	21	43	85
$j_n =$	$-1/2$	2	1	5	7	17	31	65	127	257

At various points of the developments below, identities involving elements of these sequences are needed. Although many of these are well-known, it is in all cases straightforward to give a direct proof by substitution of some well-known formulas:

$$\begin{aligned} F_n &= \frac{{\lambda_+}^n - {\lambda_-}^n}{\lambda_+ - \lambda_-}, \quad \lambda_\pm = (1 \pm \sqrt{5})/2 \\ L_n &= {\lambda_+}^n + {\lambda_-}^n \\ J_n &= \frac{2^n - (-1)^n}{2-(-1)} \\ j_n &= 2^n + (-1)^n. \end{aligned} \tag{1.3}$$

2. Proposition 1

Proposition 1. *If the $y_n \in \mathcal{S}_1$ are subject to (1.1) with x, a, b, y_0, y_1 all fixed real numbers, then the determinants $T_n = T_n(x, y_0, y_1, a, b) \in \mathcal{S}_2$ are related by the difference equation*

$$T_n = (y_1 + ax)T_{n-1} + bx(y_0 + x)T_{n-2}. \tag{2.1}$$

Proof. Expand T_n on its first column, getting

$$T_n = y_1 T_{n-1} + x R_{n-1}, \tag{2.2}$$

where R_{n-1} is an $(n-1) \times (n-1)$ determinant, whose first row is

$$y_2, y_3, y_4, \dots, y_{n-2}, y_{n-1}, y_n.$$

Apply (1.1) to each element of this row. This gives $R_{n-1} = aT_{n-1} + bS_{n-1}$, where S_{n-1} is another $(n-1) \times (n-1)$ determinant whose first row is

$$y_0, y_1, y_2, \dots, y_{n-4}, y_{n-3}, y_{n-2}.$$

Since $S_{n-1} = (y_0 + x)T_{n-2}$, (2.1) follows. □

3. Examples

3.1. Example 1.

Define $\mathcal{S}_1 = \{0, 1, 0, 1, \dots\}$. Thus

$$y_0 = 0, \quad y_1 = 1, \quad y_n = y_{n-2}, \quad a = 0, \quad b = 1. \tag{3.1}$$

Making the choice $x = 1$, the low n members $T_n = T_n(1, 0, 1, 0, 1) \in \mathcal{S}_2$ take the form

$$T_1 = 1, \quad T_2 = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}, \quad T_3 = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}, \quad T_4 = \begin{vmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix}.$$

Proposition 1 yields the result

$$T_n = T_{n-1} + T_{n-2}. \tag{3.2}$$

Since $T_1 = 1, \quad T_2 = 1$, it follows that

$$T_n = F_n. \tag{3.3}$$

3.2. **Example 2.** Define $\mathcal{S}_1 = \{0, 1, 2, 3, 4, \dots\}$, so that $y_0 = 0,\ y_1 = 1$, and

$$y_n = 2y_{n-1} - y_{n-2}, \quad a = 2, \quad b = -1. \tag{3.4}$$

Taking $x = 1$, the first few $T_n = T_n(x, 0, 1, 2, -1) \in \mathcal{S}_2$ can be seen to be

$$T_1 = 1, \quad T_2 = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}, \quad T_3 = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix}, \qquad T_4 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{vmatrix}.$$

Proposition 1 yields the result

$$T_n = 3T_{n-1} - T_{n-2}. \tag{3.5}$$

Comparing (3.5) with the easily proved result

$$F_{2n} = 3F_{2n-2} - F_{2n-4}, \tag{3.6}$$

and noting $T_1 = 1 = F_2, \quad T_2 = 3 = F_4$, it follows that

$$T_n = F_{2n}. \tag{3.7}$$

In equations (3.3) and (3.7), we have two families of determinantal formulas for Fibonacci numbers. These examples are among the simplest illustrations, but they by no means exhaust the possibilities.

3.3. **Example 3.** Set out from an identity valid for each fixed integer k:

$$F_{n+k} = L_k F_n - (-1)^k F_{n-k}. \tag{3.8}$$

Set $n = rk$ and $y_r = F_{rk}$. Then

$$y_{r+1} = L_k y_r - (-1)^k y_{r-1}, \tag{3.9}$$

so that

$$y_0 = 0, \quad y_1 = F_k, \quad a = L_k, \quad b = (-1)^{k+1}. \tag{3.10}$$

Proposition 1 shows that

$$T_n = T_n(x = 1, F_0, F_k, L_k, (-1)^{k+1}) \in \mathcal{S}_2 \tag{3.11}$$

obeys

$$\begin{aligned} T_n &= (F_k + L_k)T_{n-1} + (-1)^{k+1}T_{n-2} \\ &= 2F_{k+1}T_{n-1} + (-1)^{k+1}T_{n-2}. \end{aligned} \tag{3.12}$$

Here the identity $y_1 + ax = F_k + L_k = 2F_{k+1}$ has been used. Also

$$T_1 = F_k, \quad T_2 = {F_k}^2 + F_{2k} = F_k(F_k + L_k) = 2F_kF_{k+1}. \tag{3.13}$$

Note also that $T_0 = 0$ is compatible with (3.12) and (3.13).

To evaluate T_n, we note that the equation

$$s^2 - 2F_{k+1}s + (-1)^k = 0 \tag{3.14}$$

has roots

$$\begin{aligned} s_\pm &= F_{k+1} \pm \sqrt{{F_{k+1}}^2 - (-1)^k} \\ &= F_{k+1} \pm \sqrt{F_kF_{k+2}}. \end{aligned} \tag{3.15}$$

Hence (3.12) can be solved subject to the initial conditions $T_0 = 0$; $T_1 = F_k$. The answer is

$$\begin{aligned} T_n &= F_k \frac{{s_+}^n - {s_-}^n}{s_+ - s_-} \\ &= F_k \sum_{\substack{r=1 \\ r \text{ odd}}} \binom{n}{r} (F_k F_{k+2})^{(r-1)/2} (F_{k+1})^{n-r}. \end{aligned} \tag{3.16}$$

In the last result the power $(r-1)/2$ takes on only integer values. It is easy to check for low enough n, that (3.16) gives expressions consistent with output from direct use of (3.12).

3.4. **Example 4.** There is nice variant of Example 3 in which $k = 2$, but the definition of $\mathcal{S}_1$ involves a shift by one of the n-value: $\mathcal{S}_1 = \{F_{-2} = -1, F_0 = 0, F_2, F_4, \ldots\}$. Hence, $y_n = F_{2n-2}$ obeys

$$y_n = 3y_{n-1} - y_{n-2}. \tag{3.17}$$

Proposition 1 now implies that $T_n = T_n(1, F_{-2}, F_0, 3, -1) \in \mathcal{S}_2$ satisfies

$$T_n = 3T_{n-1}, \quad n \geq 3. \tag{3.18}$$

Also, $T_1 = F_0 = 0$, $T_2 = F_2 = 1$, $T_3 = F_4 = 3$, so that

$$T_n = 3^{n-2}, \quad n \geq 2. \tag{3.19}$$

Explicit formulas for some further low n-values are displayed because the determinants T_n defined initially can systematically be simplified.

$$T_3 = \begin{vmatrix} 0 & F_2 & F_4 \\ -1 & 0 & F_2 \\ 0 & -1 & 0 \end{vmatrix} = F_4 = 3, \qquad T_4 = \begin{vmatrix} 0 & F_2 & F_4 & F_6 \\ -1 & 0 & F_2 & F_4 \\ 0 & -1 & 0 & F_2 \\ 0 & 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} F_2 & F_6 \\ -1 & F_2 \end{vmatrix} = 9, \tag{3.20}$$

$$T_5 = \begin{vmatrix} 0 & F_2 & F_4 & F_6 & F_8 \\ -1 & 0 & F_2 & F_4 & F_6 \\ 0 & -1 & 0 & F_2 & F_4 \\ 0 & 0 & -1 & 0 & F_2 \\ 0 & 0 & 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} F_2 & F_4 & F_8 \\ -1 & 0 & F_4 \\ 0 & -1 & F_2 \end{vmatrix} = 27. \tag{3.21}$$

3.5. **Example 5.** Take $y_n = F_n$, so that

$$y_0 = 0, \quad y_1 = 1, \quad a = 1, \quad b = 1. \tag{3.22}$$

If $x = -\frac{1}{2}$,

$$T_n = T(-\frac{1}{2}, 0, 1, 1, 1) \in \mathcal{S}_1, \tag{3.23}$$

obeys

$$T_n = \frac{1}{2} T_{n-1} + \frac{1}{4} T_{n-2} \tag{3.24}$$

and

$$T_1 = 1, \quad T_2 = \frac{1}{2}. \tag{3.25}$$

Setting $T_n = G_n/(2^{n-1})$ so that

$$G_n = G_{n-1} + G_{n-2}, \quad G_1 = 1, \quad G_2 = 1 \tag{3.26}$$

yields $G_n = F_n$ and

$$T_n = F_n/(2^{n-1}). \tag{3.27}$$

It is necessary to write (3.27) out in full to study it. Interesting results, illustrated for $n = 2, 4, 6, 8$, arise only for even n:

$$0 = F_2 - F_1^2. \tag{3.28}$$
$$0 = F_4 - 2F_3F_1 - F_2^2 + 6F_2F_1^2 - 4F_1^4. \tag{3.29}$$
$$\begin{aligned} 0 = {} & F_6 - 2F_5F_1 - 2F_4F_2 + 6F_4F_1^2 - F_3^2 + 12F_3F_2F_1 \\ & -16F_3F_1^3 + 2F_2^3 - 24F_2^2F_1^2 + 40F_2F_1^4 - 16F_1^6. \end{aligned} \tag{3.30}$$
$$\begin{aligned} 0 = {} & F_8 - 2F_7F_1 - 2F_6F_2 + 6F_6F_1^2 - 2F_5F_3 + 12F_5F_2F_1 \\ & -16F_5F_1^3 - F_4^2 + 12F_4F_3F_1 + 6F_4F_2^2 - 48F_4F_2F_1^2 + 40F_4F_1^4 \\ & +6F_3^2F_2 - 24F_3^2F_1^2 - 48F_3F_2^2F_1 + 160F_3F_2F_1^3 - 96F_3F_1^5 \\ & -4F_2^4 + 80F_2^3F_1^2 - 240F_2^2F_1^4 + 224F_2F_1^6 - 64F_1^8. \end{aligned} \tag{3.31}$$

The terms seen here are in exact correspondence with the partitions of n in each case, there being $2, 5, 11, 22$ distinct partitions for $n = 2, 4, 6, 8$. Defining the weight of the product $F_{a_1}F_{a_2}\ldots F_{a_n}$ to be $\sum_{i=1}^{n} a_i$, then it is true for each of $n = 2, 4, 6, 8$, and in general that each of the results is a linear relation among all the possible products of the F_n of weight n. No such result emerges for odd n; in fact F_n cancels out of (3.27) for odd n.

It may be checked that putting the well-known values of the F_n into the right sides of (3.28) etc., does give the answer zero.

3.6. **More Examples.** The example just treated is just one, perhaps the nicest, of the type wherein $\mathcal{S}_2$ is forced to involve the same sequence of numbers as has already been used to define $\mathcal{S}_1$.

Define $\mathcal{S}_1$ by means of $y_n = F_{n+2}$, so that $a = b = 1,\ y_0 = 1,\ y_1 = 2$. Then, referring to Proposition 1, require that x and f satisfy

$$y_1 + xa = 2 + x = f, \quad bx(y_0 + x) = x(1 + x) = f^2. \tag{3.32}$$

This fixes the values $x = -\frac{4}{3},\ f = \frac{2}{3}$, so that Proposition 1 implies that $T_n = T_n(x, 1, 2, 1, 1) \in \mathcal{S}_1$ satisfies

$$T_{n+2} = fT_{n+1} + f^2T_n.$$

Then an obvious change of variable leads to the result

$$T_n(-4/3, F_2, F_3, 1, 1) = 3f^nF_{n-2}, \quad f = \frac{2}{3}. \tag{3.33}$$

Similarly, defining $\mathcal{S}_1$ via $y_n = F_n$, the method just described gives $x = -\frac{1}{2},\ f = \frac{1}{2}$, so that

$$T_n(-1/2, F_0, F_1, 1, 1) = 2f^nF_n, \quad f = 1/2.$$

This is just (3.27) again.

Leaving results for the J_n as a possible exercise, we note also

$$\begin{aligned} T_n(-\frac{9}{5}, L_1, L_2, 1, 1) &= \frac{5}{4} f^n L_n, \quad f = \frac{6}{5} \\ T_n(-\frac{4}{5}, L_{-1}, L_0, 1, 1) &= \frac{5}{9} f^n L_{n+1}, \quad f = \frac{6}{5} \\ T_n(-\frac{25}{9}, j_1, j_2, 1, 2) &= \frac{9}{8} f^n j_{n-1}, \quad f = \frac{20}{9} \\ T_n(-\frac{8}{9}, j_{-1}, j_0, 1, 2) &= \frac{9}{25} f^n j_{n+1}, \quad f = \frac{10}{9}. \end{aligned} \tag{3.34}$$

3.7. **An Example from Cellular Automaton Theory.** Define $\mathcal{S}_1$ using the Jacobsthal numbers J_n by means of $y_n = J_{n+1}$, so that $y_0 = y_1 = 1$, $a = 1$, $b = 2$. Then for $T_n = T_n(1, 1, 1, 1, 2)$, $n \geq 1$, for low values of n

$$T_1 = 1, \quad T_2 = 2 \cdot 2, \quad T_3 = 4 \cdot 3, \quad T_4 = 8 \cdot 5.$$

It follows from Proposition 1 that

$$T_n = 2T_{n-1} + 4T_{n-2}, \quad n \geq 1. \tag{3.35}$$

Set $T_n = 2^{n-1} G_n$. Then (3.35) reduces to

$$G_n = G_{n-1} + G_{n-2}, \quad n \geq 1. \tag{3.36}$$

Hence, $G_n = F_{n+1}$, $n \geq 1$. This gives a determinantal formula for F_n in terms of Jacobsthal numbers J_n:

$$F_n = \frac{4}{2^n} \begin{vmatrix} J_2 & J_3 & J_4 & \cdots & \cdots & J_{n-1} & J_n \\ -1 & J_2 & J_3 & \cdots & \cdots & J_{n-2} & J_{n-1} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & J_2 & J_3 \\ 0 & 0 & 0 & \cdots & \cdots & -1 & J_2 \end{vmatrix}. \tag{3.37}$$

Studies of one-dimensional cellular automata have their origin in the seminal paper [23]. The present paper emerged because results on this subject in [17, 16] can be rearranged to give the result (3.37). See equations (4) and (7) in [17]. Equation (4) shows that $2^N F_{N+2}$ is equal to the sum of 2^N quantities of the type given by Equation (7). With the natural variables χ_n of the cellular automaton context used in (7) related to the Jacobsthal numbers via $\chi_n = J_{n+2}$, the passage from the equations (4) and (7) of [17] to (3.37) can be checked out. It may also be noted that the numbers F_n, L_n, J_n appear regularly in the results relating to cellular automata.

4. More General Sequences $\mathcal{S}_1$

There is no reason at all for restricting $\mathcal{S}_1$ to sequences governed by second order difference equations.

Consider a sequence $\mathcal{S}_1$ for which y_0, y_1, y_2 are given. Then all other y_n can be determined using the difference equation

$$y_n = a y_{n-1} + b y_{n-2} + c y_{n-3}. \tag{4.1}$$

Proposition 2. *$T_n = T_n(x, y_0, y_1, y_2, a, b, c)$, defined by the right side of (1.2), satisfies*

$$T_n = (y_1 + xa)T_{n-1} + x\left[b(y_0 + x) + cy_{-1}\right]T_{n-2} + cx^2(y_0 + x)T_{n-3}. \tag{4.2}$$

This is proved by adapting the method of proof of Proposition 1.

In general the T_n obey a third order difference equation. But it is clearly possible to define the y_n so that the T_n satisfy one of lower order, whenever $y_0 = -x$.

Only one example will be given.

Define the sequence $\mathcal{S}_1 = \{1, 0, 1, 1, 0, 1, 1 \ldots\}$, $y_{n+1} = F_n \pmod 2$. This is governed by the difference equation $y_{n+3} = y_n$, and the initial conditions $y_0, y_1, y_2 = 1, 0, 1$, Proposition 2 indicates that

$$T_n = T_n(-1, 1, 0, 1, 0, 0, 1) \in \mathcal{S}_2$$

satisfies $T_n = -T_{n-2}$ and $T_n = -1, 1, 1, -1, \ -1, 1, 1, -1, \ \ldots$, for $n \geq 2$.

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MSC2000: 11B39, 39A10

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