MULTIPLICATIVE IDENTITIES FOR BINOMIAL COEFFICIENTS

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Abstract. Identities between binomial coefficients have been extensively studied (see the references at the end of this paper). Almost without exception these are *regular multiplicative* identities, a term which is defined here. The simplest of these are called "Star of David" identities, which assert the equality of two products, each of three binomial coefficients. The purpose of this paper is to show that every regular multiplicative identity can be obtained by taking products of a suitable set Star of David identities. Starting from any given identity, a geometrical method (RMI-diagrams) is used to determine the corresponding product of Star of David identities and several examples are given.

For all integers $r, s \geq 0$ and $t = r + s$ the coefficient of $x^r y^s$ in the expansion of $(x+y)^t$ is $\frac{t!}{r!s!}$. This is a *binomial coefficient* and it will be denoted by $\begin{pmatrix} & t & \\ r & & s \end{pmatrix}$ (in preference to other notations such as $\binom{t}{r}$ and ${}^tC_r$ that have been proposed by other authors). This notation will clarify the exposition throughout. We shall refer to t as the *upper index* and r and s as the *left and right lower indices*, respectively. Notice that our notation is only meaningful if r, s, and t are non-negative integers and the upper index is the sum of the two lower indices. Since $\begin{pmatrix} & t & \\ r & & s \end{pmatrix} = \begin{pmatrix} & t & \\ s & & r \end{pmatrix}$, we may interchange the lower indices at will. Binomial coefficients have also been defined for $t < 0$ [2, p. 154], but we shall not consider these here.

Numerous papers were published at the end of the twentieth century and beginning of the twenty-first century that exhibit identities between binomial coefficients (see the list of references). For example [2] contains dozens of identities of many different kinds. Here we are concerned with *multiplicative identities*, that is, identities of the type

$$\begin{pmatrix} & t_1 & \\ r_1 & & s_1 \end{pmatrix}\begin{pmatrix} & t_2 & \\ r_2 & & s_2 \end{pmatrix}\cdots\begin{pmatrix} & t_n & \\ r_n & & s_n \end{pmatrix} = \begin{pmatrix} & w_1 & \\ u_1 & & v_1 \end{pmatrix}\begin{pmatrix} & w_2 & \\ u_2 & & v_2 \end{pmatrix}\cdots\begin{pmatrix} & w_m & \\ u_m & & v_m \end{pmatrix}. \tag{1}$$

We distinguish two sorts: first there are the *regular (multiplicative) identities* in which

(a) there is an equal number of terms on each side of the equality sign, that is $m = n$,
(b) the sets of upper indices $\{t_1, t_2, \ldots, t_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ are permutations of each other, and
(c) the sets of lower indices $\{r_1, s_1, r_2, s_2, \ldots, r_n, s_n\}$ and $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ are permutations of each other.

An example of a regular (multiplicative) identity is

$$\begin{aligned}&\begin{pmatrix} & 8 & \\ 1 & & 7 \end{pmatrix}\begin{pmatrix} & 11 & \\ 5 & & 6 \end{pmatrix}\begin{pmatrix} & 14 & \\ 4 & & 10 \end{pmatrix}\begin{pmatrix} & 9 & \\ 3 & & 6 \end{pmatrix}\begin{pmatrix} & 12 & \\ 2 & & 10 \end{pmatrix}\begin{pmatrix} & 10 & \\ 1 & & 9 \end{pmatrix}\begin{pmatrix} & 13 & \\ 5 & & 8 \end{pmatrix} \\ &\quad = \begin{pmatrix} & 12 & \\ 5 & & 7 \end{pmatrix}\begin{pmatrix} & 10 & \\ 4 & & 6 \end{pmatrix}\begin{pmatrix} & 13 & \\ 3 & & 10 \end{pmatrix}\begin{pmatrix} & 8 & \\ 2 & & 6 \end{pmatrix}\begin{pmatrix} & 11 & \\ 1 & & 10 \end{pmatrix}\begin{pmatrix} & 14 & \\ 5 & & 9 \end{pmatrix}\begin{pmatrix} & 9 & \\ 1 & & 8 \end{pmatrix}.\end{aligned} \tag{2}$$

Another example is

$$\binom{t-1}{r-1\quad s}\binom{t}{r+1\quad s-1}\binom{t+1}{r\quad s+1}=\binom{t-1}{r\quad s-1}\binom{t}{r-1\quad s+1}\binom{t+1}{r+1\quad s}. \tag{3}$$

for all r, $s \geq 1$ and $t = r + s$. This is especially important for reasons which will appear later (Section 3). It is sometimes called a *Star of David identity* [1, 3], and it will be denoted by $\mathfrak{D}(r, s)$. In general, regular multiplicative identities will be denoted by Fraktur letters.

Secondly, there exist multiplicative identities which satisfy (a) above but not (b) or (c) and so are not regular. An example is

$$\binom{r}{t\quad r-t}\binom{t}{s\quad t-s}=\binom{r}{s\quad r-s}\binom{r-s}{r-t\quad t-s} \tag{4}$$

for any integers $r \geq t \geq s \geq 0$ [2, p. 168]. (See also the comments on non-regular identities in Section 4.)

Regular and non-regular identities must be carefully distinguished since, as we shall see later, they have essentially different properties. Both regular and non-regular identities are easy to prove by substituting the values of the binomial coefficients in terms of factorials.

1. Introduction

A convenient way to display binomial coefficients is by means of a triangular array of integers called the *Pascal Triangle*

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ .&&.&&.&&.&&.&&. \end{array} \tag{5}$$

Here the $(r+1)$st term in row t of the triangle is $\binom{t}{r\quad s}$. Because of the (non-multiplicative identity often known as *Pascal's Rule*:

$$\binom{t-1}{r-1\quad s}+\binom{t-1}{r\quad s}=\binom{t}{r\quad s} \tag{6}$$

for all r, $s \geq 1$, $t = r + s$, (an equation also easily verified by substituting factorials), each entry in the Pascal Triangle is the sum of the two entries diagonally above it. Equality (3) can be displayed as part of the Pascal Triangle as in (7).

$$\begin{array}{ccccccccccccc} .&&.&&.&&.&&.&&.&&. \\ &.&&.&&\mathcal{L}&&\mathcal{R}&&.&&.& \\ .&&.&&\mathcal{R}&&\bullet&&\mathcal{L}&&.&&. \\ &.&&.&&\mathcal{L}&&\mathcal{R}&&.&&.& \\ .&&.&&.&&.&&.&&.&&. \end{array} \tag{7}$$

The term $\begin{pmatrix} & t & \\ r & & s \end{pmatrix}$ is denoted by $\bullet$, the terms on the left of (3) are denoted by $\mathcal{L}$ and terms on the right by $\mathcal{R}$. The six vertices of the two triangles form a hexagon around the entry $\bullet$. It should now be evident why (3) is sometimes called a Star of David identity.

Other regular multiplicative identities can be displayed in a similar manner. For example, consider the following example of Usiskin [7, p. 204(b)]. Here again the point $\bullet$ is $\begin{pmatrix} & t & \\ r & & s \end{pmatrix}$.

$$
\begin{array}{ccccccccccccccc}
. & & . & & . & & . & & . & & . & & . & & . \\
 & . & & . & & \mathcal{L} & & \mathcal{R} & & . & & . & & . & \\
. & & . & & . & & . & & \mathcal{L} & & \mathcal{R} & & . & & . \\
 & . & & \mathcal{R} & & . & & \bullet & & . & & \mathcal{L} & & . & \\
. & & . & & \mathcal{L} & & \mathcal{R} & & . & & . & & . & & . \\
 & . & & . & & . & & \mathcal{L} & & \mathcal{R} & & . & & . & \\
. & & . & & . & & . & & . & & . & & . & & .
\end{array} \tag{8}
$$

The points marked $\mathcal{L}$ correspond to the binomial coefficients

$$\begin{pmatrix} & t-2 & \\ r-2 & & s \end{pmatrix}\begin{pmatrix} & t+1 & \\ r-1 & & s+2 \end{pmatrix}\begin{pmatrix} & t-1 & \\ r & & s-1 \end{pmatrix}\begin{pmatrix} & t+2 & \\ r+1 & & s+1 \end{pmatrix}\begin{pmatrix} & t & \\ r+2 & & s-2 \end{pmatrix}$$

and those marked $\mathcal{R}$ to the coefficients

$$\begin{pmatrix} & t & \\ r-2 & & s+2 \end{pmatrix}\begin{pmatrix} & t-2 & \\ r-1 & & s-1 \end{pmatrix}\begin{pmatrix} & t+1 & \\ r & & s+1 \end{pmatrix}\begin{pmatrix} & t-1 & \\ r+1 & & s-2 \end{pmatrix}\begin{pmatrix} & t+2 & \\ r+2 & & s \end{pmatrix},$$

where, as always, $t = r + s$. Putting these two products equal yields a regular multiplicative identity which we shall refer to as (9).

2. RMI-diagrams

We now introduce *regular multiplicative diagrams,* or RMI-diagrams for short. Pascal Diagrams and RMI-diagrams are, in a sense, interchangeable, but an RMI-diagram is more easily used since many of the terms (such as x-axis, y-axis quadrant) are already familiar. This avoids numerous definitions for the corresponding elements in the Pascal diagram. Essentially each point (r, s) of the diagram in the Cartesian plane corresponds to the binomial coefficient $\begin{pmatrix} & t & \\ r & & s \end{pmatrix}$ in the given identity. As we shall see, the use of RMI-diagrams plays a central role in the proof of the Main Theorem 20.

Definition. *An RMI-diagram is a finite set of points with integer coordinates in the non-negative quadrant of the Cartesian plane, each point of the set having an integer label. The label attached to the point (r, s) will be denoted by $p(r, s)$. For such a labelled set to be an RMI-diagram it must satisfy the following three conditions:*

(i) *For each value of* $s \geq 0, \sum_{r=0}^{\infty} p(r,s) = 0$, *that is, the sum of the elements in each row is zero.*

(ii) *For each value of* $r \geq 0, \sum_{s=0}^{\infty} p(r,s) = 0$, *that is, the sum of the elements in each row is zero.*

(iii) *For each value of* $k \geq 0, \sum_{i=0}^{k} p(k-i,i) = 0$, *that is, the sum of the elements in each diagonal* $x + y = k$ *is zero.*

(We shall refer to the diagonal $x + y = k$ as the kth diagonal.)

(10). *Every RMI-diagram corresponds to a regular multiplicative identity.*

Explicitly, this identity is obtained from

$$\prod_{r=0}^{\infty} \prod_{s=0}^{\infty} \binom{r+s}{r \quad s}^{p(r,s)} = 1 \tag{11}$$

by taking all the terms with negative coefficients over to the right side of the equality sign. Denote the resulting identity by $\mathfrak{R}$.

To prove (10) we insert the expressions for the binomial coefficients in terms of factorials. For each term r, the term $r!$ occurs in the denominators on the the left of (11) $\sum_{s=0}^{\infty} p(r,s) = 0$ times. Hence in the expression $\mathfrak{R}$ the term $r!$ occurs the same number of times in the denominators on the left and on the right of the equality sign. The same is true for all the terms involving $s!$. Finally, the terms $(r+s)!$ for which $r + s = k$ (that is, lie on the kth diagonal) occur $\sum_{i=0}^{k} p(k-i,i) = 0$ times in (11), and so the term $k!$ occurs the same number of times on the left as on the right of $\mathfrak{R}$. We deduce $\mathfrak{R}$ is a regular multiplicative identity.

Notice that if an RMI-diagram is left invariant by the operation of reflecting in the line $x = y$ and then altering the signs of all its labels, the resulting identity is trivial in that all its terms cancel. We shall call such a diagram *antisymmetric*.

In Figure 1a we show an RMI-diagram which leads to the identity (2), and also one (Figure 1b) which leads to the identity $\mathfrak{D}(r,s)$ in (3). The latter is minimal in the sense that it has

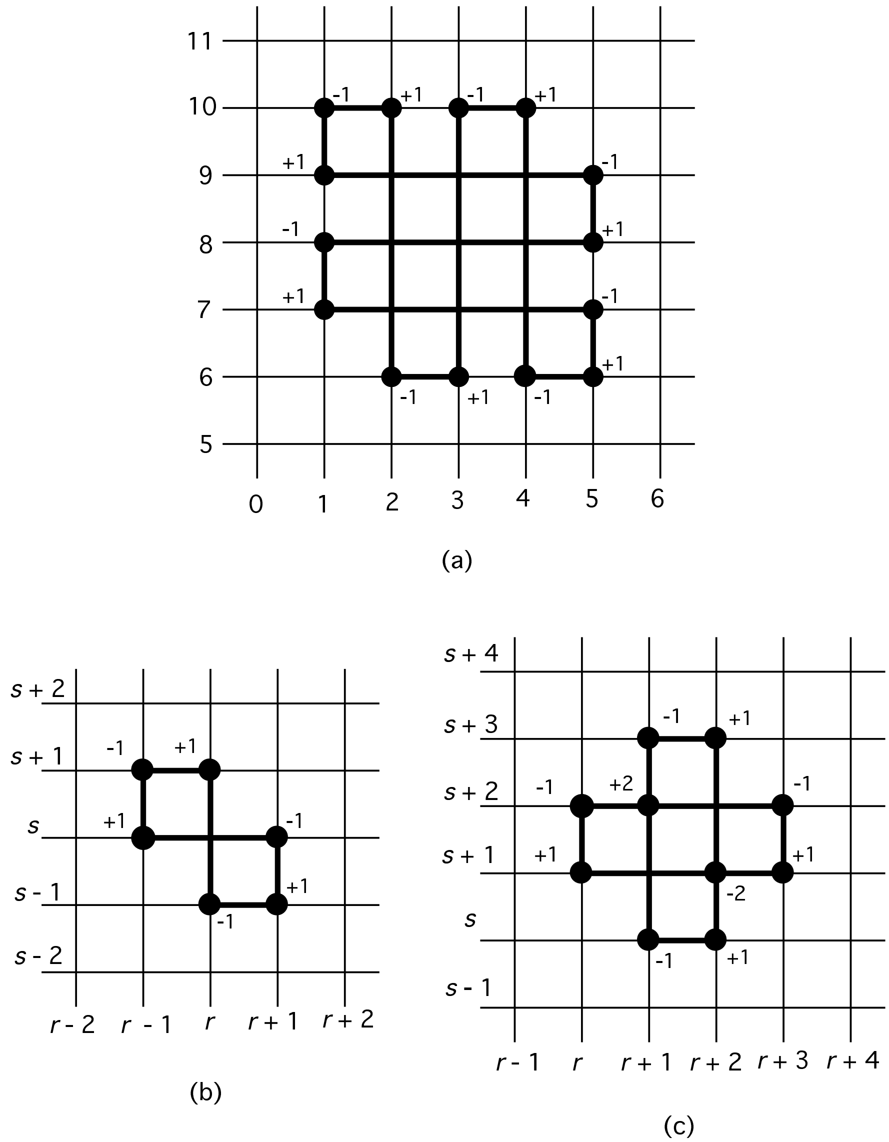

Figure 1

the smallest possible number of non-zero labels (six) and also the points lie on the smallest number of k diagonals. It is easily verified that the points of any RMI-diagram must lie on more than two diagonals, so $k \geq 3$. We also show in Figure 1c an RMI-diagram in which one point is labelled $+2$ and one point is labelled -2. This corresponds to the identity

$$\begin{aligned}\binom{t+1}{r \quad s+1}\binom{t+3}{r+1 \quad s+2}^2\binom{t+5}{r+2 \quad s+3}\binom{t+4}{r+3 \quad s+1}\binom{t+2}{r+2 \quad s}& \\ = \binom{t+2}{r \quad s+2}\binom{t+4}{r+1 \quad s+3}\binom{t+3}{r+2 \quad s+1}^2\binom{t+5}{r+3 \quad s+2}\binom{t+1}{r+1 \quad s}&\end{aligned} \tag{12}$$

where $t = r + s$. The second powers of two of the binomial coefficients correspond to the labels 2 in the RMI-diagram. In all parts of Figure 1 ignore the heavy lines at present.

As we have seen, the proof of (10) is straightforward. The converse is slightly more difficult.

(13). *Every regular multiplicative identity corresponds to an RMI-diagram.*

The difficulty here is that we cannot simply copy down the lower indices in the given identity and interpret them as coordinates of points in an RMI-diagram. For example, in (1), with $m = n$ the points (r_1, s_1), $(r_2, s_2), \ldots, (r_n, s_n)$, (u_1, v_1), $(u_2, v_2), \ldots, (u_n, v_n)$ are *not*, in general, the points of an RMI-diagram. A specific numerical example is the diagram corresponding to the regular identity

$$\binom{4}{3 \quad 1}\binom{5}{3 \quad 2}\binom{6}{2 \quad 4} = \binom{4}{2 \quad 2}\binom{5}{4 \quad 1}\binom{6}{3 \quad 3}. \tag{14}$$

This is shown in Figure 2 with $p(3,1) = +1$, $p(3,2) = +1$, $p(2,4) = +1$, $p(2,2) = -1$, $p(4,1) = -1$, $p(3,3) = -1$. This is not an RMI-diagram because conditions (i) and (ii) are violated, but (14) can be trivially modified (in this case by interchanging the numbers 1 and 3 in the first term and the numbers 4 and 1 in the fifth term) to yield (15) which corresponds to an RMI-diagram, namely $\mathfrak{D}(2,\ 3)$ (see Figure 1(b)).

$$\binom{4}{1 \quad 3}\binom{5}{3 \quad 2}\binom{6}{2 \quad 4} = \binom{4}{2 \quad 2}\binom{5}{1 \quad 4}\binom{6}{3 \quad 3}. \tag{15}$$

We shall show that every regular multiplicative identity can be modified in such a way (by interchanging lower indices) that the corresponding diagram is an RMI-diagram.

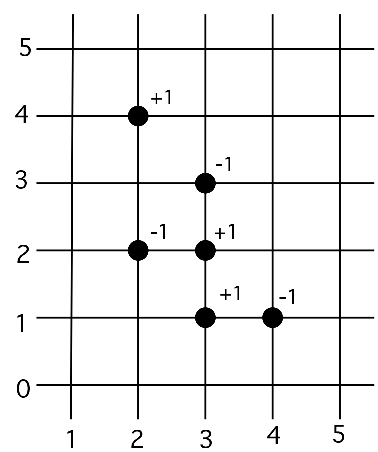

Figure 2

Consider the identity $\mathfrak{R}$ corresponding to equation (1) with $m = n$. The first pair of lower indices on the left of (1) is $(r_1,\ s_1)$. Since the lower indices on the right of the equality form a permutation of those on the left, s_1 must occur as a lower index on the right. Assume it is a right lower index (if it is a left lower index, simply interchange the lower indices). Renumbering, if necessary, denote the pair containing s_1 by $(u_1,\ v_1)$ with $s_1 = v_1$. Now u_1 must occur as a lower index on the left of the equality. Assume it is a left lower index, and renumbering if necessary, denote the pair containing u_1 by $(r_2,\ s_2)$. Proceed in a similar manner to obtain a sequence of pairs

$$(r_1, s_1), (u_1, v_1), (r_2, s_2), (u_2, v_2), (r_3, s_3), \ldots, \tag{16}$$

with $s_1 = v_1$, $u_1 = r_2$, $s_2 = v_2$, $u_2 = r_3$, etc. This sequence may terminate in the sense that the first pair $(r_1,\ s_1)$ appears again. In this case pick a pair of lower indices that has not yet been used, proceed as before. We obtain a number of sequences of type (16) containing $2n$ points in all. For each $i = 1, 2, \ldots, n$, assign labels $p(r_i, s_i) = +1$ and $p(u_i, v_i) = -1$ and then each of the sequences in (16) lists, in order, the vertices of a polygon. Each such polygon has its edges alternately parallel to the x- and y-axes and the ends of each edge are labelled $+1$ and -1. The union of these polygons is a (possibly non-connected) $2n$-gon which clearly satisfies conditions (i) and (ii) of an RMI-diagram. Condition (iii) is also satisfied since the set of upper indices on the left of (1) is a permutation of those on the right. Hence this union is an RMI-diagram corresponding to the identity $\mathfrak{R}$ from which we started. We have proved (13)

A regular identity in form (1) is said to be in *standard form* if the lower indices on the two sides are in the form (16), so the identity leads to polygons as described above.

Together with (10), statement (13) establishes a correspondence between regular identities and RMI-diagrams. This is not, however, a bijection since although an RMI-diagram corresponds to a unique multiplicative identity, the converse is not true. For example, if an identity $\mathfrak{R}$ corresponds to an RMI-diagram R, then R' (the reflection of R in the line $x = y$) also corresponds to the same identity $\mathfrak{R}$ but with the lower indices in each term interchanged. Similarly the diagram $-\mathsf{R}$ (in which the signs of all the labels are changed) also corresponds to $\mathfrak{R}$ but with the two sides of the equality interchanged. Not only this, but if S is an RMI-diagram which is antisymmetric, then the corresponding identity S is trivial and $\mathsf{R} + \mathsf{S}$ corresponds to the same identity $\mathfrak{R}$.

In Figure 1a the heavy lines indicate the polygon constructed as in the proof of (16) from the identity (2). It is not hard to see that every RMI-diagram can be written as the union of polygons of the type described above, though sometimes this is not unique. We can now explain the heavy lines in the RMI-diagrams of Figures 1, 3 and 4; these are the sides of the polygons whose union is the RMI-diagram under consideration.

3. The Set of RMI-diagrams and the Main Theorem

Given an RMI-diagram R with labels $p(r, s)$ we define a *translate* R' of R by the vector (a, b) as the RMI-diagram of points $(r+a, s+b)$ (for all $(r, s) \in \mathsf{R}$) and labels $p'(r+a, s+b) = p(r, s)$. Of course a and b must be chosen so the translated diagram lies in the non-negative quadrant. In Figures 1b and 1c the various diagrams obtained by varying r and s are clearly translates of each other. Thus every RMI-diagram corresponds, by translation, to an infinity of regular multiplicative identities. In general, non-regular multiplicative identities do not have this property; the transformation $r \to r + a, s \to s + b, t \to t + a + b$ applied to a non-regular identity usually leads to a statement which is false.

The *scalar product* of an RMI-diagram R by an integer c is the same diagram with the same points as R but with the label $cp(r, s)$ attached to each point of R. The *sum* of two RMI-diagrams R_1 and R_2 is the RMI-diagram of which the set of points is the union of the set of points of R_1 and the set of points of R_2 and the label of each point is the sum of the labels of these points in R_1 and R_2. It is worth noting that the sum of the diagrams for R_1 and R_2 corresponds to the identity obtained by equating the product of the left sides of the identities $\mathfrak{R}_1$ and $\mathfrak{R}_2$ with the right sides of these two identities. We shall describe this simply as the *product* of the identities $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

With the operations of addition and scalar multiplication, RMI-diagrams form a lattice which we shall denote by $\mathbb{L}$. We now determine a set of elements which generate this lattice, and hence determine a basis for $\mathbb{L}$.

Theorem 17. *Every RMI-diagram can be expressed as the sum of* D*-diagrams* $\mathsf{D}(r,s)$ *(translates of the Star of David diagram).*

Three examples (21), (23), and (25) illustrating this theorem are given at the end of this section.

Proof. Let $\mathsf{S}(r,s)$ denote the square arrangement of labelled points shown in Figure 3. The top right corner of the square is (r,s). $\mathsf{S}(r,s)$ is <u>*not*</u> an RMI-diagram since it does not satisfy

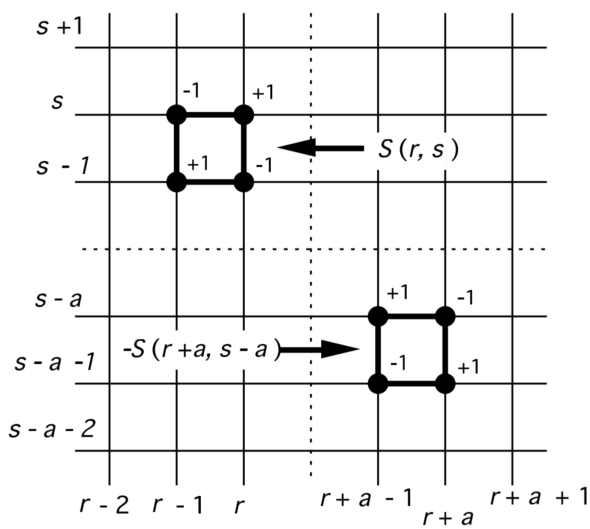

Figure 3

condition (iii) but may be made into one by adding another square, situated diagonally to the right and below it, namely $-\mathsf{S}(r+a,s-a)$ where $a<s$, see Figure 3. Further, we note that if $a=1$ then in $\mathsf{S}(r,s)-\mathsf{S}(r+1,s-1)$ the labels at the point $(r,s-1)$ cancel and we obtain an RMI-diagram, namely the D-diagram $\mathsf{D}(r,s-1)$. Further, for any $a<s$,

$$\begin{aligned}\mathsf{S}(r,s)-\mathsf{S}(r+a,s-a) &= \sum_{k=0}^{a-1}[\mathsf{S}(r+k,s-k)-\mathsf{S}(r+k+1,s-k-1)]\\ &= \sum_{k=0}^{a-1}\mathsf{D}(r+k,s-k-1).\end{aligned}$$

In particular, for the largest value of a, namely $a=s-1$, $\mathsf{S}(r,s)-\mathsf{S}(r+s-1,1)$ is a sum of D-diagrams.

Let R be any given RMI-diagram. Its points lie on a finite set of diagonals, say $x+y=b$, $x+y=b+1$, $x+y=b+2,\ldots,x+y=c$, and no others. Note that $c-b\geq 2$ since, as remarked above, when defining the Star of David identity, there is no RMI-diagram all of whose points lie on two consecutive diagonals. Let $\mathsf{T}(r,s,c)$ be the RMI-diagram defined by

$$\begin{aligned} -\mathsf{T}(r,s,c) &= \sum_{r+s=c} p(r,s)(\mathsf{S}(r,s) - \mathsf{S}(r+s-1)) \\ &= \sum_{r+s=c} p(r,s)\mathsf{S}(r,s) \end{aligned}$$

since $\sum\limits_{r+s=c} p(r,s)$, which is the coefficient of $S(r+s-1)$, is zero by condition (iii) in the definition of an RMI-diagram. Now $\mathsf{T}(r,s,c)+\mathsf{R}$ has no points on $x+y=c$ since each point of R lying on $x+y=c$ has label $p(r,s)$ and this is cancelled by the same point in $\mathsf{T}(r,s,c)$ which has label $-p(r,s)$. Thus we have eliminated all the points of R lying on the diagonal $x+y=c$ (though in doing so we may have introduced extra points on the diagonals $x+y=c-1$ and $x+y=c-2$). Now eliminate all the points of R lying on $x+y=c-1$ (together with any extra points introduced by the operation just described) by using $\mathsf{T}(r,s,c-1)$ in a similar manner. Proceed thus, eliminating the points of R lying on $x+y=c$, $x+y=c-1$, $x+y=c-2,\ldots,x+y=b+2$. We can go no further since, at this stage, all the points lie on the diagonals $x+y=b$ and $x+y=b+1$. But this is vacuous since there is no RMI-diagram with all points lying on two consecutive diagonals. Hence we have expressed our given RMI-diagram as a sum of T-diagrams and so to the sum of D-diagrams. This proves Theorem 17. □

Since summation of RMI-diagrams corresponds to the products of the corresponding identities, we deduce the following.

(18)**.** *Every regular multiplicative identity can be expressed as the product of a set of* D-*identities.*

Thus D-diagrams generate the lattice $\mathbb{L}$ of RMI-diagrams. They are not, however, independent since $\mathsf{D}(r,s) = -\mathsf{D}(s,r)$. Therefore we may restrict attention to diagrams D for which $r \geq s$. We shall show that, with this restriction, the diagrams D are independent.

Suppose that there exist non-zero integers a_i such that the D-diagram

$$\mathsf{R} = \sum_{i=1}^{n} a_i \mathsf{D}(r_i, s_i) \tag{19}$$

is vacuous, that is, contains no point with non-zero coefficient. Let $r_i + s_i = c_i$ and choose terms $\mathsf{D}(r_i, s_i)$ for which c_i is as large as possible. From all such terms choose that for which r_i is as small as possible. Denote this term by $\mathsf{D}(r', s')$. Then $\mathsf{D}(r', s')$ contains the point $(r', s'+1)$ with label $p(r', s'+1) = 1$. Moreover, by choice of c_i and r_i no other terms in (19) contain this point. Hence in R, the label $p(r', s'+1) = a_i \neq 0$, and so R is not vacuous. From this contradiction we deduce that the D-diagrams $\mathsf{D}(r,s)$ with $r \geq s$ are independent. Hence we have the following theorem.

Main Theorem 20. *The* D-*diagrams* $\mathsf{D}(r,s)$ $(r \geq s \geq 0)$ *form a basis for the lattice* $\mathbb{L}$ *of RMI-diagrams. Consequently all regular multiplicative identities for binomial coefficients can be written as a unique product of Star of David identities* $\mathsf{D}(r,s)$ *with* $(r \geq s \geq 0)$.

This theorem is illustrated by the following three examples.

Example 21. *The RMI-diagram of Figure 1c is clearly the sum of* D*-diagrams* $\mathsf{D}(r+1, s+1)$ *and* $\mathsf{D}(r+2, s+2)$*. We deduce that the identity* (12) *is the product of the two* D*-identities* $\mathfrak{D}(r+1, s+1)$ *and* $\mathfrak{D}(r+2, s+2)$ *namely*

$$\binom{t+1}{r \quad s+1}\binom{t+2}{r+2 \quad s}\binom{t+3}{r+1 \quad s+2} = \binom{t+1}{r+1 \quad s}\binom{t+2}{r \quad s+2}\binom{t+3}{r+2 \quad s+1}$$

and

$$\binom{t+3}{r+1 \quad s+2}\binom{t+4}{r+3 \quad s+1}\binom{t+5}{r+2 \quad s+3} = \binom{t+3}{r+2 \quad s+1}\binom{t+4}{r+1 \quad s+3}\binom{t+5}{r+3 \quad s+2}.$$

That the product of these is (12) is easily verified.

Example 22. *The RMI-diagram* R *in Figure 4 corresponds to the regular identity*

$$\binom{10}{1 \quad 9}\binom{8}{2 \quad 6}\binom{11}{3 \quad 8}\binom{9}{4 \quad 5}\binom{12}{5 \quad 7} = \binom{11}{2 \quad 9}\binom{9}{3 \quad 6}\binom{12}{4 \quad 8}\binom{10}{5 \quad 5}\binom{8}{1 \quad 7}. \tag{23}$$

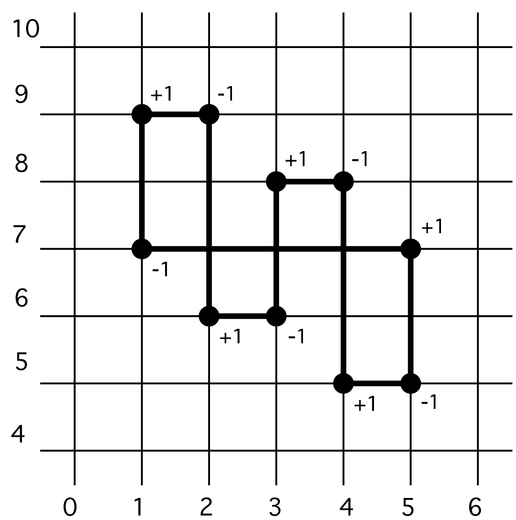

Figure 4

Using the procedure described in the proof of Theorem 17, R can be written as the sum of five D-diagrams:

$$\mathsf{R} = \mathsf{D}(4, 7) + \mathsf{D}(2, 8) + \mathsf{D}(3, 7) + \mathsf{D}(4, 6) + \mathsf{D}(2, 7).$$

The corresponding D-identities are:

$$\begin{aligned}
\mathfrak{D}(4,7)\text{ is }&\binom{11}{3\quad 8}\binom{10}{4\quad 6}\binom{12}{5\quad 7}=\binom{12}{4\quad 8}\binom{11}{5\quad 6}\binom{10}{3\quad 7}\\
\mathfrak{D}(2,8)\text{ is }&\binom{10}{1\quad 9}\binom{9}{2\quad 7}\binom{11}{3\quad 8}=\binom{11}{2\quad 9}\binom{10}{3\quad 7}\binom{9}{1\quad 8}\\
\mathfrak{D}(3,7)\text{ is }&\binom{10}{2\quad 8}\binom{9}{3\quad 6}\binom{11}{4\quad 7}=\binom{11}{3\quad 8}\binom{10}{4\quad 6}\binom{9}{2\quad 7}\\
\mathfrak{D}(4,6)\text{ is }&\binom{10}{3\quad 7}\binom{9}{4\quad 5}\binom{11}{5\quad 6}=\binom{11}{4\quad 7}\binom{10}{5\quad 5}\binom{9}{3\quad 6}\\
\mathfrak{D}(2,7)\text{ is }&\binom{9}{1\quad 8}\binom{8}{2\quad 6}\binom{10}{3\quad 7}=\binom{10}{2\quad 8}\binom{9}{3\quad 6}\binom{8}{1\quad 7}.
\end{aligned}\tag{24}$$

The product of these five identities contains fifteen terms on each side of the equality sign, but ten of these cancel, namely

$$\binom{9}{1\quad 8},\binom{9}{2\quad 7},\binom{9}{3\quad 6},\binom{10}{4\quad 6},\binom{10}{3\quad 7}^2,\binom{10}{2\quad 8},\binom{11}{5\quad 6},\binom{11}{3\quad 8}\text{ and }\binom{11}{4\quad 7},$$

and the result is the identity (23).

Example 25. *Identity* $\mathfrak{R}$ *of* (2) *corresponds to the RMI-diagram* R *of Figure 1a, and it is easy to see that*

$$\mathsf{R}=\mathsf{D}(2,9)+\mathsf{D}(4,9)+\mathsf{D}(2,7)+\mathsf{D}(4,7).$$

Hence $\mathfrak{R}$ can be expressed as the product of four corresponding Star of David identities, $\mathfrak{D}(2,9),\mathfrak{D}(4,9),\mathfrak{D}(2,7)$ and $\mathfrak{D}(4,7)$. Here five terms cancel from each side of the identity.

4. History and Comments

Because of his famous triangle, it is often thought that the study of binomial coefficients started with Blaise Pascal (1623–1662). This is not so, for the triangle has been discovered many times, for example by Omar Khayyám (1048–1131) in Persia, Yang Hui (1258–1298) in China and by Petrus Apianus (1495–1552) in Leipzig. The name of Pascal was not associated with the triangle until the 18th century. For more history see [6].

Recent works seem to be concerned, almost exclusively, with regular identities. Gould [1] states, and proves, about twenty such identities each involving the equality of three, four, or five binomial coefficients. That these are regular is not immediately apparent due to the notation that he uses. He remarks on the following interesting generalization: Let $A_1, A_2, A_3, \dots,$ be any sequence of non-zero numbers, and write

$$[n]! = A_1A_2\cdots A_{n-1}A_n, \quad [0]! = 1.$$

Then if we substitute $[n]!$ for $n!$ in the definition of binomial coefficients, all his identities remain true. Notice that this only applies to *regular* identities but not to non-regular identities. It thus provides another illustration of the different properties of these two types of identities.

Also the idea of identities being in "standard form" seems to have not been previously mentioned. As we have seen in (13), standard form is necessary whether Pascal diagrams or RMI-diagrams are used. It is also essential for the geometrical method of checking identities mentioned in [3] to be applicable. That all regular identities can be put in standard form is the main purpose of (13).

Many authors assume that identities are independent of translations. This is because they only (tacitly) consider regular identities. Also, in diagrams based on the Pascal triangle, the possibility of points having multiplicity greater than 1 (or, in our terminology, labels greater than 1) is not mentioned. Because of this many possible identities are overlooked.

Apart from [7], the algebra of regular identities (as in our Section 3) is hardly mentioned. Usiskin [7] has a hint of such when he mentions adding two identities represented (in the Pascal triangle) by hexagons to obtain another identity.

Above, we have only dealt with questions concerning the equality of two products of n binomial coefficients. However it is possible for three products, each of n binomial coefficients to be equal. Let us call such a relation a *triple identity*. Triple identities are mentioned by Usiskin and he gives an explicit example with $n = 9$ in [7, p. 207]. He also explains how to construct r-tuple identities, that is the equality of r products of binomial coefficients, for all $r \geq 3$. All these are regular identities, but non-regular triple identities also exist, for example

$$\binom{a+b+c}{c \quad a+b}\binom{a+b}{a \quad b} = \binom{a+b+c}{a \quad b+c}\binom{b+c}{b \quad c} = \binom{a+b+c}{b \quad c+a}\binom{c+a}{c \quad a} \tag{26}$$

where a, b, c are any non-negative integers. Identity (26) is interesting because all three terms are equal to the trinomial coefficient

$$\binom{a+b+c}{a \quad b \quad c} = \frac{(a+b+c)!}{a!b!c!},$$

see [2, p.168]. However the topic of r-tuple identities with $r \geq 3$ has yet to be fully explored.

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