GENERALIZATIONS OF THE PERIODICITY OF CERTAIN RECURSIVE SEQUENCES

TREVOR MCGUIRE

Abstract. In recent years, much research has been conducted in the area of random Fibonacci sequences. Recently, it was shown that some recursive sequences can be periodic, and occur as a subset of measure 0 of the set of random Fibonacci sequences. In this paper we will generalize those results to the case of n-nacci sequences and draw similar conclusions to the $n = 2$ case.

In 2009 the present author showed in [2] that if one chose a sequence of plus signs and minus signs, then picked two seeds and used the sequence of pluses and minuses to generate a sequence of numbers using the seeds, some of those sequences turned out to be periodic. We will start with basic definitions and notations, some of which differ slightly from the original paper due to hindsight proving that some notation is better than others. After that we will restate the main results in a generalized form. We begin with an example.

Example 1. *Consider using the seeds (1,1) and using the sequence of pluses and minuses* $\{+, -\}$. *Then our sequence would begin with* $1, 1, 2, -1, 1, -2, -1, -1, -2, 1, -1, 2, 1, 1 \ldots$. *This is actually the entire sequence because this plus/minus sequence is periodic and has period 6 as shown in* [2].

This example was the driving force behind the original question already addressed. The natural extension is to wonder if similar results exist for the tribonacci sequence, the tetranacci sequence, and the n-nacci sequences in general. We begin with an example of a tribonacci-like sequence. In this case we use 3 seeds, and we use the previous 3 terms to arrive at the next term of the sequence. Consequently, our sequence of pluses and minuses will actually be a sequence of *ordered pairs* of pluses and minuses.

Example 2. *Consider the sequence with seeds* $(1, 1, 1)$ *and plus/minus sequence of* $\{(+, -), (-, -), (-, -), (+, +)\}$. *That is, if our sequence is* A_n, *then* $A_1 = A_2 = A_3 = 1$, $A_4 = A_1 + A_2 - A_3$. *Continuing,* $A_5 = A_2 - A_3 - A_4$. *So to get the* A_{n+1} *term, we simply line up* A_{n-2}, A_{n-1} *and* A_n *and insert the proper signs in between them then perform our operations left to right. As such our sequence is:*

$$1, 1, 1, 1, -1, 1, 1, -1, 1, 1, 1.$$

We will see shortly that this sequence is not only periodic for these specific seeds but is in fact periodic for all seeds.

In the following example we look at a tetranacci-like sequence. In this case our sequence of pluses and minuses will be a sequence of ordered triples and the calculations are done similarly to the previous example.

Example 3. *Consider the sequence with seeds* $(1,1,1,1)$ *and plus/minus sequence of*

$$\{(+,+,-),(+,-,-),(-,-,-),(+,+,+),(+,+,+)\}.$$

Then our sequence begins with

$$1,1,1,1,2,-1,-1,1,1,-2,1,1,1,1\ldots.$$

As before, we will see later that this sequence is also periodic for all seeds.

Now that the reader has a flavor of the objects being worked with, we can begin to formally state some of our definitions.

Consider first that our seed can be thought of as a vector in $\mathbb{Z}^n$, and each of our $(n-1)$-tuples of pluses and minuses can be thought of as an $(n-1)\times(n-1)$ matrix acting on that seed. In the case of the standard Fibonacci sequence, when $n=2$, and the only plus/minus sequence is $\{+\}$, then the only matrix we concern ourselves with is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which has an eigenvalue equal to the golden ratio. If we also consider the sequence $\{-\}$, then we must consider the matrix $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ as outlined in many linear algebra texts and in [2]. It is clear to see that as n increases the number of matrices required increase as well. The exact number of matrices needed will be 2^{n-1}. This will become clear when we generate some of the matrices needed for $n=3$.

Consider a relation R_1 defined as:

$$R_1 : X_{n+1} = X_{n-2} + X_{n-1} + X_n.$$

Let $\overrightarrow{X_n} = \begin{pmatrix} X_n \\ X_{n-1} \\ X_{n-2} \end{pmatrix}$, and find A_1 such that

$$A_1\overrightarrow{X_n} = \begin{pmatrix} X_{n+1} \\ X_n \\ X_{n-1} \end{pmatrix} = \begin{pmatrix} X_{n-2} + X_{n-1} + X_n \\ X_n \\ X_{n-1} \end{pmatrix}.$$

It is easy to see that $A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

So our matrix A_1 is the matrix associated with the ordered pair $(+,+)$. We will find the matrix for $(+,-)$ and it will become clear what each of the 4 matrices will look like and more generally what each of the 2^{n-1} matrices will look like.

Consider now the relation R_2 defined as:

$$R_1 : X_{n+1} = X_{n-2} + X_{n-1} - X_n.$$

Let $\overrightarrow{X_n} = \begin{pmatrix} X_n \\ X_{n-1} \\ X_{n-2} \end{pmatrix}$, and find A_2 such that

$$A_2\overrightarrow{X_n} = \begin{pmatrix} X_{n+1} \\ X_n \\ X_{n-1} \end{pmatrix} = \begin{pmatrix} X_{n-2} + X_{n-1} - X_n \\ X_n \\ X_{n-1} \end{pmatrix}.$$

It is easy to see that $A_2 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

We see now that the 4 matrices we will need are the two listed above and

$$A_3 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly, in the general case the only row that will change in the matrix will be the top row with the rightmost entry remaining 1 and the other entries taking on all possible arrangements of ± 1. The bottom part of each matrix will resemble the identity with an extra column of zeros on the right.

It will prove useful to have a standard way of referring to these matrices in the general case. To do this we introduce the following notation. In the general case, when we have 2^{n-1} matrices, we will denote each matrix by $M_{n,k}$. The k will be the base 10 representation of the binary number generated by associating + with 1 and - with 0. For example, if $n = 6$ and our 5-tuple is $(+,+,-,-,+)$, then we associate that with the number 11001 in base 2, which is 25 in base 10, so that matrix will be called $M_{6,25}$. From our matrices above, $A_1 = M_{3,3}$ and $A_4 = M_{3,0}$.

Definition 1. *Let $\Omega_n = \{f \mid f : \mathbb{N} \longrightarrow \{0, 1, \ldots, n-1\}\}$. We say $\{X_i\}_{i=1}^{\infty}$ is a random n-nacci sequence if*

(1) $f \in \Omega_n$,
(2) $\overrightarrow{X_k} = M_{n,f(k)} \overrightarrow{X}_{k-1}$.

Definition 2. *With Ω_n as before, and $f \in \Omega_n$, we say σ is a motif if $\sigma = \prod_{i=m}^{1} M_{n,f(i)}$. We will use $|\sigma|$ to denote the length of the motif, which is the number of matrices in the product, or more simply, m.*

Note that the indices on the product are backwards from usual notation. This is because of the author's preference for left multiplication with matrices.

Theorem 1. *Suppose σ is a motif, then σ is periodic if and only if there exists a $k \in \mathbb{N}$ such that $\sigma^k = I$, where I is the identity matrix.*

Proof. A sequence is periodic if it repeats itself after a finite number of steps. If $\sigma^k = I$, then for all seeds $\overrightarrow{X_n}$, $\sigma^k \overrightarrow{X_n} = I\overrightarrow{X_n} = \overrightarrow{X_n}$. Thus, after k steps, the sequence repeats itself. □

For notation purposes, we will say $\rho(\sigma) = k$ is the period of σ if k is the smallest such positive integer for which $\sigma^k = I$. For a fixed n, let $\mathcal{M}_n$ be the set of all periodic motifs.

We can now clean up the previous examples using our new notation. In Example 1, our motif was $\sigma_1 = M_{2,0}M_{2,1}$. In Example 2, our motif was $\sigma_2 = M_{3,3}M_{3,0}M_{3,0}M_{3,2}$. Finally, in Example 3, we had the motif $\sigma_3 = M_{4,7}M_{4,7}M_{4,0}M_{4,4}M_{4,6}$. Notice that $\sigma_1^6 = I$, $\sigma_2^2 = I$, and $\sigma_3^2 = I$. Thus, all the motifs were periodic.

Proposition 1. *Suppose $\sigma \in \mathcal{M}_n$, $f \in \Omega_n$, and $\sigma = \prod_{i=m}^{1} M_{n,f(i)}$. If $\tau = (\prod_{i=l}^{1} M_{n,f(i)}) (\prod_{j=m}^{l-1} M_{n,f(j)})$, then $\tau \in \mathcal{M}_n$. This simply means that given any periodic motif, one can begin anywhere in that motif and cycle back to the beginning when the end is reached.*

Proof. The proof is identical to Proposition 2 of [2]. □

Lemma 1. *Suppose $\sigma \in \mathcal{M}_n$, $\rho(\sigma) = k$, then there exists $\tau \in \mathcal{M}_n$ such that $\rho(\tau) = d(k)$, where $d(k)$ is a divisor of k, for all $d(k)$.*

Proof. The proof is identical to Lemma 1 of [2]. □

Lemma 2. *The determinant of $M_{n,k}$ is $(-1)^{n+1}$ for all n, k.*

Proof. We calculate. Notice that we can perform a cofactor expansion on the nth column of $M_{n,k}$ easily because it consists of the column vector of zeros with a 1 in the top position. When we expand here, we get $(-1)^{n+1}\det(I) = (-1)^{n+1}$ since we only have the identity matrix remaining after eliminating the top row and far right column. The rest of the terms are zero. □

Proposition 2. *Suppose $\sigma \in \mathcal{M}_n$ where n is odd, then $\rho(\sigma) \neq 3$.*

Proof. First notice that if $\rho(\sigma) = 3$, then any eigenvalues of σ must necessarily be 3^{rd} roots of unity, since they equal 1 when cubed. In rectangular form, the 3 roots would be 1 and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. As such, the characteristic polynomial of σ must be of the form

$$(\lambda - 1)^a(\lambda - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i))^b(\lambda - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i))^c$$

where $a + b + c = n$. For simplicity, let $s = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $t = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Recall that the constant term in any characteristic polynomial is equal to the determinant of the matrix. Since n is odd, then, by Lemma 2, $\det(\sigma) = 1$. The constant term in our polynomial is clearly $(-1)^a(-s)^b(-t)^c$. Notice that since s and t are third roots of unity, and $(-1)^2 = 1$, we have

$$(-1)^a(-s)^b(-t)^c = (-1)^{a \pmod 2}(-s)^{b \pmod 3}(-t)^{c \pmod 3}. \tag{1}$$

With the restriction that (1)=$\det(\sigma) = 1$, we have only a few options for a, b and c. Notice that if $b \not\equiv 0 \pmod 3$ or if $c \not\equiv 0 \pmod 3$, then we can't possibly have $(1) = 1$. Thus, assume $b \equiv c \equiv 0 \pmod 3$. Notice also that $b = c$ since they are complex conjugates. Our only choices for a are 0 and 1. If $a = 1$, again we have that $(1) \neq 1$, so $a \equiv 0 \pmod 2$ is the only option.

So we have that $a \equiv 0 \pmod 2$, and $b = c \equiv 0 \pmod 3$. Thus, $a + b + c \equiv 0 \pmod 2$. However, this contradicts the fact that $a+b+c = n \equiv 1 \pmod 2$. Thus, we have the desired result by contradiction. □

Definition 3. *Define the matrix $V_{(k,n)}$ as*

$$V_{(k,n)} = \begin{pmatrix} 2^{k-1} & 2^{k-1} & \cdots & 2^{k-1} & 2^{k-1} \\ 2^{k-2} & 2^{k-2} & \cdots & 2^{k-2} & 2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 2 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

where $V_{(k,n)}$ is a $k \times n$ matrix.

Notice that $V_{(k,n)}$ is similar to a Vandermonde matrix, hence the letter designation.

Definition 4. *Define the matrix B_k as*

$$B_k = \begin{pmatrix} 2^k - 1 & 2^k - 2 & \cdots & 2^k - 2^{k-2} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} - 1 & \cdots & 2^{k-1} - 2^{k-3} & 2^{k-2} \\ 2^{k-2} & 2^{k-2} & \cdots & 2^{k-2} - 2^{k-4} & 2^{k-3} \\ \vdots & \vdots & & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Definition 5. *Define the matrix A_n as*

$$A_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Notice that A_n is an $n \times n$ matrix with an $(n-1) \times (n-1)$ identity as a submatrix, and an extra row of 1's at the top with 0's filling in the rightmost column.

Example 4. *One of each of the previous 3 matrices is shown here in order:*

$$V(3,5) = \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad B_4 = \begin{pmatrix} 15 & 14 & 12 & 8 \\ 8 & 7 & 6 & 4 \\ 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 3. *With* $A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$, $A_n^k = \begin{pmatrix} V_{(k-1,n-k+1)} & B_{k-1} \\ A_{n-k+1} & E_{(n-k+1,k-1)} \end{pmatrix}$ *for $k \geq 2$. Where $E_{(n-k+1,k-1)}$ is an $(n-k+1) \times (k-1)$ matrix consisting of one row of 1's in the top row, and 0's for the remaining entries.*

Proof. The proof is by induction. Consider $A_n A_n$:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \left(\begin{array}{ccccc|c} 2 & 2 & \cdots & 2 & 2 & 1 \\ \hline 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{array}\right).$$

Thus, $A_n A_n = \begin{pmatrix} V_{(1,n-1)} & B_1 \\ A_{n-1} & E_{(n-1,1)} \end{pmatrix}$ as required.

Now assume $A_n^{k-1} = \begin{pmatrix} V_{(k-2,n-k+2)} & B_{k-2} \\ A_{n-k+2} & E_{n-k+2,k-2} \end{pmatrix}$. We will calculate $A_n^{k-1}A_n$.

$$\left(\begin{array}{ccc|cccc} 2^{k-2} & \cdots & 2^{k-2} & 2^{k-2}-1 & 2^{k-2}-2 & \cdots & 2^{k-3} \\ 2^{k-3} & \cdots & 2^{k-3} & 2^{k-3} & 2^{k-3}-1 & \cdots & 2^{k-4} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 2 & 2 & \cdots & 1 \\ \hline 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right) \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Careful calculation shows this equals

$$\left(\begin{array}{ccccc|cccc} 2^{k-1} & 2^{k-1} & \cdots & 2^{k-1} & 2^{k-1} & 2^{k-1}-1 & 2^{k-1}-2 & \cdots & 2^{k-2} \\ 2^{k-2} & 2^{k-2} & \cdots & 2^{k-2} & 2^{k-2} & 2^{k-2} & 2^{k-2}-1 & \cdots & 2^{k-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 2 & 2 & \cdots & 2 & 2 & 2 & 2 & \cdots & 1 \\ \hline 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{array}\right).$$

All that remains to show is that the sizes of the partitions are correct. It is easy to see that the number of rows in the upper left partition is $k-1$. The number of columns in the upper right partition is also clearly $k-1$. Thus, the number of columns in the upper left partition is $n-(k-1)$, and as such, that partition must be $V_{(k-1,n-k+1)}$ and the upper right partition is necessarily B_{k-1}. With this established, it is a trivial matter to show that the bottom left partition is in fact A_{n-k+1} and the bottom right partition is $E_{(n-k+1,k-1)}$. This is the desired result. □

Corollary 1. *If we consider the entries modulo 2,* $A_n^{n+1} \equiv I$.

Proof. By Lemma 3, we have that $A_n^{n+1} = B_n$. This happens because

$$A_n^{n+1} = \begin{pmatrix} V_{(n,0)} & B_n \\ A_0 & E_{0,n} \end{pmatrix},$$

and each of the 0 subscripts mean the submatrix has either 0 rows or 0 columns. Notice $B_{n_{i,j}} \equiv \begin{cases} 0 \pmod 2 & \text{for } i \neq j, \\ 1 \pmod 2 & \text{for } i = j, \end{cases}$ which is the desired result. □

Theorem 2. *Consider* $\sigma \in \mathcal{M}_n$ *such that* $\rho(\sigma) = p$, *and* $|\sigma| = m$, *then* $mp \equiv 0 \pmod{n+1}$.

Proof. Since $\sigma \in \mathcal{M}_n$, then σ is the finite product of some $M_{n,k}$. However, if we consider the entries of $M_{n,k}$ modulo 2, we find that $M_{n,k} \equiv A_n$ for all n, k. Consequently, if $|\sigma| = m$, then $\sigma \equiv A_n^m \pmod 2$. Thus, by Lemma 3, we have that if σ is periodic, then $\sigma^{mp} = I$, but this can only happen when $mp \equiv 0 \pmod{n+1}$. □

We have managed to generalize some of the most interesting results from the case of $n = 2$ here, excluding one. For the case of $n = 2$, we showed in [2] that not only was the product of the period and the length of the motif congruent to 0 modulo 3, but we also showed that the only possible periods were 1, 2, 3, and 6. Early computational results seem to suggest that for the case of $n = 3$, the only possible periods are 1, 2, and 4, but this has yet to be proven. The higher values of n are pure guesswork at this point. There is a wonderful result in a 2002 paper [1] concerning possible values of the periods, but future research will be required to determine which periods are actually attainable. Proposition 2 was a sample of what those results might look like. Other topics of current research include calculating exactly how many distinct periodic motifs exist for a given length and necessary and sufficient conditions under which a given matrix represents a periodic motif, which has been solved for the $n = 2$ case, and further results are expected.

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References

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Department of Mathematics, Louisiana State University, Baton Rouge, LA
E-mail address: `TMcgui1@LSU.edu`