THE CASSINI IDENTITY AND ITS RELATIVES

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ABSTRACT. We make use of an observation connected to three term recurrence relations to obtain Cassini-like formulas for Fibonacci and Lucas numbers in addition to similar identities for Fibonacci, Lucas, Jacobsthal, Morgan-Voyce, Chebyshev and Dickson polynomials. A general result for three term recurrences is also given.

1. Introduction

The Cassini identity for Fibonacci numbers F_n, namely that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, is one of the facts about the Fibonacci numbers that one might call common mathematical knowledge. We will in the following aim at presenting the identity in a more general context and in the process obtain similar results for related sequences.

2. Theoretical Framework

We start off by defining a few basic and well-known notions to avoid confusion. Our objects of study will be the *three term recurrence relations*, defined by

$$X_n = \alpha X_{n-1} + \beta X_{n-2}, \tag{2.1}$$

where α and β are (possibly complex) constants. A sequence $\{X_i\}_{i=0}^{\infty}$ is called a *solution* of (2.1) if its elements satisfy this equality for all $i \in \mathbb{N}$. The set of all solutions of (2.1) is a linear space, as seen in [3], and consequently we have that if $\{X_n\}$ is a solution of (2.1), then $\{aX_n\}$ is also a solution for any fixed number $a \in \mathbb{C}$. In addition, if $\{X_n\}$ and $\{Y_n\}$ are solutions of (2.1), then $\{X_n\} + \{Y_n\} = \{X_n + Y_n\}$ is also a solution of (2.1). It is also worth pointing out that if $\{X_i\}_{i=0}^{\infty}$ is a solution, then $\{X_{i+l}\}_{i=0}^{\infty}$ is also a solution for any l since α and β are constants.

Let us now assume that $\{X_m\}$ and $\{Y_m\}$ are two solutions of the recurrence (2.1) and define $\Delta_m^{(k)}$ to be given by

$$\Delta_m^{(k)} = X_{m+k}Y_{m-1} - X_{m-1}Y_{m+k}. \tag{2.2}$$

We then have that

$$\begin{aligned}
\Delta_m^{(0)} &= X_mY_{m-1} - X_{m-1}Y_m && (2.3)\\
&= (\alpha X_{m-1} + \beta X_{m-2})Y_{m-1} - X_{m-1}(\alpha Y_{m-1} + \beta Y_{m-2}) && (2.4)\\
&= \alpha(X_{m-1}Y_{m-1} - X_{m-1}Y_{m-1}) + \beta(X_{m-2}Y_{m-1} - X_{m-1}Y_{m-2}) && (2.5)\\
&= -\beta\Delta_{m-1}^{(0)} = \cdots = (-\beta)^{m-l}\Delta_l^{(0)}. && (2.6)
\end{aligned}$$

We also have that

$$\begin{aligned}
\Delta_m^{(1)} &= X_{m+1}Y_{m-1} - X_{m-1}Y_{m+1} && (2.7)\\
&= (\alpha X_m + \beta X_{m-1})Y_{m-1} - X_{m-1}(\alpha Y_m + \beta Y_{m-1}) && (2.8)\\
&= \alpha(X_m Y_{m-1} - X_{m-1}Y_m) + \beta(X_{m-1}Y_{m-1} - X_{m-1}Y_{m-1}) && (2.9)\\
&= \alpha\Delta_m^{(0)}. && (2.10)
\end{aligned}$$

Hence, for $k \geq 2$ we have

$$\begin{aligned}
\Delta_m^{(k)} &= X_{m+k}Y_{m-1} - X_{m-1}Y_{m+k} && (2.11)\\
&= (\alpha X_{m+k-1} + \beta X_{m+k-2})Y_{m-1} - X_{m-1}(\alpha Y_{m+k-1} + \beta Y_{m+k-2}) && (2.12)\\
&= \alpha(X_{m+k-1}Y_{m-1} - X_{m-1}Y_{m+k-1})\\
&\quad + \beta(X_{m+k-2}Y_{m-1} - X_{m-1}Y_{m+k-2}) && (2.13)\\
&= \alpha\Delta_m^{(k-1)} + \beta\Delta_m^{(k-2)}. && (2.14)
\end{aligned}$$

Thus, $\Delta_m^{(k)}$ is also a solution of the recurrence in (2.1). With this in mind, let us return to the recurrence in (2.1). Let $\{P_n\}$ be a solution of (2.1) where $P_{-1} = 0$ and $P_0 = 1$. We then have that

$$\begin{aligned}
P_0 &= 1 && (2.15)\\
P_1 &= \alpha && (2.16)\\
P_2 &= \alpha^2 + \beta && (2.17)\\
P_3 &= \alpha^3 + 2\alpha\beta && (2.18)\\
&\vdots
\end{aligned}$$

and in general, by [1, p. 168], we have that

$$P_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} \alpha^{k-2i}\beta^i \qquad (2.19)$$

for $k \geq 0$. Now, with $\Delta_m^{(0)}$ and $\Delta_m^{(1)}$ as starting values in (2.1), we observe the following:

$$\begin{aligned}
\Delta_m^{(1)} &= \alpha\Delta_m^{(0)} = P_1\Delta_m^{(0)} && (2.20)\\
\Delta_m^{(2)} &= \alpha\Delta_m^{(1)} + \beta\Delta_m^{(0)} = (\alpha^2 + \beta)\Delta_m^{(0)} = P_2\Delta_m^{(0)} && (2.21)\\
\Delta_m^{(3)} &= \alpha\Delta_m^{(2)} + \beta\Delta_m^{(1)} = (\alpha^3 + 2\alpha\beta)\Delta_m^{(0)} = P_3\Delta_m^{(0)} && (2.22)\\
\Delta_m^{(4)} &= \alpha\Delta_m^{(3)} + \beta\Delta_m^{(2)} = (\alpha^4 + 2\alpha^2\beta + \beta^2)\Delta_m^{(0)} = P_4\Delta_m^{(0)} && (2.23)\\
&\vdots\,.
\end{aligned}$$

Thus we have that

$$\Delta_m^{(k)} = P_k\Delta_m^{(0)} = P_k(-\beta)^{m-l}\Delta_l^{(0)}. \qquad (2.24)$$

If we now set $l = 1$, we have proven the following theorem.

Theorem 2.1. *If $\{X_n\}$ and $\{Y_n\}$ are two solutions of the recurrence* (2.1) *then the identity*

$$X_{m+k}Y_{m-1} - X_{m-1}Y_{m+k} = P_k(-\beta)^{m-1}[X_1Y_0 - X_0Y_1] \qquad (2.25)$$

holds.

If in 2.1 we set $Y_n = X_{n+l}$ and X_n is a solution such that $P_k = X_{k+1}$ we see that we obtain

$$X_{m+k}X_{m+l-1} - X_{m-1}X_{m+l+k} = X_{k+1}X_l(-\beta)^{m-1}. \tag{2.26}$$

By setting $k = 0$ and $l = 1$ and use the fact that $P_k = X_{k+1}$, we obtain the identity

$$X_m^2 - X_{m-1}X_{m+1} = (-\beta)^{m-1}. \tag{2.27}$$

We have thus proven the following corollary.

Corollary 2.2. *If $\{X_n\}$ is a solution of the recurrence* (2.1) *such that $P_k = X_{k+1}$ then the following identities hold:*

$$X_{m+k}X_{m+l-1} - X_{m-1}X_{m+l+k} = X_{k+1}X_l(-\beta)^{m-1} \tag{2.28}$$

$$X_m^2 - X_{m-1}X_{m+1} = (-\beta)^{m-1}. \tag{2.29}$$

Application of these results to various recurrences will be the focus of the rest of the paper.

3. Identities

We will now use Theorem 2.1 to derive identities connected to various recurrences of the type given in (2.1).

3.1. **Fibonacci and Lucas Identities.** We start off by giving a few identities related to the Fibonacci and Lucas numbers. We recall that the Fibonacci and Lucas numbers are given by setting $\alpha = \beta = 1$ in the recurrence in (2.1) and that the starting values for the Fibonacci sequence is $F_0 = 0$ and $F_1 = 1$ and that the starting values for the Lucas sequence is $L_0 = 2$ and $L_1 = 1$. We then observe that $P_k = F_{k+1}$. Now, we easily obtain the identity

$$F_{m+k}L_{m-1} - F_{m-1}L_{m+k} = 2F_{k+1}(-1)^{(m-1)} \tag{3.1}$$

by setting $X_n = F_n$ and $Y_n = L_n$ in Theorem 2.1. But we can do more. We may apply Corollary 2.2 to obtain the identity

$$F_{m+k}F_{m+l-1} - F_{m-1}F_{m+l+k} = F_{k+1}F_l(-1)^{m-1} \tag{3.2}$$

from which we also obtain the Cassini identity. We can also set $X_n = L_n$ and $Y_n = X_{n+l}$ in Theorem 2.1 to obtain

$$L_{m+k}L_{m+l-1} - L_{m-1}L_{m+l+k} = F_{k+1}(-1)^{m-1}[L_l - 2L_{l+1}]. \tag{3.3}$$

Hence, if we set $k = 0$ and $l = 1$ we have a Cassini-like formula for the Lucas numbers, namely

$$L_m^2 - L_{m-1}L_{m+1} = 5(-1)^m. \tag{3.4}$$

3.2. **Fibonacci and Lucas Polynomials.** We can also apply Theorem 2.1 to the Fibonacci and Lucas polynomials. The recurrence (2.1) is the same for both polynomials, with $\alpha = x$ and $\beta = 1$. Starting values for the Fibonacci and Lucas polynomials are $f_1(x) = 1$, $f_2(x) = x$, $l_0(x) = 2$ and $l_1(x) = x$, respectively. We also observe that $P_k = f_{k+1}(x)$ and that $f_0(x) = 0$ and thus, by Theorem 2.1, we obtain the identity

$$f_{m+k}(x)l_{m-1}(x) - f_{m-1}(x)l_{m+k}(x) = 2f_{k+1}(x)(-1)^{m-1}. \tag{3.5}$$

Furthermore, by application of Corollary 2.2 we obtain an identity similar to the one given in (3.2), namely

$$f_{m+k}(x)f_{m+l-1}(x) - f_{m-1}(x)f_{m+k+l}(x) = f_{k+1}(x)f_l(x)(-1)^{m-1} \tag{3.6}$$

which again yields the Cassini-formula. We may also set $X_t = l_t(x)$ and $Y_t = X_{t+n}$ in Theorem 2.1 to obtain

$$l_{m+k}(x)l_{m+n-1}(x) - l_{m-1}(x)l_{m+n+k}(x) = f_{k+1}(x)(-1)^{m-1}[l_1(x)l_n(x) - l_0(x)l_{n+1}(x)]. \tag{3.7}$$

From this we may also obtain the Cassini-like identity

$$l_m(x)^2 - l_{m-1}(x)l_{m+1}(x) = (-1)^m(x^2+4) \tag{3.8}$$

by setting $k = 0$ and $n = 1$.

3.3. **Jacobsthal Polynomials.** We can also apply Theorem 2.1 to the Jacobsthal polynomials. The Jacobsthal polynomials are given by a recurrence as in (2.1) where $\alpha = 1$ and $\beta = x$. The initial values for the Jacobsthal polynomials are $J_1(x) = J_2(x) = 1$. With $X_n = J_n(x)$ and $Y_n = X_{n+l}$ in Theorem 2.1 we have that $P_k = J_{k+1}(x)$ and hence that the identity

$$J_{m+k}(x)J_{m+l-1}(x) - J_{m-1}(x)J_{m+k+l}(x) = J_{k+1}(x)(-x)^{m-1}[J_l(x) - J_0(x)J_{l+1}(x)] \tag{3.9}$$

holds. Application of Corollary 2.2 yields the identity

$$J_{m+k}(x)J_{m+l-1}(x) - J_{m-1}(x)J_{m+k+l}(x) = J_{k+1}(x)J_l(x)(-x)^{m-1}. \tag{3.10}$$

From this we also obtain the Cassini-like formula

$$J_m(x)^2 - J_{m-1}(x)J_{m+1}(x) = (-x)^{m-1}. \tag{3.11}$$

In [1, p. 471] the polynomial $K_n(x)$ is defined in the same way as the Jacobsthal polynomials, but with starting values $K_1(x) = 1$ and $K_2(x) = x$. We now apply Theorem 2.1 with $\{X_n\} = J_n(x)$ and $\{Y_n\} = K_{n+1}(x)$ to obtain the identity

$$J_{m+k}K_m - J_{m-1}K_{m+k+1} = J_{k+1}(x)(-x)^{m-1}.$$

With $X_n = K_{n+1}(x)$ and $Y_n = X_{n+l}$ in Theorem 2.1 we have that

$$\begin{aligned} &K_{m+k+1}(x)K_{m+l}(x) - K_m(x)K_{m+k+l+1}(x) \\ &\qquad = J_{k+1}(x)(-x)^{m-1}[K_2(x)K_{l+1}(x) - K_1(x)K_{l+2}(x)]. \end{aligned} \tag{3.12}$$

Also, if $k = 0$ and $l = 1$ we have that

$$K_{m+1}(x)K_{m+1}(x) - K_m(x)K_{m+2}(x) = J_{k+1}(x)(-x)^{m-1}(x^2-2x). \tag{3.13}$$

3.4. **Morgan-Voyce Polynomials.** We also apply Theorem 2.1 and Corollary 2.2 to the Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$, with initial values $B_0 = 1$, $B_1 = x+2$, $b_0 = 1$ and $b_1 = x+1$, and $\alpha = x+2$, $\beta = -1$ in (2.1). We easily obtain the identities

$$B_{m+k}(x)b_{m-1}(x) - B_{m-1}(x)b_{m+k}(x) = B_k(x) \tag{3.14}$$

$$B_{m+k}(x)B_{m+l-1}(x) - B_{m-1}(x)B_{m+k+l}(x) = B_k(x)B_{l-1}(x) \tag{3.15}$$

$$B_m(x)^2 - B_{m-1}(x)B_{m+1}(x) = 1 \tag{3.16}$$

$$b_{m+k}(x)b_{m+l-1}(x) - b_{m-1}(x)b_{m+k+l}(x) = B_k(x)[(x+1)b_l(x) - b_{l+1}(x)] \tag{3.17}$$

$$b_m(x)^2 - b_{m-1}(x)b_{m+1}(x) = -x. \tag{3.18}$$

Special cases of some of these identities are found in [4] and [5].

3.5. **Chebyshev Polynomials.** Similarly, we may apply Theorem 2.1 and Corollary 2.2 to the Chebyshev polynomials of the first kind, $T_n(x)$, and second kind, $U_n(x)$, with initial values $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$ and $U_1(x) = 2x$, and $\alpha = 2x$, $\beta = -1$ in (2.1). Hence we obtain the identities

$$U_{m+k}(x)T_{m-1}(x) - U_{m-1}(x)T_{m+k}(x) = xU_k(x) \tag{3.19}$$

$$U_{m+k}(x)U_{m+l-1}(x) - U_{m-1}(x)U_{m+k+l}(x) = U_k(x)U_{l-1}(x) \tag{3.20}$$

$$U_m(x)^2 - U_{m-1}(x)U_{m+1}(x) = 1 \tag{3.21}$$

$$T_{m+k}(x)T_{m+l-1}(x) - T_{m-1}(x)T_{m+k+l}(x) = U_k(x)[xT_l(x) - T_{l+1}(x)] \tag{3.22}$$

$$T_m(x)^2 - T_{m-1}(x)T_{m+1}(x) = 1 - x^2 \tag{3.23}$$

since $P_k = U_k(x)$.

3.6. **Dickson Polynomials.** Finally, we apply Theorem 2.1 and Corollary 2.2 to the Dickson polynomials of the first kind, $D_n(x, \alpha)$, and second kind, $E_n(x, \alpha)$, with initial values $D_0(x, \alpha) = 2$, $D_1(x, \alpha) = x$, $E_0(x, \alpha) = 1$, and $E_1(x, \alpha) = x$, and $\alpha = x$, $\beta = -\alpha$ in (2.1). Hence we obtain the identities

$$E_{m+k}(x)D_{m-1}(x) - E_{m-1}(x)D_{m+k}(x) = x\alpha^{m-1}E_k(x) \tag{3.24}$$

$$E_{m+k}(x)E_{m+l-1}(x) - E_{m-1}(x)E_{m+k+l}(x) = E_k(x)E_{l-1}(x)\alpha^m \tag{3.25}$$

$$E_m(x)^2 - E_{m-1}(x)E_{m+1}(x) = \alpha^m \tag{3.26}$$

$$D_{m+k}(x)D_{m+l-1}(x) - D_{m-1}(x)D_{m+k+l}(x) = E_k(x)[xD_l(x) - 2D_{l+1}(x)] \tag{3.27}$$

$$D_m(x)^2 - D_{m-1}(x)D_{m+1}(x) = \alpha^{m-1}(2\alpha - x^2). \tag{3.28}$$

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