MULTIDIMENSIONAL ZECKENDORF REPRESENTATIONS

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Abstract. We generalize Zeckendorf's Theorem to represent points in $\mathbb{Z}^{k-1}$, uniquely, as sums of elements of order-k linear recurrences.

1. Background and Definitions

Throughout this paper, $k \geq 2$ is a fixed integer.

Definition 1. *The k-bonacci sequence $\{X_n\}$ is given by the recurrence*

$$\begin{aligned} X_n &= 0 \quad \text{for } -k+2 \leq n \leq 0, \\ X_1 &= 1, \\ X_n &= \sum_{i=1}^{k} X_{n-i} \quad \text{for all } n \in \mathbb{Z}\ . \end{aligned} \tag{1}$$

When $k = 2$, $\{X_n\}$ is the Fibonacci sequence, when $k = 3$ the tribonacci sequence, and so on. Our purpose herein is to generalize the following well-known theorem [5] (see also [2, 3, 4] [1].

Theorem 1. Zeckendorf's Theorem. *Every nonnegative number, n, is a unique sum of distinct k-bonacci numbers:*

$$n = \sum_{i \geq 2} c_i X_i$$

such that $c_i \in \{0, 1\}$ for all i, and no string of k consecutive c_i's are equal to 1.

Definition 2. Call a sequence $\{c_i\}$ satisfying the constraints of Theorem 1 a *satisfying sequence* and such a representation a *satisfying representation (SR).*

Definition 3. *The k-bonacci vectors, $\vec{\mathbf{X}}_i \in \mathbb{Z}^{k-1}$, are given by the recurrence*

$$\begin{aligned} \vec{\mathbf{X}}_0 &= \vec{\mathbf{0}}, \\ \vec{\mathbf{X}}_{-i} &= \vec{\mathbf{e}}_i \quad \text{for } 1 \leq i \leq k-1 \quad \text{(the standard unit vectors)}, \\ \vec{\mathbf{X}}_n &= \sum_{i=1}^{k} \vec{\mathbf{X}}_{n-i} \quad \text{for all } n \in \mathbb{Z}. \end{aligned} \tag{2}$$

We use the $\vec{\mathbf{X}}_n$ with $n \leq 0$. For this use—i.e., working backwards—rewrite the above recurrence, Equation (2), as the following

$$\vec{\mathbf{X}}_n = \vec{\mathbf{X}}_{n+k} - \sum_{i=1}^{k-1} \vec{\mathbf{X}}_{n+i}. \tag{3}$$

[1] Strictly speaking, Zeckendorf's Theorem applies to the Fibonacci numbers ($k = 2$), but the proof via greedy change-making applies without change to k-bonacci numbers.

Here is a list of the first few *tribonacci vectors* (i.e., $k = 3$). See also Figure 2.

$$\begin{aligned}
\vec{\mathbf{X}}_0 &= (0,0)\\
\vec{\mathbf{X}}_{-1} &= (1,0)\\
\vec{\mathbf{X}}_{-2} &= (0,1)\\
\vec{\mathbf{X}}_{-3} &= (-1,-1)\\
\vec{\mathbf{X}}_{-4} &= (2,0)\\
\vec{\mathbf{X}}_{-5} &= (-1,2)\\
\vec{\mathbf{X}}_{-6} &= (-2,-3)\\
\vec{\mathbf{X}}_{-7} &= (5,1)\\
\vec{\mathbf{X}}_{-8} &= (-4,4)\\
\vec{\mathbf{X}}_{-9} &= (-3,-8)
\end{aligned}$$

2. Main Theorem

Theorem 2. *Every $\vec{\mathbf{v}} \in \mathbb{Z}^{k-1}$ has a unique SR in the sense of Theorem 1, namely, $\vec{\mathbf{v}} = \sum_{i\geq 1} c_i \vec{\mathbf{X}}_{-i}$.*

Before proving Theorem 2, we establish some machinery.

Definition 4. *For $n \geq k-2$, $S_n : \mathbb{Z}^{k-1} \to [0, X_n)$ is the scalar product*

$$S_n(\vec{\mathbf{v}}) = \vec{\mathbf{v}} \cdot (X_{n-1}, \ldots, X_{n-(k-1)}) \pmod{X_n}.$$

Lemma 3. $S_n(\sum_{i=1}^{p} c_i \vec{\mathbf{X}}_{-i}) \equiv \sum_{i=1}^{p} c_i X_{n-i} \pmod{X_n}$.

Proof. $S_n(\vec{\mathbf{X}}_{-i}) \equiv X_{n-i} \pmod{X_n}$ for $0 \leq i \leq k-1$, by definition, and for $i \geq k$, by the recursive definitions of X_i and $\vec{\mathbf{X}}_i$. The proof of this Lemma then follows by linearity. □

Definition 5. A *nearly satisfying representation (NSR)* for $\vec{\mathbf{v}} \in \mathbb{Z}^{k-1}$ is a sum, $\vec{\mathbf{v}} = \sum_{i\geq 1} c_i \vec{\mathbf{X}}_{-i}$, for which $c_i \in \{0, 1, 2\}$ for all i, such that

1. the blocks of consecutive non-zero values of c_i all have length less than k, and
2. only a single such block contains any 2's.

Our proof of Theorem 2 involves manipulating the coefficients of NSRs using analogs of the grade-school arithmetic *carrying* and *borrowing* concepts.

Definition 6. If $\vec{\mathbf{v}} = \sum_{i\geq 1} c_i \vec{\mathbf{X}}_{-i}$ is any representation of $\vec{\mathbf{v}}$ then

1. *Carrying into c_i* increments c_i by 1 and decrements $c_{i+1}, \ldots, c_{i+k}$ by 1.
2. *Borrowing from c_i*, conversely, decrements c_i by 1 and increments $c_{i+1}, \ldots, c_{i+k}$ by 1.

Both operations leave the sum unchanged.

We will carry into c_i when $c_i = 0$ and $c_{i+1}, \ldots, c_{i+k}$ are all positive which will shorten the lengths of blocks of non-zero coefficients. We will borrow from c_i when $c_i = 2$ (which will necessitate future carrying).

Here is an illustration of carrying and borrowing. We start with the SR of $(2, -2) = \vec{\mathbf{X}}_{-2} + \vec{\mathbf{X}}_{-3} + \vec{\mathbf{X}}_{-6} + \vec{\mathbf{X}}_{-7}$ (line 1 in the following table), add $(-1, -1) = \vec{\mathbf{X}}_{-3}$, to get an NSR (line 2), borrow once and carry twice, achieving the SR $(1, -3) = \vec{\mathbf{X}}_{-1} + \vec{\mathbf{X}}_{-4} + \vec{\mathbf{X}}_{-6}$ (line 5).

line	c_1	c_2	c_3	c_4	c_5	c_6	c_7	comment
1	0	1	1	0	0	1	1	$(2,-2) = \vec{\mathbf{X}}_{-2} + \vec{\mathbf{X}}_{-3} + \vec{\mathbf{X}}_{-6} + \vec{\mathbf{X}}_{-7}$
2	0	1	2	0	0	1	1	$(2,-2) + \vec{\mathbf{X}}_{-3} = (1,-3)$
3	0	1	1	1	1	2	1	Result of borrowing from c_3
4	1	0	0	0	1	2	1	Result of carrying into c_1
5	1	0	0	1	0	1	0	Result of carrying into c_4

Lemma 4. *Suppose* $\vec{\mathbf{v}} = \sum_{i\geq 1} c_i \vec{\mathbf{X}}_{-i}$ *is* any *representation of* $\vec{\mathbf{v}} \in \mathbb{Z}^{k-1}$ *with* $c_i \geq 0$ *for all* i. *Then there is a representation* $\vec{\mathbf{v}} = \sum_{i\geq 1} c'_i \vec{\mathbf{X}}_{-i}$ *with* $c'_i \geq 0$ *for all* i *such that every block of positive coefficients has length less than* k.

Proof. Iteratively locate any block of positive coefficients $(c_{i+1}, \ldots, c_{i+j})$ with $j \geq k$ and $c_i = 0$ (we can always assume $c_0 = 0$), and carry into c_i. Since each carrying reduces $\sum_{i\geq 1} c_i$, the process terminates. □

We are now prepared to prove Theorem 2.

Proof. Uniqueness. It is easy to see that the function S_n is one-to-one on satisfying representations of the form $\sum_{i=1}^{n-1} c_i \vec{\mathbf{X}}_{-i}$ (Lemma 3), thus two such different representations cannot be equal.

Existence. We use induction as follows: $\vec{\mathbf{0}} \in \mathbb{Z}^{k-1}$ has an SR, and whenever $\vec{\mathbf{v}}$ has an SR, then, as we shall show, so do $\vec{\mathbf{v}} + \vec{\mathbf{e}}_i$ for $1 \leq i \leq k-1$ (which proves that any vector with non-negative coordinates has an SR) and $\vec{\mathbf{v}} - \vec{\mathbf{e}}_1 - \cdots - \vec{\mathbf{e}}_{k-1}$ (which then proves that all vectors have an SR). Because, from Definition 3, $\vec{\mathbf{X}}_{-i} = \vec{\mathbf{e}}_i$, for $1 \leq i \leq k-1$, and, from Equation (3), $\vec{\mathbf{X}}_{-k} = -\vec{\mathbf{e}}_1 - \cdots - \vec{\mathbf{e}}_{k-1}$, these inductive steps involve incrementing some coefficient in a satisfying representation by one, which we repair using carrying and borrowing.

In case this increment only changes a 0 to a 1, Lemma 4 applies and iterated carrying yields an SR.

Otherwise the increment yields an NSR, which we repair as follows. Denote the block of 1's and 2's by $(c_{i+1}, \ldots, c_{i+j})$ such that $c_i = c_{i+j+1} = 0$. Borrow from c_{i+p}, such that $c_{i+p} = 2$, and for any $q > i+p$, $c_q < 2$. This borrowing will create a block of k or more positive coefficients, so next carry into c_i, and continue carrying into lower-subscripted coefficients if necessary to assure all blocks of positive coefficients with lower coefficients than i are shorter than k. This borrowing and carrying can have three different outcomes:

(1) The borrowing changes no 1's to 2's beyond the coefficients in the block $(c_{i+1}, \ldots, c_{i+j})$. In this case, the coefficients have been transformed in an SR, and the process terminates.
(2) The borrowing creates a block of positive coefficients of length at least $2k$. In this case, two carrying operations remove all the 2's, and the process terminates. (This occurs in our illustration above.)
(3) The borrowing followed by one carrying step leaves at least one $c_m = 2$, for some $m > i+j$. Denote the new block of 1's and 2's by $(c_{i'+1}, \ldots, c_{i'+j'})$, where $c_{i'} = c_{i'+j'+1} = 0$ and $i'+1 \leq m \leq i'+j'$. Let M denote the largest index such that $c_M > 0$. We have moved our block of 1's and 2's closer to M; i.e., $M - (i'+j') < M - (i+j)$. Hence by induction, this process must terminate.

□

Corollary 5. *Suppose* $\vec{\mathbf{v}} = \sum_1^M c_i \vec{\mathbf{X}}_{-i}$ *and* $\vec{\mathbf{v}}' = \vec{\mathbf{v}} + \vec{\mathbf{X}}_{-p} = \sum_1^{M'} c_i' \vec{\mathbf{X}}_{-i}$ *are two SRs and* $p \leq M$*. Then* $M' - M \leq k$.

Proof. The final borrowing operation used to convert an NSR to an SR can only extend non-zero values at most k positions past c_M. □

Figure 2 suggests Corollary 5. Region D_M is completely surrounded by region D_{M+3} for $0 \leq M \leq 6$.

Theorem 2 generalizes M. W. Bunder's result [1]: "Every integer can be represented uniquely as a sum of nonconsecutive negatively subscripted Fibonacci numbers."

3. Illustrations with the Tribonacci Sequence

The sequence $\{c_i\}$ in Theorem 2 is essentially a k-Zeckendorf representation for non-negative integers so that the theorem gives a natural one-to-one correspondence between $\mathbb{Z}^{k-1}$ and $\mathbb{Z}^+$. The following table for $k = 3$ shows this, matching Figures 1–3. ($Z(n)$ is the tribonacci Zeckendorf representation of n.)

n	Z(n)	$\{c_i\}$	vector $\in \mathbb{Z}^2$
0	0	00000...	$(0,0)$
1	1	10000...	$(1,0)$
2	10	01000...	$(0,1)$
3	11	11000...	$(1,1)$
4	100	00100...	$(-1,-1)$
5	101	10100...	$(0,-1)$
6	110	01100...	$(-1,0)$
7	1000	00010...	$(2,0)$

In this Section $k = 3$, X_i is the ith tribonacci number and $\vec{\mathbf{X}}_{-n}$ we call the $-n$th *tribonacci vector*.

Definition 7. *Let* $D_n = \{\vec{\mathbf{v}} \in \mathbb{Z}^2 \mid \vec{\mathbf{v}} = \sum_{i=1}^n c_i \vec{\mathbf{X}}_{-i}\}$*, i.e., the points with an* n*-bit representation. By this definition, the number of points in* D_n *is* $|D_n| = X_{n+2}$.

Figure 1 illustrates domains $D_1, \ldots, D_7$. Figure 2 gives another view of $D_0, D_1, \ldots, D_9$ along with a spiral connecting the vectors $\vec{\mathbf{X}}_0, \vec{\mathbf{X}}_{-1}, \ldots, \vec{\mathbf{X}}_{-9}$. Figure 3 shows how these regions reflect the tribonacci recurrence: $X_n = X_{n-1} + X_{n-2} + X_{n-3}$.

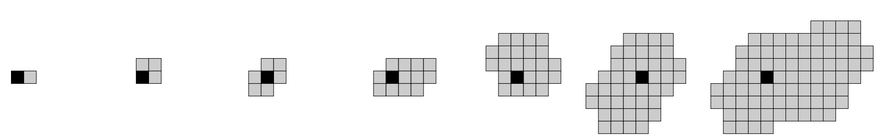

Figure 1. Regions $D_1, \ldots, D_7$. The black squares indicate $\vec{\mathbf{0}} \in \mathbb{Z}^2$.

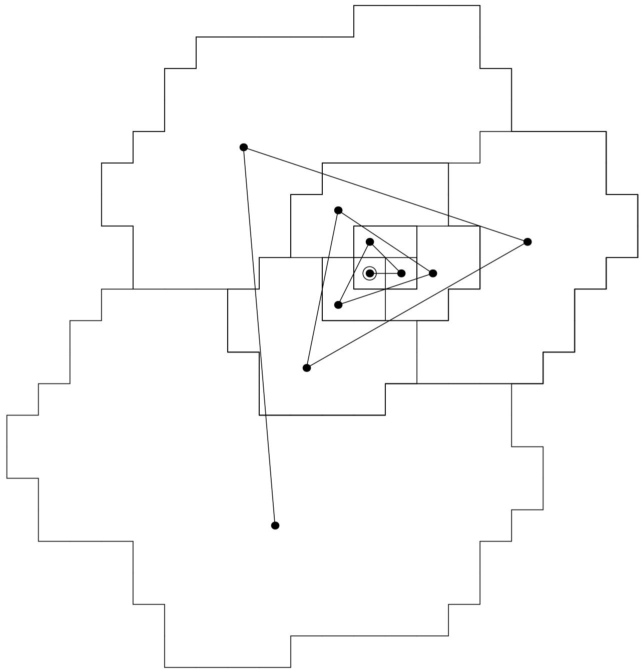

FIGURE 2. Regions $D_0, D_1, \ldots, D_9$ and the spiral connecting $\vec{\mathbf{X}}_0$, $\vec{\mathbf{X}}_{-1}$, $\ldots$, $\vec{\mathbf{X}}_{-9}$, which are indicated by black dots. The bulls-eye indicates $\vec{\mathbf{0}} = \vec{\mathbf{X}}_0$.

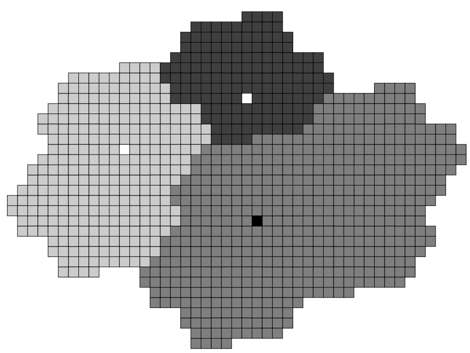

FIGURE 3. Illustrating the tribonacci recurrence: $D_{14} = D_{13} \sqcup (\vec{\mathbf{X}}_{-13} + D_{12}) \sqcup (\vec{\mathbf{X}}_{-13} + \vec{\mathbf{X}}_{-12} + D_{11})$. The black square indicates $\vec{\mathbf{0}} \in \mathbb{Z}^2$. The white squares indicate the translation vectors $\vec{\mathbf{X}}_{-13}$ and $\vec{\mathbf{X}}_{-13} + \vec{\mathbf{X}}_{-12}$.

4. Acknowledgement

We are very grateful to the anonymous referee whose thoughtful comments, suggestions, and criticism improved our paper immensely.

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MSC2010: 11B34, 11B37, 11B39

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