ON CERTAIN COMBINATIONS OF HIGHER POWERS OF FIBONACCI NUMBERS

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Abstract. We present identities that we feel can be regarded as higher order analogues of the well-known identity $F_n^2 + F_{n+1}^2 = F_{2n+1}$. We give three theorems corresponding to the powers 4, 6, and 8. We also state two conjectures that give the form of similar identities that involve higher powers.

1. Introduction

To put our main results in context, we begin with the well-known identity

$$F_n^2 + F_{n+1}^2 = F_{2n+1}. \tag{1.1}$$

The generalization

$$F_n^2 + (-1)^{n+k-1} F_k^2 = F_{n-k} F_{n+k} \tag{1.2}$$

appears as I_{19} in [2, page 59]. Replacing k by $n+k$ in (1.2), we obtain

$$(-1)^{k+1} F_n^2 + F_{n+k}^2 = F_k F_{2n+k}. \tag{1.3}$$

The form of (1.1) motivated us to discover

$$F_n^4 + 4F_{n+1}^4 + 4F_{n+2}^4 + F_{n+3}^4 = 6F_{2n+3}^2. \tag{1.4}$$

We then found

$$F_n^4 - 6F_{n+2}^4 - 6F_{n+4}^4 + F_{n+6}^4 = 56F_{2n+6}^2 + 20, \tag{1.5}$$

$$F_n^4 + 19F_{n+3}^4 + 19F_{n+6}^4 + F_{n+9}^4 = 1224F_{2n+9}^2 - 480, \tag{1.6}$$

and

$$F_n^4 - 46F_{n+4}^4 - 46F_{n+8}^4 + F_{n+12}^4 = 20304F_{2n+12}^2 + 8100. \tag{1.7}$$

In the opinion of this writer, identities (1.5)-(1.7) appear unusual due to the presence of the constants on the far right side. In fact (1.4)-(1.7) belong to an infinite family of similar identities. We present this family in Section 2 and in an analogous manner to 1.3.

We also discovered

$$F_n^6 + 8F_{n+1}^6 + 8F_{n+2}^6 + F_{n+3}^6 = 10F_{2n+3}^3, \tag{1.8}$$

which is part of an infinite family of similar identities. We also present this family in Section 2. In the sequel we discuss the situation for higher powers.

2. Main Results and a Method of Proof

Our first main result, which is presented in the theorem that follows, has (1.4)-(1.7) as special cases.

Theorem 2.1. *Let n and k be integers. Then*

$$\begin{aligned} &F_n^4 + \left((-1)^{k+1}L_{2k}+1\right)F_{n+k}^4 + \left((-1)^{k+1}L_{2k}+1\right)F_{n+2k}^4 + F_{n+3k}^4 \\ &= F_k L_{2k} F_{3k} F_{2n+3k}^2 + 10(-1)^k F_{k-1}F_k^4 F_{k+1}. \end{aligned} \tag{2.1}$$

Our next theorem gives a family of identities that includes (1.8) as a special case.

Theorem 2.2. *Let n and k be integers. Then*

$$\begin{aligned} &(-1)^{k+1}F_n^6 + (L_{4k}+1)F_{n+k}^6 + (-1)^{k+1}(L_{4k}+1)F_{n+2k}^6 + F_{n+3k}^6 \\ &= F_k F_{3k} F_{5k} F_{2n+3k}^3 + 15(-1)^k F_{k-1}F_k^4 F_{k+1}F_{3k}F_{2n+3k}. \end{aligned} \tag{2.2}$$

Each result in this paper can be proved with the use of a method introduced by Dresel [1]. We have found Dresel's method of proof extremely useful in past work, employing it, for instance, in [3, 4], and [5]. To illustrate, we prove Theorem 2.2.

As Dresel explains, since $(-1)^n = (\alpha\beta)^n$, where α and β are the roots of $x^2 - x - 1 = 0$, then $(-1)^n$ is of degree 2 in the variable n. Therefore, we insert $(-1)^{2n}$ into the product on the far right of (2.2). In the terminology of Dresel, this makes (2.2) homogeneous of degree 6 in the variable n.

Next we look at the variable k. Write the left side as

$$(-1)^{9k+1}F_n^6 + (-1)^{4k}\left(L_{4k} + (-1)^{2k}\right)F_{n+k}^6 + (-1)^{k+1}\left(L_{4k} + (-1)^{2k}\right)F_{n+2k}^6 + F_{n+3k}^6,$$

and the right side as

$$F_k F_{3k} F_{5k} F_{2n+3k}^3 + 15(-1)^{3k}(-1)^{2n} F_{k-1}F_k^4 F_{k+1}F_{3k}F_{2n+3k}.$$

This makes (2.2) homogeneous of degree 18 in the variable k.

We noted above that (2.2) is homogeneous of degree 6 in the variable n. Therefore, to prove (2.2) with the **verification theorem** of Dresel [1, page 171], we need only verify its validity for seven distinct values of n. Accordingly, we write down the cases that correspond to $n =$ 1, 2, 3, 4, 5, 6, and 7. We are required to prove each of these seven cases. Now, each of these seven cases is an identity that is homogeneous of degree 18 in the variable k. Therefore, to prove any one of these seven cases, we need only verify its validity for nineteen distinct values of k; say $k = 1, 2, \ldots, 19$. We are required to verify (2.2) for 7×19 distinct ordered pairs (n, k). We managed to perform these verifications, and thereby complete the proof of Theorem 2.2, in a matter of seconds with the use of the computer algebra system *Mathematica* 6.0.

3. Higher Powers

Similar identities for higher powers exist and are very lengthy. To illustrate we discuss the next case in which the powers on the left are 8. For convenience we define $a_i = a_i(k)$ and

$b_i = b_i(k)$ as

$$\begin{aligned}
a_0 = a_5 &= 1; \\
a_1 = a_4 &= (-1)^{k+1}(L_{6k} + L_{2k}) + 1; \\
a_2 = a_3 &= L_{8k} + (-1)^{k+1}L_{6k} + L_{4k} + (-1)^{k+1}L_{2k} + 2; \\
b_0 &= F_k F_{3k} L_{4k} F_{5k} F_{7k}; \\
b_1 &= 4(-1)^k F_k^4 F_{3k} F_{5k}\left(L_{8k} + 3(-1)^k L_{6k} + 6L_{4k} + 10(-1)^k L_{2k} + 9\right); \\
b_2 &= 2F_k^7 F_{3k}\left(L_{10k} + 6(-1)^k L_{8k} + 21L_{6k} + 56(-1)^k L_{4k} + 99L_{2k} + 117(-1)^k\right).
\end{aligned}$$

We are now able to state our third theorem.

Theorem 3.1. *Let a_i, $0 \le i \le 5$, and b_i, $0 \le i \le 2$, be as defined above. Then*

$$\begin{aligned}
&a_0F_n^8 + a_1F_{n+k}^8 + a_2F_{n+2k}^8 + a_3F_{n+3k}^8 + a_4F_{n+4k}^8 + a_5F_{n+5k}^8 \\
&= b_0F_{2n+5k}^4 + b_1F_{2n+5k}^2 + b_2.
\end{aligned} \tag{3.1}$$

The proof of Theorem 3.1 follows the same lines as the proof of Theorem 2.2. By inserting appropriate powers of $(-1)^n$, we may regard (3.1) as being homogeneous of degree 8 in the variable n. We also note that in (3.1) the terms $a_4F_{n+4k}^8$, $a_5F_{n+5k}^8$, and $b_0F_{2n+5k}^4$ are each of degree 40 in the variable k. Therefore, by inserting appropriate powers of $(-1)^k$ into each of the remaining six terms we may regard (3.1) as being homogeneous of degree 40 in the variable k. We are therefore required to verify (3.1) for 9×41 distinct ordered pairs (n, k) in the same manner described for the proof of Theorem 2.2. Once again, we have performed these verifications with the use of *Mathematica* 6.0.

We have not found the general form of such identities where the powers on the left are 10 or higher, since such identities become unwieldy very quickly. However, by examining specific cases, we are able to say something about the structure of such identities. Our observations are contained in the two conjectures that follow.

Conjecture 3.2. *For $p \in \{2, 4, 6, 8, \ldots\}$ and any positive integer k, there exist integers $a_0, a_1, \ldots, a_{p+1}$, and integers $b_0, b_1, \ldots, b_p$, such that*

$$\sum_{i=0}^{p+1} a_i F_{n+ki}^{2p} = \sum_{i=0}^{p} b_i F_{2n+(p+1)k}^i.$$

Furthermore, $a_i = a_{p+1-i}$ for $i = 0, 1, \ldots, p/2$, with $a_0 = a_{p+1} = 1$. Also, $b_1 = b_3 = \cdots = b_{p-1} = 0$.

Conjecture 3.3. *For $p \in \{3, 5, 7, 9, \ldots\}$ and any positive integer k, there exist integers $a_0, a_1, \ldots, a_p$, and integers $b_0, b_1, \ldots, b_p$, such that*

$$\sum_{i=0}^{p} a_i F_{n+ki}^{2p} = \sum_{i=0}^{p} b_i F_{2n+pk}^i.$$

Furthermore, $a_i = (-1)^{k+1}a_{p-i}$ for $i = 0, 1, \ldots, (p-1)/2$, with $a_0 = (-1)^{k+1}$ and $a_p = 1$. Also, $b_0 = b_2 = \cdots = b_{p-1} = 0$.

In order to illustrate Conjectures 3.2 and 3.3, we present an instance of each. For $(p, k) = (4, 2)$ an instance of Conjecture 3.2 is

$$\begin{aligned} &F_n^8 - 328F_{n+2}^8 + 1927F_{n+4}^8 + 1927F_{n+6}^8 - 328F_{n+8}^8 + F_{n+10}^8 \\ &= 7796360F_{2n+10}^4 + 6219840F_{2n+10}^2 + 617168. \end{aligned} \tag{3.2}$$

For $(p, k) = (3, 2)$ an instance of Conjecture 3.3 is

$$-F_n^6 + 48F_{n+2}^6 - 48F_{n+4}^6 + F_{n+6}^6 = 440F_{2n+6}^3 + 240F_{2n+6}. \tag{3.3}$$

We invite the reader to check the validity of (3.2) and (3.3), and to also check that in each case the stated conditions on the a_i and b_i are satisfied. The reader may also wish to construct further instances of Conjectures 3.2 and 3.3.

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